

## A CLOSED-FORM EXPRESSION FOR SECOND-ORDER RECURRENCES

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ABSTRACT. This paper introduces a closed-form expression for the second-order recurrence relation  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ , in which  $c_1$  and  $c_2$  are fixed constants and the value of two arbitrary terms  $a_{n-p}$  and  $a_{n-q}$  are known where  $p$  and  $q$  are positive integers and  $p > q$ . This expression is

$$a_n = \frac{S(p+1)}{S(p-q+1)} a_{n-q} - (-c_2)^{(p-q)} \frac{S(q+1)}{S(p-q+1)} a_{n-p}$$

where

$$S(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} c_1^{n-2i} c_2^i.$$

### 1. INTRODUCTION

The Fibonacci and Lucas numbers are famous for possessing fabulous properties. For example, the ratio of Fibonacci numbers converges to the golden ratio, and the sum and differences of Fibonacci numbers are Fibonacci numbers. Nevertheless, many of these properties can be stated and proved for a much more general class of sequences, namely, second-order recurrences [1, 2], defined by  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ , in which  $c_1$  and  $c_2$  are fixed constants. Lucas in his foundation paper [3] studied these sequences.

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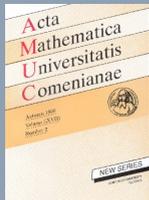


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One of the difficulties in using recurrence relations is the need to compute all intermediate values up to the required term. In general, to overcome this for a  $d$ -order linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d}$$

we can state  $a_n = \sum_{i=1}^d k_i r_i^n$ , where  $r_i$  are the roots of the characteristic polynomial

$$P(t) = t^d - c_1 t^{d-1} - c_2 t^{d-2} - \dots - c_d = \prod_{i=1}^d (t - r_i)$$

and the constants  $k_i$  are evaluated by looking at the initial values [4]. For the Fibonacci numbers, this solution reduces to the well-known Binet formula.

We recall the polynomial identity

$$\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} (xy)^i (x+y)^{n-2i} = x^n + x^{n-1}y + \dots + y^n = \frac{x^{n+1} - y^{n+1}}{x - y}$$

observed in [7]. Sury stated that since  $\sum_{i \geq 0} \binom{n-i}{i}$  satisfies the same recursion as the Fibonacci sequence and starts with  $F_2$  and  $F_3$ , it follows by induction that  $\sum_{i \geq 0} \binom{n-i}{i} = F_{n+1}$ , and he used this polynomial to obtain the Binet formula [6]. He also used this identity to derive trigonometric expressions for the Fibonacci and Lucas numbers [5].

In this paper, the second-order recurrences  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$  are studied and this polynomial identity is applied to the general solution  $a_n = k_1 r_1 + k_2 r_2$  to derive an explicit expression for  $a_n$  in terms of two arbitrary known terms  $a_{n-p}$  and  $a_{n-q}$ .

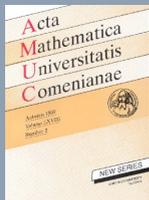


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## 2. CLOSED-FORM EXPRESSION FOR SECOND-ORDER RECURRENCES

Let  $c_1$  and  $c_2$  be fixed constants. Consider the linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

For this recursion, the characteristic polynomial is  $p(t) = t^2 - c_1 t - c_2$ . Let  $\alpha$  and  $\beta$  be its roots; hence  $a_n = k_1 \alpha^n + k_2 \beta^n$  for certain constants  $k_1$  and  $k_2$ . From the equalities

$$a_{n-p} = k_1 \alpha^{n-p} + k_2 \beta^{n-p}$$

$$a_{n-q} = k_1 \alpha^{n-q} + k_2 \beta^{n-q}$$

we can solve for  $k_1$  and  $k_2$  and obtain finally that

$$a_n = \frac{\alpha^p - \beta^p}{\alpha^{p-q} - \beta^{p-q}} a_{n-q} - \frac{(\alpha\beta)^{p-q} (\alpha^q - \beta^q)}{\alpha^{p-q} - \beta^{p-q}} a_{n-p}.$$

In order to simplify this expression further, we rearrange it to form the aforementioned polynomial identity. We also apply the equality  $\alpha\beta = -c_2$ ; thus

$$a_n = \frac{\alpha^p - \beta^p}{\alpha^{p-q} - \beta^{p-q}} a_{n-q} - \frac{(-c_2)^{p-q} \frac{\alpha^q - \beta^q}{\alpha - \beta}}{\frac{\alpha^{p-q} - \beta^{p-q}}{\alpha - \beta}} a_{n-p}.$$

As  $\alpha + \beta = c_1$  and  $\alpha\beta = -c_2$ , the polynomial identity results in

$$\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} c_1^{n-2i} c_2^i.$$

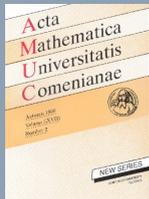


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Thus if we denote  $S(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} c_1^{n-2i} c_2^i$ , we have

$$a_n = \frac{S(p+1)}{S(p-q+1)} a_{n-q} - (-c_2)^{(p-q)} \frac{S(q+1)}{S(p-q+1)} a_{n-p}.$$

1. Benjamin A. T. and Quinn J. J., *The Fibonacci Numbers Exposed More Discretely*, Mathematics Magazine, **76** (2003), 182–192.
2. Kalman D. and Mena R., *The Fibonacci Numbers-Exposed*, Mathematics Magazine, **76** (2003), 167–181.
3. Lucas E., *Théorie des fonctions numériques simplement périodiques*, Amer. J. Math. **1** (1878), 184–240, 289–321.
4. Niven I., Zuckerman H. S. and Montgomery H. L., *An Introduction to the Theory of Numbers*, John Wiley and Sons, Inc., New York, Chichester, Brisbane, Toronto and Singapore, (1991), 197–206.
5. Sury B., *Trigonometric Expressions for Fibonacci and Lucas Numbers*, Acta Math. Univ. Comenianae, **79** (2010), 199–208.
6. ———, *A parent of Binet's formula?*, Mathematics Magazine, **77** (2004), 308–310.
7. ———, *On a curious polynomial identity*, Nieuw Arch. Wisk., **11** (1993), 93–96.

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