Acta Math. Univ. Comenianae
Vol. LXXX, 2 (2011), pp. 277–285

ISOMETRIES AND ISOMORPHISMS IN QUASI-BANACH ALGEBRAS

ZHIHUA WANG and WANXIONG ZHANG

Abstract. In this paper, we prove the Hyers-Ulam-Rassias stability of isometries and of homomorphisms for additive functional equations in quasi-Banach algebras. This is applied to investigate isomorphisms between quasi-Banach algebras.

1. Introduction and preliminaries

Stability is investigated when one concerns whether a small error of parameters causes a large deviation of the solution. Generally speaking, given a function which satisfies a functional equation approximately called an *approximate solution*, we ask: Is there a solution of this equation which is close to the approximate solution in some accuracy? An ealier work was done by Hyers [**11**] in order to answer Ulam's question ([**20**]) on approximately additive mappings. Later there have been given lots of results on stability in the Hyers-Ulam sense or some generalized sense (see books and papers [**1, 3, 8, 9, 12, 17, 18**] and references therein).

G. Z. Eskandani [**7**] established the general solution and investigated the Hyers-Ulam-Rassias stability of the following functional equation

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} mx_i\right) \tag{1.1}$$

in quasi-Banach spaces, where $m \in \mathbb{N}$ and $m \geq 2$. The stability of isometries in norms spaces and Banach spaces was investigated in several papers [**4, 6, 10, 13**]. However, C. Park and Th. M. Rassias [**15**] proved the Hyers-Ulam stability of isometric additive functional equations in quasi-Banach spaces. C. Park [**16**] studied the Hyers-Ulam stability of homomorphisms in quasi-Banach algebras. Recently, M. S. Moslehian and Gh. Sadeghi [**14**] have proved the Hyers-Ulam-Rassias stability of linear mappings in quasi-Banach modules associated to the Cauchy functional equation and a generalized Jensen functional equation.

Received January 25, 2011.
2010 *Mathematics Subject Classification.* Primary 46B03, 47B48, 39B72.
Key words and phrases. Hyers-Ulam-Rassias stability; isometry; isomorphism; quasi-Banach algebra.
Project No.CDJZR10 10 00 08 supported by the Fundamental Research Funds for the Central Universities.

The main purpose of this paper is to study the Hyers-Ulam-Rassias stability of equation (1.1). More precisely, we prove the Hyers-Ulam-Rassias stability of isometric additive functional equations (1.1) in quasi-Banach algebras. Furthermore, we investigate the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to additive functional equations (1.1). This is applied to investigate isomorphisms between quasi-Banach algebras.

We now give some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 (cf. [**5, 19**]). Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and for all $x \in X$.
(3) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\|+\|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a *quasi-norm* on X. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete *quasi-normed space.*

A *quasi-norm* $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a *quasi-Banach space* is called a *p-Banach space.*

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem [**19**] (see also [**5**]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Definition 1.2 (cf. [**2**]). Let $(X, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(X, \|\cdot\|)$ is called a *quasi-normed algebra* if X is an algebra and there is a constant $C > 0$ such that $\|xy\| \leq C\|x\|\|y\|$ for all $x, y \in X$.

A *quasi-Banach algebra* is a complete *quasi-normed algebra.* If the quasi-norm $\|\cdot\|$ is a p-norm, then the *quasi-Banach algebra* is called a *p-Banach algebra.*

Definition 1.3 (cf. [**15**]). Let X and Y be quasi-Banach algebras with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. An additive mapping $A\colon X \to Y$ is called an isometric additive mapping if the additive mapping $A\colon X \to Y$ satisfies

$$\|A(x) - A(y)\|_Y = \|x - y\|_X$$

for all $x, y \in X$.

2. Stability of isometric additive mappings in quasi-Banach algebras

Throughout this section and Section 3, assume that X is a quasi-normed algebra with quasi-norm $\|\cdot\|_X$ and that Y is a p-Banach algebra with p-norm $\|\cdot\|_Y$. Let

K be the modulus of concavity of $\|\cdot\|_Y$. For convenience, we use the following abbreviation for a given mapping $f\colon X \to Y$:

$$Df(x_1, \cdots, x_m) = \sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) - 2f\left(\sum_{i=1}^{m} mx_i\right)$$

for all $x_j \in X$ $(1 \le j \le m)$. We prove the Hyers-Ulam-Rassias stability of the isometric additive functional equation (1.1) in quasi-Banach algebras.

Theorem 2.1. *Let* $\varphi\colon X^m \to [0, \infty)$ *be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{m^n} \varphi(m^n x_1, \cdots, m^n x_m) = 0 \tag{2.1}$$

$$\tilde{\varphi}(x) := \sum_{i=0}^{\infty} \frac{1}{m^{ip}} (\varphi(m^i x, 0, \cdots, 0))^p < \infty \tag{2.2}$$

for all $x, x_j \in X$ $(1 \le j \le m)$. *Suppose that a mapping* $f\colon X \to Y$ *satisfies*

$$\|Df(x_1, \cdots, x_m)\|_Y \le \varphi(x_1, \cdots, x_m) \tag{2.3}$$

$$|\ \|f(x)\|_Y - \|x\|_X\ | \le \varphi(\underbrace{x, \cdots, x}_{m-times}) \tag{2.4}$$

for all $x, x_j \in X$ $(1 \le j \le m)$. *Then there exists a unique isometric additive mapping* $A\colon X \to Y$ *such that*

$$\|f(x) - A(x)\|_Y \le \frac{1}{m} [\tilde{\varphi}(x)]^{\frac{1}{p}} \tag{2.5}$$

for all $x \in X$.

Proof. By the Eskandani's theorem [**7**, Theorem 2.2], it follows from (2.1), (2.2) and (2.3) that there exists a unique additive mapping $A\colon X \to Y$ satisfying (2.5). The additive mapping $A\colon X \to Y$ is given by

$$A(x) := \lim_{n\to\infty} \frac{1}{m^n} f(m^n x) \tag{2.6}$$

for all $x \in X$.

It follows from (2.4) that

$$\begin{aligned} |\ \|\frac{1}{m^n} f(m^n x)\|_Y - \|x\|_X\ | &\le \frac{1}{m^n}\ |\ \|f(m^n x)\|_Y - \|m^n x\|_X\ | \\ &\le \frac{1}{m^n} \varphi(\underbrace{m^n x, \cdots, m^n x}_{m-times}) \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So

$$\|A(x)\|_Y = \lim_{n\to\infty} \|\frac{1}{m^n} f(m^n x)\|_Y = \|x\|_X$$

for all $x \in X$. Since $A\colon X \to Y$ is additive,

$$\|A(x) - A(y)\|_Y = \|A(x - y)\|_Y = \|x - y\|_X$$

for all $x \in X$. So the mapping $A\colon X \to Y$ is an isometry. Thus the mapping $A\colon X \to Y$ is a unique isometric additive mapping satisfying (2.5). This completes the proof of the theorem. □

Theorem 2.2. *Let $\phi\colon X^m \to [0,\infty)$ be a mapping such that*

$$\lim_{n\to\infty} m^n \phi(\frac{x_1}{m^n},\cdots,\frac{x_m}{m^n}) = 0 \tag{2.7}$$

$$\tilde{\phi}(x) := \sum_{i=1}^{\infty} m^{ip}(\phi(\frac{x}{m^i},0,\cdots,0))^p < \infty \tag{2.8}$$

for all $x, x_j \in X$ $(1 \le j \le m)$. Suppose that a mapping $f\colon X \to Y$ satisfies

$$\|Df(x_1,\cdots,x_m)\|_Y \le \phi(x_1,\cdots,x_m) \tag{2.9}$$

$$|\ \|f(x)\|_Y - \|x\|_X\ | \le \phi(\underbrace{x,\cdots,x}_{m-times}) \tag{2.10}$$

for all $x, x_j \in X$ $(1 \le j \le m)$. Then there exists a unique isometric additive mapping $A\colon X \to Y$ such that

$$\|f(x) - A(x)\|_Y \le \frac{1}{m}[\tilde{\phi}(x)]^{\frac{1}{p}} \tag{2.11}$$

for all $x \in X$.

Proof. By the Eskandani's theorem [**7**, Theorem 2.3], it follows from (2.7), (2.8) and (2.9) that there exists a unique additive mapping $A\colon X \to Y$ satisfying (2.11). The additive mapping $A\colon X \to Y$ is given by

$$A(x) := \lim_{n\to\infty} m^n f(\frac{x}{m^n}) \tag{2.12}$$

for all $x \in X$.

By (2.10), we have

$$\begin{aligned} |\ \|m^n f(\frac{x}{m^n})\|_Y - \|x\|_X\ | &\le m^n\ |\ \|f(\frac{x}{m^n})\|_Y - \|\frac{x}{m^n}\|_X\ | \\ &\le m^n \varphi(\underbrace{\frac{x}{m^n},\cdots,\frac{x}{m^n}}_{m-times}) \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. By (2.12), we obtain

$$\|A(x)\|_Y = \lim_{n\to\infty} \|m^n f(\frac{x}{m^n})\|_Y = \|x\|_X$$

for all $x \in X$. Hence

$$\|A(x) - A(y)\|_Y = \|A(x-y)\|_Y = \|x-y\|_X$$

for all $x \in X$. So the additive mapping $A\colon X \to Y$ is an isometry. This completes the proof of the theorem. □

Corollary 2.1. *Let θ, r_j $(1 \le j \le m)$ be non-negative real numbers such that $r_j > 1$ or $0 < r_j < 1$. Suppose that a mapping $f\colon X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_m)\|_Y \le \theta \sum_{i=1}^{m} \|x_i\|_X^{r_i}$$
$$| \, \|f(x)\|_Y - \|x\|_X \, | \le \theta \sum_{i=1}^{m} \|x\|_X^{r_i}$$

for all $x, x_j \in X$ $(1 \le j \le m)$. Then there exists a unique isometric additive mapping $A\colon X \to Y$ such that

$$\|f(x) - A(x)\|_Y \le \frac{\theta}{|m^p - m^{pr_1}|^{\frac{1}{p}}} \|x\|_X^{r_1}$$

for all $x \in X$.

Proof. The result follows from the proofs of Theorems 2.1 and 2.2. □

3. Stability of homomorphisms in quasi-Banach algebras

We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras, associated to the additive functional equation (1.1).

Theorem 3.1. *Suppose that a mapping $f\colon X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_m)\|_Y \le \varphi(x_1, \cdots, x_m) \tag{3.1}$$

$$\|f(xy) - f(x)f(y)\|_Y \le \psi(x, y) \tag{3.2}$$

for all $x, y, x_j \in X$ $(1 \le j \le m)$, where $\varphi\colon X^m \to [0, \infty)$ satisfies (2.1) and (2.2), and $\psi\colon X \times X \to [0, \infty)$ satisfies the following

$$\lim_{n\to\infty} \frac{1}{m^n} \psi(m^n x, m^n y) = 0 \tag{3.3}$$

for all $x, y \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H\colon X \to Y$ such that

$$\|f(x) - H(x)\|_Y \le \frac{1}{m} [\tilde{\varphi}(x)]^{\frac{1}{p}} \tag{3.4}$$

for all $x \in X$.

Proof. By Theorem 2.1, there exists a unique additive mapping $H\colon X \to Y$ satisfying (3.4). The additive mapping $H\colon X \to Y$ is given by

$$H(x) := \lim_{n\to\infty} \frac{1}{m^n} f(m^n x) \tag{3.5}$$

for all $x \in X$. By the same reasoning as in the proof of Theorem of [**17**], the mapping $H\colon X \to Y$ is $\mathbb{R}$-linear.

It follows from (3.2) that

$$\begin{aligned}\|H(xy)-H(x)H(y)\|_Y &= \lim_{n\to\infty}\frac{1}{m^{2n}}\|f(m^{2n}xy)-f(m^nx)f(m^ny)\|_Y\\ &\le \lim_{n\to\infty}\frac{1}{m^{2n}}\psi(m^nx,m^ny)=0\end{aligned}$$

for all $x, y \in X$. Hence, we get

$$H(xy) = H(x)H(y)$$

for all $x, y \in X$. Thus the mapping $H\colon X \to Y$ is a unique homomorphism satisfying (3.4). This completes the proof of the theorem. □

Theorem 3.2. *Suppose that a mapping $f\colon X \to Y$ satisfies*

$$\|Df(x_1,\cdots,x_m)\|_Y \le \phi(x_1,\cdots,x_m) \tag{3.6}$$

$$\|f(xy)-f(x)f(y)\|_Y \le \Psi(x,y) \tag{3.7}$$

for all $x, y, x_j \in X$ $(1 \le j \le m)$, where $\phi\colon X^m \to [0,\infty)$ satisfies (2.7) *and* (2.8), *and $\Psi\colon X\times X \to [0,\infty)$ satisfies the following*

$$\lim_{n\to\infty} m^n\Psi(\frac{x}{m^n},\frac{y}{m^n}) = 0 \tag{3.8}$$

for all $x, y \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H\colon X \to Y$ such that

$$\|f(x)-H(x)\|_Y \le \frac{1}{m}[\tilde{\phi}(x)]^{\frac{1}{p}} \tag{3.9}$$

for all $x \in X$.

Proof. By Theorem 2.2, there exists a unique additive mapping $H\colon X \to Y$ satisfying (3.9). The additive mapping $H\colon X \to Y$ is given by

$$H(x) := \lim_{n\to\infty} m^n f(\frac{x}{m^n}) \tag{3.10}$$

for all $x \in X$. By the same reasoning as in the proof of Theorem of [**17**], the mapping $H\colon X \to Y$ is $\mathbb{R}$-linear.

It follows from (3.8) that

$$\begin{aligned}\|H(xy)-H(x)H(y)\|_Y &= \lim_{n\to\infty} m^{2n}\|f(\frac{xy}{m^n\cdot m^n})-f(\frac{x}{m^n})f(\frac{y}{m^n})\|_Y\\ &\le \lim_{n\to\infty} m^{2n}\Psi(\frac{x}{m^n},\frac{y}{m^n}) = 0\end{aligned}$$

for all $x, y \in X$. Hence, we get

$$H(xy) = H(x)H(y)$$

for all $x, y \in X$. Thus the mapping $H\colon X \to Y$ is a unique homomorphism satisfying (3.9). This completes the proof of the theorem. □

Corollary 3.1. *Let θ, δ be non-negative real numbers and let r_j $(1 \le j \le m)$, s_1, s_2 be non-negative real numbers such that $r_j > 1$, $s_1, s_2 > 2$ or $0 < r_j < 1$, $s_1, s_2 < 2$. Suppose that a mapping $f\colon X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_m)\|_Y \le \theta \sum_{i=1}^{m} \|x_i\|_X^{r_i} \tag{3.11}$$

$$\|f(xy) - f(x)f(y)\|_Y \le \delta(\|x\|_X^{s_1} + \|y\|_X^{s_2}) \tag{3.12}$$

for all $x, y, x_j \in X$ $(1 \le j \le m)$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then there exists a unique homomorphism $H\colon X \to Y$ such that

$$\|f(x) - H(x)\|_Y \le \frac{\theta}{|m^p - m^{pr_1}|^{\frac{1}{p}}} \|x\|_X^{r_1}$$

for all $x \in X$.

Proof. The result follows from the proofs of Theorems 3.1 and 3.2. □

Corollary 3.2. *Let θ, δ be non-negative real numbers and let r_j $(1 \le j \le m)$, s_1, s_2 be non-negative real numbers such that $\sum_{i=1}^{m} r_i > 1$, $s_1 + s_2 > 2$ or $\sum_{i=1}^{m} r_i < 1$, $s_1 + s_2 < 2$ and $r_j \ne 0$ for some j $(2 \le j \le m)$. Suppose that a mapping $f\colon X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_m)\|_Y \le \theta \prod_{i=1}^{m} \|x_i\|_X^{r_i} \tag{3.13}$$

$$\|f(xy) - f(x)f(y)\|_Y \le \delta \|x\|_X^{s_1} \|y\|_X^{s_2} \tag{3.14}$$

for all $x, y, x_j \in X$ $(1 \le j \le m)$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping $f\colon X \to Y$ is a homomorphism.

Proof. The result follows from the proofs of Theorems 3.1 and 3.2. □

4. Isomorphisms between quasi-Banach algebras

Throughout this section, assume that X is a quasi-Banach algebra with quasi-norm $\|\cdot\|_X$ and unit e and that Y is a p-Banach algebra with p-norm $\|\cdot\|_Y$ and unit e'. Let K be the modulus of concavity of $\|\cdot\|_Y$.

We investigate isomorphisms between quasi-Banach algebras associated to the additive functional equation (1.1).

Theorem 4.1. *Suppose that $f\colon X \to Y$ is a bijective mapping satisfying* (3.1) *such that*

$$f(xy) = f(x)f(y) \tag{4.1}$$

for all $x, y \in X$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim_{n\to\infty} \frac{1}{m^n} f(m^n e) = e'$, then the mapping $f\colon X \to Y$ is an isomorphism.

Proof. By Theorem 3.1, there exists a homomorphism $H\colon X \to Y$ satisfying (3.4). The mapping $H\colon X \to Y$ is given by

$$H(x) := \lim_{n\to\infty} \frac{1}{m^n} f(m^n x) \tag{4.2}$$

for all $x \in X$.

By (4.1), we have

$$\begin{aligned} H(x) = H(ex) &= \lim_{n\to\infty} \frac{1}{m^n} f(m^n ex) = \lim_{n\to\infty} \frac{1}{m^n} f(m^n e \cdot x) \\ &= \lim_{n\to\infty} \frac{1}{m^n} f(m^n e) f(x) = e' f(x) = f(x) \end{aligned}$$

for all $x \in X$. So the bijective mapping $f\colon X \to Y$ is an isomorphism. This completes the proof of the theorem. □

Theorem 4.2. *Suppose that $f\colon X \to Y$ is a bijective mapping satisfying (3.6) and (4.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim\limits_{n\to\infty} m^n f(\frac{e}{m^n}) = e'$, then the mapping $f\colon X \to Y$ is an isomorphism.*

Proof. By Theorem 3.2, there exists a homomorphism $H\colon X \to Y$ satisfying (3.9). The mapping $H\colon X \to Y$ is given by

$$H(x) := \lim_{n\to\infty} m^n f(\frac{x}{m^n}) \tag{4.3}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 4.1. This completes the proof of the theorem. □

Corollary 4.1. *Let θ, r_j $(1 \le j \le m)$ be non-negative real numbers such that $r_j > 1$ or $0 < r_j < 1$. Suppose that a bijective mapping $f\colon X \to Y$ satisfies (3.11) and (4.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$ and $\lim\limits_{n\to\infty} m^n f(\frac{e}{m^n}) = e'$ or $\lim\limits_{n\to\infty} \frac{1}{m^n} f(m^n e) = e'$, then the mapping $f\colon X \to Y$ is an isomorphism.*

Proof. The result follows from the proofs of Theorems 4.1 and 4.2. □

Acknowledgment. The authors are very grateful to the referees for their helpful comments and suggestions.

References

1. Agarwal R. P., Xu B. and Zhang W., *Stability of functional equations in single variable*, J. Math. Anal. Appl. **288** (2003), 852–869.
2. Almira J. M. and Luther U., *Inverse closedness of approximation algebras*, J. Math. Anal. Appl. **314** (2006), 30-44.
3. Aoki T., *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
4. Baker J., *Isometries in normed spaces*, Amer. Math. Monthly **78** (1971), 655–658.
5. Benyamini Y. and Lindenstrauss J., *Geometric Nonlinear Functional Analysis*, vol. 1, Amer. Math. Soc. Colloq. Publ., vol. 48, Amer. Math. Soc., Providence, RI, 2000.
6. Dolinar G., *Generalized stability of isometries*, J. Math. Anal. Appl. **242** (2000), 39–56.

7. Eskandani G. Z., *On the Hyers-Ulam-Rassias stability of an additive functional equation in quasi-Banach spaces*, *J. Math. Anal. Appl.* **345** (2008), 405–409.
8. Forti G. L., *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), 143–190.
9. Găvruta P., *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
10. Gevirtz J., *Stability of isometries on Banach spaces*, Proc. Amer. Math. Soc. **89** (1983), 633–636.
11. Hyers D. H., *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
12. Jung S. M., *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.
13. Kalton N., *An elementary example of a Banach space not isomorphic to its complex conjugate*, Canad. Math. Bull. **38** (1995), 218–222.
14. Moslehian M. S. and Sadeghi Gh., *Stability of linear mappings in quasi-Banach modules*, Math. Inequal. Appl. **11** (2008), 549–557.
15. Park C. and Rassias Th. M., *Isometric additive mappings in quasi-Banach spaces*, Nonlinear Funct. Anal. Appl. **12** (2007), 377–385.
16. Park C., *Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras*, Bull. Sci. Math. **132** (2008), 87–96.
17. Rassias Th. M., *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
18. ______, *Functional Equations, Inequalities and Applications*, Kluwer Academic, Dordrecht, 2003.
19. Rolewicz S., *Metric Linear Spaces*, PWN-Polish Sci. Publ., Warszawa, Reidel, Dordrecht, 1984.
20. Ulam S. M., *Problems in Modern Mathematics, Chapter VI*, Science Editions, Wiley, New York, 1964.

Zhihua Wang, School of Science, Hubei University of Technology, Wuhan, Hubei 430068, P. R. China

Wanxiong Zhang, College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, P. R. China