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# AN INTEGRODIFFERENTIAL EQUATION WITH FRACTIONAL DERIVATIVES IN THE NONLINEARITIES

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Abstract. An integrodifferential equation with fractional derivatives in the nonlinearities is studied in this article, and some sufficient conditions for existence and uniqueness of a solution for the equation are established by contraction mapping principle.

## 1. Introduction

This article is concerned with the existence and uniqueness of a solution of the following integrodifferential equation with fractional derivatives in the nonlinearities:

$$
\begin{aligned}
u''(t) = {} & Au(t) + f\Big(t, u(t), {}^c D^{\alpha_1} u(t), \cdots, {}^c D^{\alpha_m} u(t)\Big) \\
& + \int_0^t g\Big(t, s, u(s), {}^c D^{\beta_1} u(s), \cdots, {}^c D^{\beta_n} u(s)\Big) \mathrm{d}s, \quad t > 0, \qquad (1) \\
u(0) = {} & u_0 \in X, \qquad u'(0) = u_1 \in X,
\end{aligned}
$$

where $A$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \geq 0$ of bounded linear operators on a Banach space $X$ with norm $\|\cdot\|$, $f$ and $g$ are nonlinear mappings from $\mathbb{R}^+ \times X^m$ to $X$ and $\mathbb{R}^+ \times \mathbb{R}^+ \times X^n$ to $X$, respectively, $0 < \alpha_i, \beta_j < 1$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$, $u_0$ and $u_1$ are given initial data in $X$.

Recently, fractional order differential equations and systems have been payed much attention, of examples, the monograph of Kilbas et al. [**10**], and the papers by Anguraj et al. [**1**], Benchohra et al. [**2**]–[**4**], Guo and Liu [**5**]–[**7**], Hernandez [**8**], Hernandez et al. [**9**] Kirane et al. [**11**], Tatar [**12**]–[**15**] and the references therein.

Applying the Banach contraction principle, we obtain a result of uniqueness of a solution for problem (1). To simplify our task, we will treat the following simpler

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problem

$$
\begin{aligned}
u''(t) = {} & Au(t) + f\Big(t, u(t), {}^c D^{\alpha} u(t)\Big) \\
& + \int_0^t g\Big(t, s, u(s), {}^c D^{\beta} u(s)\Big) \mathrm{d}s, \quad t > 0, \\
u(0) = {} & u_0 \in X, \qquad u'(0) = u_1 \in X.
\end{aligned} \tag{2}
$$

The general case can be derived easily.

## 2. Preliminaries

Let us recall a basic definition in fractional calculus, which can be found in the literature.

**Definition 2.1.** The Caputo fractional derivative of order $0 < \alpha < 1$ is defined by

$$
{}^c D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t) dt, \tag{3}
$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Now list the following hypotheses for convenience

(H1) $A$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators in the Banach space $X$.

The associated sine family $S(t), t \in \mathbb{R}$ is defined by

$$
S(t)x := \int_0^t C(s) x \mathrm{d}s, \qquad t \in \mathbb{R}, x \in X. \tag{4}
$$

For $C(t)$ and $S(t)$, it is known (see [**16**]) that there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$
|C(t)| \leq M \, \mathrm{e}^{\omega|t|}, \qquad |S(t) - S(t_0)| \leq M \Big| \int_{t_0}^t \mathrm{e}^{\omega|s|} \, \mathrm{d}s \Big|, \qquad t, t_0 \in \mathbb{R}. \tag{5}
$$

Let $X_A = D(A)$ endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$.

(H2) $f \colon \mathbb{R}^+ \times X_A \times X \to X$ is continuously differentiable,

(H3) $g \colon \mathbb{R}^+ \times \mathbb{R}^+ \times X_A \times X \to X$ is continuous and continuously differentiable with respect to its first variable,

(H4) $f$, $f'$ (the total derivative of $f$),$g$ and $g_1$ (the partial derivative of $g$ with respect to its first variable) are Lipschitz continuous with respect to the last two variables, that is

$$
\begin{aligned}
\|f(t, x_1, y_1) - f(t, x_2, y_2)\| &\leq L_f \big(\|x_1 - x_2\|_A + \|y_1 - y_2\|\big), \\
\|f'(t, x_1, y_1) - f'(t, x_2, y_2)\| &\leq L_{f'} \big(\|x_1 - x_2\|_A + \|y_1 - y_2\|\big), \\
\|g(t, s, x_1, y_1) - g(t, s, x_2, y_2)\| &\leq L_g \big(\|x_1 - x_2\|_A + \|y_1 - y_2\|\big), \\
\|g_1(t, s, x_1, y_1) - g_1(t, s, x_2, y_2)\| &\leq L_{g_1} \big(\|x_1 - x_2\|_A + \|y_1 - y_2\|\big)
\end{aligned} \tag{6}
$$

for some positive constants $L_f$, $L_{f'}$ ,$L_g$ and $L_{g_1}$.

**Lemma 2.2** ([**16**]). *Assume that* (H1) *is satisfied. Then*

(i) $S(t)X \subset E, \ \ t \in \mathbb{R}$,
(ii) $S(t)E \subset X_A, \ \ t \in \mathbb{R}$,
(iii) $(\mathrm{d}/\mathrm{d}t)C(t)x = AS(t)x, x \in E, \ \ t \in \mathbb{R}$,
(iv) $(\mathrm{d}^2/\mathrm{d}t^2)C(t)x = AC(t)x = C(t)Ax, \ \ x \in X_A, \ t \in \mathbb{R}$, *where*

$$E := \{x \in X : C(t)x \text{ is once continuously differentiable on } \mathbb{R}\}. \tag{7}$$

**Lemma 2.3** ([**16**]). *Assume that* (H1) *holds,* $v\colon \mathbb{R} \to X$ *is a continuously differentiable function and* $q(t) = \int_0^t S(t-s)v(s)\mathrm{d}s$. *Then,* $q(t) \in X_A$, $q'(t) = \int_0^t C(t-s)v(s)\mathrm{d}s$ *and* $q''(t) = \int_0^t C(t-s)v'(s)\mathrm{d}s + C(t)v(0) = Aq(t) + v(t)$.

**Definition 2.4.** A function $u(\cdot) \in C^2(I, X)$ is called a classical solution of problem (2) if $u(t) \in X_A$ satisfies the equation in (2) and the initial conditions are verified.

**Definition 2.5.** A continuously differentiable solution of the integrodifferential equation

$$\begin{aligned} u(t) = C(t)u_0 + S(t)u_1 &+ \int_0^t S(t-s)f\Big(s, u(s), {}^c D^\alpha u(s)\Big)\mathrm{d}s \\ &+ \int_0^t S(t-s)\int_0^s g\Big(s, \tau, u(\tau), {}^c D^\beta u(\tau)\Big)\mathrm{d}\tau\mathrm{d}s \end{aligned} \tag{8}$$

is called a mild solution of problem (2).

## 3. Main results

In this section, the theorem of existence and uniqueness of a solution for equation (2) will be given.

**Theorem 3.1.** *Assume that* (H1)–(H4) *hold. If* $u_0 \in X_A$, $u_1 \in E$ *and* $L_f < 1$, *then there exist* $T > 0$ *and a unique function* $u\colon (0,T) \to X, u \in C((0,T), X_A) \cap C^2((0,T), X)$ *which satisfies* (2).

*Proof.* For $t \in (0,T)$, define a mapping

$$\begin{aligned} (Ku)(t) := C(t)u_0 + S(t)u_1 &+ \int_0^t S(t-s)f\Big(s, u(s), {}^c D^\alpha u(s)\Big)\mathrm{d}s \\ &+ \int_0^t S(t-s)\int_0^s g\Big(s, \tau, u(\tau), {}^c D^\beta u(\tau)\Big)\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{9}$$

It follows from $u_0 \in X_A$ and $AC(t)u_0 = C(t)Au_0$ that $C(t)u_0 \in X_A$. Clearly, $S(t)u_1 \in X_A$ because $u_1 \in E$ and $S(t)E \subset X_A$ (see (ii) of Lemma 2.2). Moreover, by Lemma 2.3, (H2) and (H3), we know that both integral terms in (9) are in $X_A$.

Therefore, $Ku \in C((0,T), X_A)$. By Lemma 2.3, we have

$$
\begin{aligned}
(AKu)(t) = {} & C(t)Au_0 + AS(t)u_1 + \int_0^t C(t-s)f'\Big(s, u(s), {}^c D^\alpha u(s)\Big)\mathrm{d}s \\
& + C(t)f(0, u_0, {}^c D^\alpha u_0) - f\Big(t, u(t), {}^c D^\alpha u(t)\Big) \\
& + \int_0^t C(t-s)\Big[\int_0^s g_1\Big(s,\tau,u(\tau), {}^c D^\beta u(\tau)\Big)\mathrm{d}\tau \\
& + g\Big(s,s,u(s), {}^c D^\beta u(s)\Big)\Big]\mathrm{d}s \\
& - \int_0^t g\Big(t,\tau,u(\tau), {}^c D^\beta u(\tau)\Big)\mathrm{d}\tau, \qquad t \in (0,T).
\end{aligned} \tag{10}
$$

Differentiating (9), we get

$$
\begin{aligned}
(Ku)'(t) = {} & AS(t)u_0 + C(t)u_1 + \int_0^t C(t-s)f\Big(s, u(s), {}^c D^\alpha u(s)\Big)\mathrm{d}s \\
& + \int_0^t C(t-s)\int_0^s g\Big(s,\tau,u(\tau), {}^c D^\beta u(\tau)\Big)\mathrm{d}\tau\mathrm{d}s, \qquad t \in (0,T).
\end{aligned} \tag{11}
$$

Hence, $Ku \in C^1((0,T), X)$ and $K$ maps $C^1$ into $C^1$.

It is claimed that $K$ is a contraction on $C^1$ endowed with the metric

$$
\rho(u,v) := \sup_{0\le t\le T}\big(\|u(t)-v(t)\| + \|A(u(t)-v(t))\| + \|u'(t)-v'(t)\|\big). \tag{12}
$$

For $u, v \in C^1$, it can be derived that

$$
\begin{aligned}
& \|(Ku)(t) - (Kv)(t)\| \\
& \quad \le \int_0^t |S(t-s)|\Big[L_f\big(\|u(s)-v(s)\|_A + \|{}^c D^\alpha u(s) - {}^c D^\alpha v(s)\|\big) \\
& \qquad\qquad + \int_0^s L_g\big(\|u(\tau)-v(\tau)\|_A + \|{}^c D^\beta u(\tau) - {}^c D^\beta v(\tau)\|\big)\mathrm{d}\tau\Big]\mathrm{d}s \\
& \quad \le \int_0^t M\int_0^{t-s} \mathrm{e}^{\omega\tau}\,\mathrm{d}\tau\Big[L_f\big(\|u(s)-v(s)\|_A \\
& \qquad\qquad + \frac{1}{\Gamma(1-\alpha)}\int_0^s (s-\tau)^{-\alpha}\|u'(\tau)-v'(\tau)\|\mathrm{d}\tau\big) \\
& \qquad\qquad + \int_0^s L_g\big(\|u(\tau)-v(\tau)\|_A \\
& \qquad\qquad + \frac{1}{\Gamma(1-\beta)}\int_0^\tau (\tau-\sigma)^{-\beta}\|u'(\sigma)-v'(\sigma)\|d\sigma\big)\mathrm{d}\tau\Big]\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leq M\int_0^T \mathrm{e}^{\omega\tau}\,\mathrm{d}\tau \int_0^t \Bigg[L_f\big(\|u(s)-v(s)\|_A \\
&\qquad\qquad + \frac{s^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{0\leq t\leq T}\|u'(t)-v'(t)\|\big) \\
&\quad + \int_0^s L_g\big(\|u(\tau)-v(\tau)\|_A + \frac{\tau^{1-\beta}}{\Gamma(2-\beta)}\sup_{0\leq t\leq T}\|u'(t)-v'(t)\|\big)\mathrm{d}\tau\Bigg]\mathrm{d}s \\
&\leq M\int_0^T \mathrm{e}^{\omega\tau}\,\mathrm{d}\tau \int_0^t \Bigg[L_f \max\Big\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\Big\}\rho(u,v) \\
&\qquad\qquad + L_g \max\Big\{1,\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\}\rho(u,v)s\Bigg]\mathrm{d}s \\
&\leq M\int_0^T \mathrm{e}^{\omega\tau}\,\mathrm{d}\tau \max\Big\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)},\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\}(L_f+L_gT/2)T\rho(u,v),
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&\|(AKu)(t)-(AKv)(t)\| \\
&\quad\leq \int_0^t M\,\mathrm{e}^{\omega(t-s)}\,L_{f'}\big(\|u(s)-v(s)\|_A + \|{}^cD^\alpha u(s) -{}^c D^\alpha v(s)\|\big)\mathrm{d}s \\
&\qquad + L_f\big(\|u(t)-v(t)\|_A + \|{}^cD^\alpha u(t) -{}^c D^\alpha v(t)\|\big) \\
&\qquad + \int_0^t M\,\mathrm{e}^{\omega(t-s)}\Big[\int_0^s L_{g_1}\big(\|u(\tau)-v(\tau)\|_A+\|{}^cD^\beta u(\tau)-{}^cD^\beta v(\tau)\|\big)\mathrm{d}\tau \\
&\qquad + L_g\big(\|u(s)-v(s)\|_A + \|{}^cD^\beta u(s) -{}^c D^\beta v(s)\|\big)\Big]\mathrm{d}s \\
&\qquad + \int_0^t L_g\big(\|u(\tau)-v(\tau)\|_A + \|{}^cD^\beta u(\tau) -{}^c D^\beta v(\tau)\|\big)\mathrm{d}\tau \\
&\quad\leq \int_0^T M\,\mathrm{e}^{\omega(T-s)}\,\mathrm{d}sL_{f'}\max\Big\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\Big\}\rho(u,v) \\
&\qquad + L_f\max\Big\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\Big\}\rho(u,v) \\
&\qquad + \int_0^T M\,\mathrm{e}^{\omega(T-s)}\,\mathrm{d}s\Bigg[L_{g_1}\max\Big\{1,\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\}T \\
&\qquad + L_g\max\Big\{1,\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\}\Bigg]\rho(u,v) + L_g\max\Big\{1,\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\}T\rho(u,v) \\
&\quad\leq \max\Big\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)},\frac{T^{1-\beta}}{\Gamma(2-\beta)}\Big\} \\
&\qquad \cdot\Big[\int_0^T M\,\mathrm{e}^{\omega(T-s)}\,\mathrm{d}s\Big(L_{f'}+L_{g_1}T+L_g\Big)+L_gT+L_f\Big]\rho(u,v),
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
&\|(Ku)'(t)-(Kv)'(t)\| \\
&\quad\le \int_0^t M\,\mathrm{e}^{\omega(t-s)}\Big[L_f\big(\|u(s)-v(s)\|_A+\|{}^cD^\alpha u(s)-{}^c D^\alpha v(s)\|\big)\mathrm{d}s \\
&\quad\quad+\int_0^s L_g\big(\|u(\tau)-v(\tau)\|_A+\|{}^cD^\beta u(\tau)-{}^c D^\beta v(\tau)\|\big)\mathrm{d}\tau\Big]\mathrm{d}s \qquad (15)\\
&\quad\le \int_0^T M\,\mathrm{e}^{\omega(T-s)}\,\mathrm{d}s\max\left\{1,\frac{T^{1-\alpha}}{\Gamma(2-\alpha)},\frac{T^{1-\beta}}{\Gamma(2-\beta)}\right\}\big(L_f+L_gT\big)\rho(u,v),
\end{aligned}
$$

The above three relations (13)–(15) and condition $L_f < 1$ guarantee that for sufficiently small $T$, $K$ is a contraction on $C^1$. Therefore, there exists a unique mild solution $u \in C^1$. Clearly, $u \in C^2((0,T),X)$ and satisfies the problem (2). This completes the proof. □

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