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STABILITY IN TOTALLY NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY

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Abstract. In this paper, we use a fixed point technique and the concept of large contractions to prove asymptotic stability results of the zero solution of a class of the totally nonlinear neutral differential equation with functional delay. The study concerns the equation

$$x'(t) = -a(t)h(x(t)) + c(t)x'(t - r(t)) + b(t)G(x(t), x(t - r(t))),$$

which has proved very challenging in the theory of Liapunov's direct method. The stability results are obtained by means of Krasnoselskii-Burton's theorem and they improve and generalize the works of Burton [**7**], and Derrardjia, Ardjouni and Djoudi [**16**].

1. Introduction

The Liapunov's direct method has been the main tool for the study of stability properties of a wide variety of ordinary, functional, partial differential and Volterra integral equations for more than 100 years. Nevertheless, the application of this method to problems of stability in differential and Volterra integral equations with delay has encountered serious obstacles if the delay is unbounded or if the equation has unbounded terms (see [**3**], [**7**]–[**9**], [**11**]–[**15**], [**22**]) and it does seem that other ways need to be investigated. Recently, Burton and Furumochi have noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [**2**]–[**3**], [**7**]–[**17**], [**19**]–[**20**] and [**22**]). The most striking object is that the fixed point method does not only solve the problem on stability but has a significant advantage over Liapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [**3**]). While it remains an art to construct a Liapunov's functional when it exists, a fixed point method, in one step, yields existence, uniqueness and stability. All we need for using the fixed point method, is a complete metric space, a suitable fixed point theorem and an elementary variation of parameters formula to solve problems that have resisted to Liapunov's method.

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Below, we present a study which concerns a totally nonlinear neutral differential equation with functional delay. In our situation it is necessary to invert the differential equation to obtain a mapping equation suitable for the fixed point theory. Unfortunately, our equation does not contain a linear term and the variation of parameter cannot be used. So, we resort to the classical idea of adding and substracting a linear term for the considered equation.

Clearly, our equation is a totally nonlinear neutral differential equation with variable delay expressed as follows

$$x'(t) = -a(t)h(x(t)) + c(t)x'(t-r(t)) + b(t)G(x(t), x(t-r(t))), \quad t \geq 0, \tag{1.1}$$

with an assumed initial function

$$x(t) = \psi(t), \qquad t \in [m_0, 0],$$

where $\psi \in C([m_0, 0], \mathbb{R})$ and $m_0 = \inf\{t - r(t) : t \geq 0\}$. Throughout this paper, we assume that $a, b \in C(\mathbb{R}_+, \mathbb{R})$ with $a(t) \geq 0$, $c \in C^1(\mathbb{R}_+, \mathbb{R})$, $h \colon \mathbb{R} \to \mathbb{R}$ is continuous with respect to its argument. We assume that $h(0) = 0$ and $r \in C^2(\mathbb{R}_+, \mathbb{R})$ such that

$$r'(t) \neq 1, \qquad t \in \mathbb{R}_+. \tag{1.2}$$

We also assume that $G(x, y)$ is locally Lipschitz continuous in x and y. That is, there are positive constants N_1 and N_2 so that if $|x|, |y| \leq \sqrt{3}/3$, then

$$|G(x, y) - G(z, w)| \leq N_1 \|x - z\| + N_2 \|y - w\| \quad \text{and} \quad G(0, 0) = 0.$$

Special cases of equation (1.1) have been recently considered and studied under various conditions and with several methods. Particularly, Burton [**7**] investigated the asymptotic stability and stability by using Krasnoselskii-Burton fixed point theorem for the following equation

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t - r(t)), \qquad t \geq 0.$$

By letting $h(x) = x^3$ and $G(x, y) = y^3$ in equation (1.1), the present authors [**16**] have studied, the asymptotic stability and the stability by using Krasnoselskii-Burton fixed point theorem, under appropriate conditions, of the following equation

$$x'(t) = -a(t)x^3(t) + c(t)x'(t - r(t)) + b(t)x^3(t - r(t)), \qquad t \geq 0,$$

and improved the results claimed in [**7**].

By using Krasnoselskii-Burton fixed point theorem, our purpose here is to give asymptotic stability and stability results for the totally nonlinear neutral differential equation with variable delay (1.1).

In Section 2, we present the inversion of equation (1.1) and we state the hybrid Krasnoselskii-Burton's fixed point theorem. For details on Krasnoselskii theorem we refer the reader to [**21**]. We present our main results on stability in Section 3 and 4. The results presented in this paper improve and generalize the main results in [**7, 16**].

2. The inversion and the fixed point theorem

One crucial step in the investigation of an equation using fixed point theory involves the construction of a suitable fixed point mapping. For that end we must invert (1.1) to obtain an equivalent integral equation from which we derive the needed mapping. During the process, an integration by parts has to be performed on the neutral term $x'(t-r(t))$. Unfortunately, when doing this, a derivative $r'(t)$ of the delay appears on the way, and so we have to support it.

Lemma 1. *Suppose* (1.2) *holds.* $x(t)$ *is a solution of equation* (1.1) *if and only if*

$$
\begin{aligned}
x(t) = {} & \left[\psi(0) - \frac{c(0)}{1-r'(0)}\psi(-r(0))\right] \mathrm{e}^{-\int_0^t a(u)\mathrm{d}u} \\
& + \int_0^t a(s)(Hx)(s)\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
& + \frac{c(t)}{1-r'(t)}x(t-r(t)) - \int_0^t \mu(s)x(s-r(s))\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
& + \int_0^t b(s)G(x(s),x(s-r(s)))\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s,
\end{aligned}
\tag{2.1}
$$

where

$$
\mu(t) = \frac{(c'(t)+a(t)c(t))(1-r'(t)) + c(t)r''(t)}{(1-r'(t))^2},
\tag{2.2}
$$

and

$$
(Hx)(t) = x(t) - h(x(t)).
\tag{2.3}
$$

Proof. Let $x(t)$ be a solution of equation (1.1). Rewrite (1.1) as

$$
\begin{aligned}
x'(t) + a(t)x(t) = {} & a(t)x(t) - a(t)h(x(t)) + c(t)x'(t-r(t)) \\
& + b(t)G(x(t),x(t-r(t))).
\end{aligned}
$$

Multiply both sides of the above equation by $\mathrm{e}^{\int_0^t a(u)\mathrm{d}u}$ and integrate from 0 to t to obtain

$$
\begin{aligned}
x(t) = {} & \psi(0)\,\mathrm{e}^{-\int_0^t a(u)\mathrm{d}u} \\
& + \int_0^t a(s)(Hx)(s)\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s + \int_0^t c(s)x'(s-r(s))\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
& + \int_0^t b(s)G(x(s),x(s-r(s)))\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s,
\end{aligned}
\tag{2.4}
$$

letting

$$
\begin{aligned}
& \int_0^t c(s)x'(s-r(s))\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
& = \int_0^t \frac{c(s)}{(1-r'(s))}(1-r'(s))x'(s-r(s))\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s.
\end{aligned}
$$

By performing an integration by parts, we obtain

$$\begin{aligned}
&\int_0^t c(s)x'(s-r(s))\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&= \frac{c(t)}{1-r'(t)}x(t-r(t)) - \frac{c(0)}{1-r'(0)}\psi(-r(0))\,\mathrm{e}^{-\int_0^t a(u)\mathrm{d}u} \\
&\quad - \int_0^t \mu(s)x(s-r(s))\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s,
\end{aligned} \tag{2.5}$$

where $\mu(s)$ is given by (2.2). We obtain (2.1) by substituting (2.5) in (2.4). Since each step is reversible, the converse follows easily. This completes the proof. □

Burton studied the theorem of Krasnoselskii and observed (see [**4**]–[**6**] and [**10**]) that Krasnoselskii result can be more interesting in applications with certain changes and formulated the Theorem 2 below (see [**4**] for its proof).

Let (M, d) be a metric space and $F: M \to M$. F is said to be a large contraction if $\varphi, \psi \in M$ with $\varphi \neq \psi$, then $d(F\varphi, F\psi) < d(\varphi, \psi)$, and if for all $\varepsilon > 0$, there exists $\eta < 1$ such that

$$[\varphi, \psi \in M,\ d(\varphi, \psi) \geq \varepsilon] \quad \Rightarrow \quad d(F\varphi, F\psi) \leq \eta d(\varphi, \psi).$$

Theorem 1 (Burton)**.** *Let (M, d) be a complete metric space and F be a large contraction. Suppose there is $x \in M$ and $\rho > 0$ such that $d(x, F^n x) \leq \rho$ for all $n \geq 1$. Then F has a unique fixed point in M.*

Below, we state Krasnoselskii-Burton's hybrid fixed point theorem which enables us to establish a stability result of the trivial solution of (1.1). For more details on Krasnoselskii's captivating theorem we refer to Smart [**21**] or [**3**].

Theorem 2 (Krasnoselskii-Burton)**.** *Let M be a closed bounded convex nonempty subset of a Banach space $(S, \|.\|)$. Suppose that A, B map M into M and that*

(i) *for all $x, y \in M \Rightarrow Ax + By \in M$,*
(ii) *A is continuous and AM is contained in a compact subset of M,*
(iii) *B is a large contraction.*

Then there is $z \in M$ with $z = Az + Bz$.

Here we manipulate function spaces defined on infinite t-intervals. So for compactness, we need an extension of Arzela-Ascoli theorem. This extension is taken from [**3**, Theorem 1.2.2, p. 20] and is as follows.

Theorem 3. *Let $q: \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\varphi_n(t)\}$ is an equicontinuous sequence of $\mathbb{R}^m$-valued functions on $\mathbb{R}_+$ with $|\varphi_n(t)| \leq q(t)$ for $t \in \mathbb{R}_+$, then there is a subsequence that converges uniformly on $\mathbb{R}_+$ to a continuous function $\varphi(t)$ with $|\varphi(t)| \leq q(t)$ for $t \in \mathbb{R}_+$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^m$.*

3. Stability by Krasnoselskii-Burton's theorem

From the existence theory which can be found in Hale [**18**] or [**3**], we conclude that for each continuous initial function $\psi\colon [m_0, 0] \to \mathbb{R}$, there exists a continuous solution $x(t, 0, \psi)$ which satisfies (1.1) on an interval $[0, \sigma)$ for some $\sigma > 0$ and $x(t, 0, \psi) = \psi(t)$, $t \in [m_0, 0]$.

We need the following stability definitions taken from [**3**].

Definition 1. The zero solution of (1.1) is said to be stable at $t = 0$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\psi\colon [m_0, 0] \to (-\delta, \delta)$ implies that $|x(t)| < \varepsilon$ for $t \geq m_0$.

Definition 2. The zero solution of (1.1) is said to be asymptotically stable if it is stable at $t = 0$ and $\delta > 0$ exists such that for any continuous function $\psi\colon [m_0, 0] \to (-\delta, \delta)$, the solution $x(t)$ with $x(t) = \psi(t)$ on $[m_0, 0]$ tends to zero as $\to \infty$.

To apply Theorem 2, we have to choose carefully a Banach space depending on the initial function ψ and construct two mappings, a large contraction and a compact operator which obey the conditions of the theorem. So let S be the Banach space of continuous bounded functions $\varphi\colon [m_0, \infty) \to \mathbb{R}$ with the supremum norm $\|\cdot\|$. Let $L > 0$ and define the set

$$\begin{aligned} S_\psi := \{\varphi \in S \mid \varphi \text{ is Lipschitzian, } |\varphi(t)| \leq L,\ t \in [m_0, \infty)\,, \\ \varphi(t) = \psi(t) \text{ if } t \in [m_0, 0] \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}. \end{aligned}$$

Clearly, if $\{\varphi_n\}$ is a sequence of $k-$Lipschitzian functions converging to a function φ, then

$$\begin{aligned} |\varphi(u) - \varphi(v)| &\leq |\varphi(u) - \varphi_n(u)| + |\varphi_n(u) - \varphi_n(v)| + |\varphi_n(v) - \varphi(v)| \\ &\leq \|\varphi - \varphi_n\| + k\,|u - v| + \|\varphi - \varphi_n\|\,. \end{aligned}$$

Consequently, as $n \to \infty$, we see that φ is $k-$Lipschitzian. It is clear that S_ψ is convex, bounded and complete endowed with $\|\cdot\|$.

For $\varphi \in S_\psi$ and $t \geq 0$, define the maps $A,\ B$ and C on S_ψ as follows:

$$\begin{aligned} (A\varphi)(t) := \frac{c(t)}{1 - r'(t)}\varphi(t - r(t)) + \int_0^t b(s)G(\varphi(s), \varphi(s - r(s)))\, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\, \mathrm{d}s \\ - \int_0^t \mu(s)\varphi(s - r(s))\, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\, \mathrm{d}s, \end{aligned} \tag{3.1}$$

$$\begin{aligned} (B\varphi)(t) := \left[\psi(0) - \frac{c(0)}{1 - r'(0)}\psi(-r(0))\right] \mathrm{e}^{-\int_0^t a(u)\mathrm{d}u} \\ + \int_0^t a(s)(H\varphi)(s)\, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\, \mathrm{d}s, \end{aligned} \tag{3.2}$$

and

$$(C\varphi)(t) := (A\varphi)(t) + (B\varphi)(t). \tag{3.3}$$

If we are able to prove that C possesses a fixed point φ on the set S_ψ, then $x(t,0,\psi) = \varphi(t)$ for $t \geq 0$, $x(t,0,\psi) = \psi(t)$ on $[m_0, 0]$, $x(t,0,\psi)$ satisfies (1.1) when its derivative exists and $x(t,0,\psi) \to 0$ as $t \to \infty$.

Let $\alpha(t) = \dfrac{c(t)}{1-r'(t)}$ and assume that there are constants $k_1, k_2, k_3 > 0$ such that for $0 \leq t_1 < t_2$,

$$\left|\int_{t_1}^{t_2} a(u)\mathrm{d}u\right| \leq k_1 |t_2 - t_1|, \tag{3.4}$$

$$|r(t_2) - r(t_1)| \leq k_2 |t_2 - t_1|, \tag{3.5}$$

and

$$|\alpha(t_2) - \alpha(t_1)| \leq k_3 \left|t_2 - t_1\right|. \tag{3.6}$$

Suppose that for $t \geq 0$,

$$|\mu(t)| \leq \delta a(t), \tag{3.7}$$

$$(N_1 + N_2)\left|b(t)\right| \leq \beta a(t), \tag{3.8}$$

$$\sup_{t\geq 0} |\alpha(t)| = \alpha_0, \tag{3.9}$$

and that

$$J(\alpha_0 + \beta + \delta) < 1, \tag{3.10}$$

$$\max(\left|H(-L)\right|, \left|H(L)\right|) \leq \frac{2L}{J}, \tag{3.11}$$

where α, β, δ and J are constants with $J > 3$.

Choose $\gamma > 0$ small enough and such that

$$\left(1 + \left|\frac{c(0)}{1-r'(0)}\right|\right)\gamma + \frac{3L}{J} \leq L. \tag{3.12}$$

The chosen γ in the relation (3.12) is used below in Lemma 3 to show that if $\varepsilon = L$ and if $\|\psi\| < \gamma$, then the solutions satisfy $|x(t,0,\psi)| < \varepsilon$.

Assume further that

$$t - r(t) \to \infty \text{ as} t \to \infty \quad \text{and} \quad \int_0^t a(u)\mathrm{d}u \to \infty \text{ as } t \to \infty, \tag{3.13}$$

$$\alpha(t) \to 0 \quad \text{as} \quad t \to \infty, \tag{3.14}$$

$$\frac{\mu(t)}{a(t)} \to 0 \quad \text{as} \quad t \to \infty, \tag{3.15}$$

and

$$\frac{b(t)}{a(t)} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.16}$$

We begin with the following theorem (see [**1**]) and for convenience we present its proof below. In the next proposition, we prove that for a well chosen function h, the mapping H given by (2.3) is a large contraction on the set S_ψ. So let us make the following assumptions on the function $h\colon \mathbb{R} \to \mathbb{R}$.

(H1) $h\colon \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,
(H2) the function h is strictly increasing on $[-L, L]$,
(H3) $\sup_{t\in(-L,L)} h'(t) \le 1$.

Theorem 4. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying* (H1)–(H3). *Then the mapping H in* (2.3) *is a large contraction on the set S_ψ.*

Proof. Let $\phi, \varphi \in S_\psi$ with $\phi \ne \varphi$. Then $\phi(t) \ne \varphi(t)$ for some $t \in \mathbb{R}$. Let us denote the set of all such t by $D(\phi, \varphi)$, i.e.,

$$D(\phi, \varphi) = \{t \in \mathbb{R} : \phi(t) \ne \varphi(t)\}.$$

For all $t \in D(\phi, \varphi)$, we have

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &= |\phi(t) - h(\phi(t)) - \varphi(t) + h(\varphi(t))| \\ &= |\phi(t) - \varphi(t)| \left| 1 - \left(\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} \right) \right|. \end{aligned} \tag{3.17}$$

Since h is a strictly increasing function, we have

$$\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > 0 \qquad \text{for all } t \in D(\phi, \varphi). \tag{3.18}$$

For each fixed $t \in D(\phi, \varphi)$, define the interval $U_t \subset [-L, L]$ by

$$U_t = \begin{cases} (\varphi(t), \phi(t)) & \text{if } \phi(t) > \varphi(t), \\ (\phi(t), \varphi(t)) & \text{if } \phi(t) < \varphi(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\phi, \varphi)$, there exists a real number $c_t \in U_t$ such that

$$\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} = h'(c_t).$$

By $(H2)$ and $(H3)$, we have

$$\begin{aligned} 0 \le \inf_{u\in(-L,L)} h'(u) &\le \inf_{u\in U_t} h'(u) \le h'(c_t) \\ &\le \sup_{u\in U_t} h'(u) \le \sup_{u\in(-L,L)} h'(u) \le 1. \end{aligned} \tag{3.19}$$

Hence, by (3.17)–(3.19), we obtain

$$|(H\phi)(t) - (H\varphi)(t)| \le \left| 1 - \inf_{u\in(-L,L)} h'(u) \right| |\phi(t) - \varphi(t)| \tag{3.20}$$

for all $t \in D(\phi, \varphi)$. Then by $(H3)$, we have

$$\|H\phi - H\varphi\| \le \|\phi - \varphi\|.$$

Now, choose a fixed $\varepsilon \in (0,1)$ and assume that ϕ and φ are two functions in S_ψ satisfying

$$\varepsilon \leq \sup_{t \in D(\phi,\varphi)} |\phi(t) - \varphi(t)| = \|\phi(t) - \varphi(t)\|.$$

If $|\phi(t) - \varphi(t)| \leq \frac{\varepsilon}{2}$ for some $t \in D(\phi,\varphi)$, then by (3.19) and (3.20), we get

$$|(H\phi)(t) - (H\varphi)(t)| \leq |\phi(t) - \varphi(t)| \leq \frac{1}{2} \|\phi - \varphi\|. \tag{3.21}$$

Since h is continuous and strictly increasing, the function $h(u + \frac{\varepsilon}{2}) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$.

Thus, if $\frac{\varepsilon}{2} \leq |\phi(t) - \varphi(t)|$ for some $t \in D(\phi,\varphi)$, then by $(H2)$ and $(H3)$, we conclude that

$$1 \geq \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2L} \min \left\{ h(u + \frac{\varepsilon}{2}) - h(u) \mid u \in [-L, L] \right\} > 0.$$

Hence, (3.17) implies

$$|(H\phi)(t) - (H\varphi)(t)| \leq (1 - \lambda) \|\phi(t) - \varphi(t)\|. \tag{3.22}$$

Consequently, combining (3.21) and (3.22), we obtain

$$|(H\phi)(t) - (H\varphi)(t)| \leq \eta \|\phi - \varphi\|,$$

where

$$\eta = \max \left\{ \frac{1}{2}, 1 - \lambda \right\} < 1.$$

The proof is complete. □

By step we will prove the fulfillment of (i), (ii) and (iii) in Theorem 2.

Lemma 2. *Suppose that* (3.7)–(3.10) *and* (3.13) *are true. For A defined in* (3.1), *if $\varphi \in S_\psi$, then $|(A\varphi)(t)| \leq L/J < L$. Moreover, $(A\varphi)(t) \to 0$ as $t \to \infty$.*

Proof. Using the conditions (3.7)–(3.10) and the expression (3.1) of the map A, we get

$$\begin{aligned}
|(A\varphi)(t)| &\leq \left| \frac{c(t)}{1 - r'(t)} \varphi(t - r(t)) \right| + \int_0^t |b(s)| \, |G(\varphi(s), \varphi(s - r(s)))| \, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \, \mathrm{d}s \\
&\quad + \int_0^t |\mu(s)| \, |\varphi(s - r(s))| \, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \, \mathrm{d}s \\
&\leq \alpha_0 L + \int_0^t (N_1 + N_2) \, |b(s)| \, L \, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \, \mathrm{d}s + L \int_0^t |\mu(s)| \, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \, \mathrm{d}s \\
&\leq L \left\{ \alpha_0 + \int_0^t \beta a(s) \, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \, \mathrm{d}s + \int_0^t \delta a(s) \, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \, \mathrm{d}s \right\} \\
&\leq L(\alpha_0 + \beta + \delta) \leq \frac{L}{J} < L.
\end{aligned}$$

So AS_ψ is bounded by L as required.

Let $\varphi \in S_\psi$ be fixed. We will prove that $(A\varphi)(t) \to 0$ as $t \to \infty$. Due to the conditions $t - r(t) \to \infty$ as $t \to \infty$ in (3.13) and (3.9), it is obvious that the first term on the right hand side of A tends to 0 as $t \to \infty$. That is

$$\left|\frac{c(t)}{1 - r'(t)}\varphi(t - r(t))\right| \le \alpha_0 |\varphi(t - r(t))| \to 0 \text{ as } t \to \infty.$$

It is left to show that the two remaining integral terms of A go to zero as $t \to \infty$.

Let $\varepsilon > 0$ be given. Find T such that $|\varphi(t - r(t))| < \varepsilon$ for $t \ge T$. Then we have

$$\begin{aligned}
&\left|\int_0^t \mu(s)\varphi(s - r(s))\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s\right| \\
&\le \int_0^T |\mu(s)\varphi(s - r(s))|\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s + \int_T^t |\mu(s)\varphi(s - r(s))|\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\le L\,\mathrm{e}^{-\int_T^t a(u)\mathrm{d}u} \int_0^T |\mu(s)|\,\mathrm{e}^{-\int_s^T a(u)\mathrm{d}u}\,\mathrm{d}s + \varepsilon \int_T^t |\mu(s)|\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\le L\delta\,\mathrm{e}^{-\int_T^t a(u)\mathrm{d}u} + \varepsilon\delta.
\end{aligned}$$

The term $L\delta\,\mathrm{e}^{-\int_T^t a(u)\mathrm{d}u}$ is arbitrarily small as $t \to \infty$, because of (3.13). The remaining integral term in A goes to zero by just a similar argument. This ends the proof. □

Lemma 3. *Let* (3.7)–(3.11) *and* (3.13) *hold. For A, B defined in* (3.1) *and* (3.2)*, if $\phi, \varphi \in S_\psi$ are arbitrary, then*

$$|B\varphi + A\phi| \le L.$$

Moreover, B is a large contraction on S_ψ with a unique fixed point in S_ψ and $B\varphi(t) \to 0$ as $t \to \infty$.

Proof. Using the definitions (3.1), (3.2) of A and B and applying (3.7)–(3.11), we obtain

$$\begin{aligned}
&|(B\varphi)(t) + (A\phi)(t)| \\
&\le \left(1 + \left|\frac{c(0)}{1 - r'(0)}\right|\right)\|\psi\|\,\mathrm{e}^{-\int_0^t a(u)\mathrm{d}u} + \alpha_0 L + L\int_0^t |\mu(s)|\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\quad + \int_0^t (N_1 + N_2)\,|b(s)|\,L\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s + \frac{2L}{J}\int_0^t a(s)\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\le \left(1 + \left|\frac{c(0)}{1 - r'(0)}\right|\right)\|\psi\| + (\alpha_0 + \beta + \delta)L + \frac{2L}{J} \\
&\le \left(1 + \left|\frac{c(0)}{1 - r'(0)}\right|\right)\|\psi\| + \frac{L}{J} + \frac{2L}{J}
\end{aligned}$$

by the monotonicity of the mapping H. So from the above inequality, by choosing the initial function ψ having small norm, say $\|\psi\| < \gamma$, then, and referring to

(3.12), we obtain

$$|(B\varphi)(t)+(A\phi)(t)| \leq \left(1+\left|\frac{c(0)}{1-r'(0)}\right|\right)\gamma+\frac{3L}{J} \leq L.$$

Since $0 \in S_\psi$, we have also proved that $|(B\varphi)(t)| \leq L$. The proof that $B\varphi$ is Lipschitzian is similar to that of the map $A\varphi$ below. To see that B is a large contraction on S_ψ with a unique fixed point, we know from Theorem 4 that $H(\varphi) = \varphi - h(\varphi)$ is a large contraction within the integrand. Thus, for any ε, from the proof of that Theorem 4, we have found $\eta < 1$ such that

$$\begin{aligned}
&|(B\varphi)(t)-(B\phi)(t)| \\
&\leq \int_0^t a(s)\,|(H\varphi)(s)-(H\phi)(s)|\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\leq \eta \int_0^t a(s)\,\|\varphi-\phi\|\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\leq \eta\,\|\varphi-\phi\|\,.
\end{aligned}$$

To prove that $(B\varphi)(t) \to 0$ as $t \to \infty$, we use (3.13) for the first term, and for the second term, we argue as above for the map A. □

Lemma 4. *Suppose* (3.7)–(3.10) *hold. Then the mapping* A *is continuous on* S_ψ.

Proof. Let $\varphi, \phi \in S_\psi$, then

$$\begin{aligned}
&|(A\varphi)(t)-(A\phi)(t)| \\
&\leq \{\alpha_0\,|\varphi(t-r(t))-\phi(t-r(t))| \\
&\quad + \left|\int_0^t b(s)\,[G(\varphi(s),\varphi(s-r(s)))-G(\phi(s),\phi(s-r(s)))]\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s\right| \\
&\quad + \left.\left|\int_0^t \mu(s)\,[\varphi(s-r(s))-\phi(s-r(s))]\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s\right|\right\} \\
&\leq \alpha_0\,\|\varphi-\phi\| + \int_0^t (N_1+N_2)\,|b(s)|\,\|\varphi-\phi\|\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\quad + \|\varphi-\phi\| \int_0^t |\mu(s)|\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\leq (\alpha_0+\beta+\delta)\,\|\varphi-\phi\| \int_0^t a(s)\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\leq (\alpha_0+\beta+\delta)\,\|\varphi-\phi\| \leq (1/J)\,\|\varphi-\phi\|\,.
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Define $\eta = \varepsilon J$. Then for $\|\varphi-\phi\| \leq \eta$, we obtain

$$\|A\varphi - A\phi\| \leq \frac{1}{J}\,\|\varphi-\phi\| \leq \varepsilon.$$

Therefore, A is continuous. □

Lemma 5. *Let* (3.4)–(3.9) *and* (3.14)–(3.16) *hold. The function* $A\varphi$ *is Lipschitzian and the operator* A *maps* S_ψ *into a compact subset of* S_ψ.

Proof. Let $\varphi \in S_\psi$ and let $0 \le t_1 < t_2$. Then

$$\begin{aligned}
&|(A\varphi)(t_2) - (A\varphi)(t_1)| \\
&\le \left| \frac{c(t_2)}{1-r'(t_2)}\varphi(t_2 - r(t_2)) - \frac{c(t_1)}{1-r'(t_1)}\varphi(t_1 - r(t_1)) \right| \\
&\quad + \left| \int_0^{t_2} \mu(s)\varphi(s-r(s))\, \mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\, \mathrm{d}s \right. \\
&\quad \left. - \int_0^{t_1} \mu(s)\varphi(s-r(s))\, \mathrm{e}^{-\int_s^{t_1} a(u)\mathrm{d}u}\, \mathrm{d}s \right| \\
&\quad + \left| \int_0^{t_2} b(s)G(\varphi(s), \varphi(s-r(s)))\, \mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\, \mathrm{d}s \right. \\
&\quad \left. - \int_0^{t_1} b(s)G(\varphi(s), \varphi(s-r(s)))\, \mathrm{e}^{-\int_s^{t_1} a(u)\mathrm{d}u}\, \mathrm{d}s \right|
\end{aligned} \tag{3.23}$$

By hypotheses (3.5)–(3.6), we have

$$\begin{aligned}
&|\alpha(t_2)\varphi(t_2 - r(t_2)) - \alpha(t_1)\varphi(t_1 - r(t_1))| \\
&\le |\alpha(t_2)|\,|\varphi(t_2 - r(t_2)) - \varphi(t_1 - r(t_1))| \\
&\quad + |\varphi(t_1 - r(t_1))|\,|\alpha(t_2) - \alpha(t_1)| \\
&\le \alpha_0 k\,|(t_2 - t_1) - (r(t_2) - r(t_1))| + Lk_3\,|t_2 - t_1| \\
&\le (\alpha_0 k + \alpha_0 k k_2 + Lk_3)\,|t_2 - t_1|\,,
\end{aligned} \tag{3.24}$$

where k is the Lipschitz constant of φ. By hypotheses (3.4) and (3.7), we have

$$\begin{aligned}
&\left| \int_0^{t_2} \mu(s)\varphi(s-r(s))\, \mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\, \mathrm{d}s - \int_0^{t_1} \mu(s)\varphi(s-r(s))\, \mathrm{e}^{-\int_s^{t_1} a(u)\mathrm{d}u}\, \mathrm{d}s \right| \\
&= \left| \int_0^{t_1} \mu(s)\varphi(s-r(s))\, \mathrm{e}^{-\int_s^{t_1} a(u)\mathrm{d}u} \left(\mathrm{e}^{-\int_{t_1}^{t_2} a(u)\mathrm{d}u} - 1 \right) \mathrm{d}s \right. \\
&\quad \left. + \int_{t_1}^{t_2} \mu(s)\varphi(s-r(s))\, \mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\, \mathrm{d}s \right| \\
&\le L \left| \mathrm{e}^{-\int_{t_1}^{t_2} a(u)\mathrm{d}u} - 1 \right| \int_0^{t_1} \delta a(s)\, \mathrm{e}^{-\int_s^{t_1} a(u)\mathrm{d}u} + L \int_{t_1}^{t_2} |\mu(s)|\, \mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\, \mathrm{d}s \\
&\le L\delta \int_{t_1}^{t_2} a(s)\mathrm{d}s + L \int_{t_1}^{t_2} \mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\, \mathrm{d}\left(\int_{t_1}^{s} |\mu(v)|\, \mathrm{d}v \right) \\
&\le L\delta \int_{t_1}^{t_2} a(s)\mathrm{d}s + L \left\{ \left[\mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u} \int_{t_1}^{s} |\mu(v)|\, \mathrm{d}v \right]_{t_1}^{t_2} \right. \\
&\quad \left. + \int_{t_1}^{t_2} a(s)\, \mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u} \int_{t_1}^{s} |\mu(v)|\, \mathrm{d}v\mathrm{d}s \right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq L\delta \int_{t_1}^{t_2} a(s)\mathrm{d}s + L\int_{t_1}^{t_2} |\mu(s)|\,\mathrm{d}s \left(1 + \int_{t_1}^{t_2} a(s)\,\mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\,\mathrm{d}s\right) \\
&\leq L\delta \int_{t_1}^{t_2} a(s)\mathrm{d}s + 2L\int_{t_1}^{t_2} |\mu(s)|\,\mathrm{d}s \\
&\leq L\delta \int_{t_1}^{t_2} a(s)\mathrm{d}s + 2L\delta\int_{t_1}^{t_2} a(s)\mathrm{d}s \\
&\leq 3L\delta k_1 \left|t_2 - t_1\right|.
\end{aligned} \tag{3.25}
$$

Similarly, by (3.4) and (3.8), we deduce

$$
\begin{aligned}
&\left|\int_0^{t_2} b(s)G(\varphi(s),\varphi(s-r(s)))\,\mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\,\mathrm{d}s\right. \\
&\quad \left. - \int_0^{t_1} b(s)G(\varphi(s),\varphi(s-r(s)))\,\mathrm{e}^{-\int_s^{t_1} a(u)\mathrm{d}u}\,\mathrm{d}s\right| \\
&= \left|\int_0^{t_1} b(s)G(\varphi(s),\varphi(s-r(s)))\,\mathrm{e}^{-\int_s^{t_1} a(u)\mathrm{d}u}\left(\mathrm{e}^{-\int_{t_1}^{t_2} a(u)\mathrm{d}u} - 1\right)\mathrm{d}s\right. \\
&\quad \left. + \int_{t_1}^{t_2} b(s)G(\varphi(s),\varphi(s-r(s)))\,\mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\,\mathrm{d}s\right| \\
&\leq L\left|\mathrm{e}^{-\int_{t_1}^{t_2} a(u)\mathrm{d}u} - 1\right| \int_0^{t_1} \beta a(s)\,\mathrm{e}^{-\int_s^{t_1} a(u)\mathrm{d}u} \\
&\quad + (N_1+N_2)L\int_{t_1}^{t_2} |b(s)|\,\mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\leq L\beta\int_{t_1}^{t_2} a(u)\mathrm{d}u + (N_1+N_2)L\int_{t_1}^{t_2} \mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\,\mathrm{d}\left(\int_{t_1}^{s} |b(v)|\,\mathrm{d}v\right) \\
&\leq L\beta\int_{t_1}^{t_2} a(u)\mathrm{d}u + (N_1+N_2)L\left\{\left[\mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\int_{t_1}^{s} |b(v)|\,\mathrm{d}v\right]_{t_1}^{t_2}\right. \\
&\quad \left. + \int_{t_1}^{t_2} a(s)\,\mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\int_{t_1}^{s} |b(v)|\,\mathrm{d}v\mathrm{d}s\right\} \\
&\leq L\beta\int_{t_1}^{t_2} a(u)\mathrm{d}u + (N_1+N_2)L\int_{t_1}^{t_2} |b(s)|\,\mathrm{d}s\left(1 + \int_{t_1}^{t_2} a(s)\,\mathrm{e}^{-\int_s^{t_2} a(u)\mathrm{d}u}\,\mathrm{d}s\right) \\
&\leq L\beta\int_{t_1}^{t_2} a(u)\mathrm{d}u + 2(N_1+N_2)L\int_{t_1}^{t_2} |b(s)|\,\mathrm{d}s \\
&\leq L\beta\int_{t_1}^{t_2} a(u)\mathrm{d}u + 2L\beta\int_{t_1}^{t_2} a(s)\mathrm{d}s \\
&\leq 3L\beta k_1\left|t_2 - t_1\right|.
\end{aligned} \tag{3.26}
$$

Thus, by substituting (3.24)–(3.26) in (3.23), we obtain

$$
\begin{aligned}
&|A\varphi(t_2) - A\varphi(t_1)| \\
&\le (\alpha_0 k + \alpha_0 k k_2 + L k_3)\,|t_2 - t_1| + 3L\delta k_1\,|t_2 - t_1| + 3L\beta k_1\,|t_2 - t_1| \\
&\le K\,|t_2 - t_1|
\end{aligned}
\tag{3.27}
$$

for a constant $K > 0$. This shows that $A\varphi$ is Lipschitzian if φ is and that AS_ψ is equicontinuous. Next, we notice that for arbitrary $\varphi \in S_\psi$, we have

$$
\begin{aligned}
&|A\varphi(t)| \\
&\le \left|\frac{c(t)}{1 - r'(t)}\varphi(t - r(t))\right| + \int_0^t |b(s)|\,|G(\varphi(s), \varphi(s - r(s)))|\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\quad + \int_0^t |\mu(s)\varphi(s - r(s))|\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\le L\,|\alpha(t)| + (N_1 + N_2)L\int_0^t a(s)\,[|b(s)|\,/a(s)]\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&\quad + L\int_0^t a(s)\,[|\mu(s)|\,/a(s)]\,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\,\mathrm{d}s \\
&:= q(t),
\end{aligned}
$$

because of (3.14)–(3.16). Using a method like the one used for the map A, we see that $q(t) \to 0$ as $t \to \infty$. By Theorem 3, we conclude that the set AS_ψ resides in a compact set. □

Theorem 5. *Let $L > 0$. Suppose that the conditions* (H1)–(H3), (1.2) *and* (3.14)–(3.16) *hold. If ψ is a given initial function which is sufficiently small, then there is a solution $x(t, 0, \psi)$ of* (1.1) *with $|x(t, 0, \psi)| \le L$ and $x(t, 0, \psi) \to 0$ as $t \to \infty$.*

Proof. From Lemmas 2 and 5 we have A is bounded by L, Lipschitzian and $(A\phi)(t) \to 0$ as $t \to \infty$. So A maps S_ψ into S_ψ. From Lemmas 3 and 5 for arbitrary, we have $\phi, \varphi \in S_\psi$, $B\varphi + A\phi \in S_\psi$ since both $A\phi$ and $B\varphi$ are Lipschitzian bounded by L and $(B\varphi)(t) \to 0$ as $t \to \infty$. From Lemmas 4 and 5, we have proved that A is continuous and AS_ψ resides in a compact set. Thus, all the conditions of Theorem 2 are satisfied. Therefore, there exists a solution of (1.1) with $|x(t, 0, \psi)| \le L$ and $x(t, 0, \psi) \to 0$ as $t \to \infty$. □

Remark 1. Theorem 5 improves and generalizes [**16**, Theorem 3.6].

4. Stability and compactness

Referring to Burton [**3**], except for the fixed point method, we know of no other way proving that solutions of (1.1) converge to zero. Nevertheless, if all we need is stability and not asymptotic stability, then we can avoid conditions (3.14)–(3.16) and still use Krasnoselskii-Burton's theorem on a Banach space endowed with a weighted norm.

Let $g\colon [m_0,\infty) \to [1,\infty)$ be any strictly increasing and continuous function with $g(m_0) = 1$, $g(s) \to \infty$ as $s \to \infty$. Let $(S, |\cdot|_g)$ be the Banach space of continuous $\varphi\colon [m_0,\infty) \to \mathbb{R}$ for which

$$|\varphi|_g := \sup_{t\geq m_0} \left|\frac{\varphi(t)}{g(t)}\right| < \infty$$

exists. We continue to use $\|\cdot\|$ as the supremum norm of any $\varphi \in S$ provided φ bounded. Also, we use $\|\psi\|$ as the bound of the initial function. Further, in a similar way as Theorem 4, we can prove that the function $H(\varphi) = \varphi - h(\varphi)$ is still a large contraction with the norm $|\cdot|_g$.

Theorem 6. *If the conditions of Theorem 5 hold, except for* (3.14)–(3.16)*, then the zero solution of* (1.1) *is stable.*

Proof. We prove the stability starting at $t_0 = 0$. Let $\varepsilon > 0$ be given such that $0 < \varepsilon < L$, then for $|x| \leq \varepsilon$, find γ^* with $|x - h(x)| \leq \gamma^*$ and choose a number γ such that

$$\gamma + \gamma^* + \frac{\varepsilon}{J} \leq \varepsilon. \tag{4.1}$$

In fact, since $x - h(x)$ is increasing on $(-L, L)$, we may take $\gamma^* = \dfrac{2\varepsilon}{J}$. Thus, inequality (4.1) allows $\gamma > 0$. Now, remove the condition $\varphi(t) \to 0$ as $t \to 0$ from S_ψ defined previously and consider the set

$$\begin{aligned} M_\psi := \{\, &\varphi \in S \mid \varphi \text{ Lipschitzian, } |\varphi(t)| \leq \varepsilon,\ t \in [m_0,\infty) \\ &\text{and } \varphi(t) = \psi(t)\ if\ t \in [m_0, 0]\ \}. \end{aligned}$$

Define A, B on M_ψ as before by (3.1), (3.2). We easily check that if $\varphi \in M_\psi$, then $|(A\varphi)(t)| < \varepsilon$, and B is a large contraction on M_ψ. Also, by choosing $\|\psi\| < \gamma$ and referring to (4.1), we verify that for $\varphi, \phi \in M_\psi$, $|(B\varphi)(t) + (A\phi)(t)| \leq \varepsilon$ and $|(B\varphi)(t)| < \varepsilon$. AM_ψ is an equicontinuous set. According to [**3**, Theorem 4.0.1], in the space $(S, |\cdot|_g)$ the set AM_ψ resides in a compact subset of M_ψ. Moreover, the operator $A : M_\psi \to M_\psi$ is continuous. Indeed, for $\varphi, \phi \in S_\psi$,

$$\begin{aligned} &\frac{|(A\varphi)(t) - (A\phi)(t)|}{g(t)} \\ &\leq \frac{1}{g(t)} \{\alpha\, |\varphi(t - r(t)) - \phi(t - r(t))| \\ &\quad + \left|\int_0^t b(s)\, [G(\varphi(s), \varphi(s - r(s))) - G(\phi(s), \phi(s - r(s)))]\, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\, \mathrm{d}s\right| \\ &\quad + \left|\int_0^t \mu(s)\, [\varphi(s - r(s)) - \phi(s - r(s))]\, \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\, \mathrm{d}s\right|\right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_0 \left|\varphi - \phi\right|_g \\
&\quad + \int_0^t |b(s)| \left(\frac{N_1 |\varphi(s) - \phi(s)|}{g(t)} + \frac{N_2 \left|\varphi(s - r(s)) - \phi(s - r(s))\right|}{g(t)} \right) \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \,\mathrm{d}s \\
&\quad + \int_0^t |\mu(s)| \frac{|\varphi(s - r(s)) - \phi(s - r(s))|}{g(t)} \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \,\mathrm{d}s \\
&\leq \alpha_0 \left|\varphi - \phi\right|_g \\
&\quad + \int_0^t |b(s)| \left[\frac{N_1 \left|\varphi(s) - \phi(s)\right|}{g(s)} \frac{g(s)}{g(t)} \right] \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \,\mathrm{d}s \\
&\quad + \int_0^t |b(s)| \left[\frac{N_2 \left|\varphi(s - r(s)) - \phi(s - r(s))\right|}{g(s - r(s))} \frac{g(s - r(s))}{g(t)} \right] \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \,\mathrm{d}s \\
&\quad + \int_0^t |\mu(s)| \frac{|\varphi(s - r(s)) - \phi(s - r(s))|}{g(t)} \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \,\mathrm{d}s \\
&\leq \alpha_0 \left|\varphi - \phi\right|_g + \left|\varphi - \phi\right|_g \int_0^t |b(s)| \left[\frac{N_1 g(s) + N_2 g(s - r(s))}{g(t)} \right] \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \,\mathrm{d}s \\
&\quad + \delta \left|\varphi - \phi\right|_g \int_0^t a(s) \frac{g(s - r(s))}{g(t)} \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \,\mathrm{d}s \\
&\leq \alpha_0 \left|\varphi - \phi\right|_g + \beta \left|\varphi - \phi\right|_g \int_0^t a(s) \,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \,\mathrm{d}s \\
&\quad + \delta \left|\varphi - \phi\right|_g \int_0^t a(s) \,\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u} \,\mathrm{d}s \\
&\leq \frac{1}{J} \left|\varphi - \phi\right|_g .
\end{aligned}
$$

The conditions of Theorem 2 are satisfied on M_ψ, and so there exists a fixed point lying in M_ψ and solving (1.1). □

Remark 2. Theorem 6 improves and generalizes [**16**, Theorem 4.1].

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