Acta Math. Univ. Comenianae
Vol. LXXXIII, 1 (2014), pp. 113–118

CONVERGENCE OF THE INCREMENTS OF A WIENER PROCESS

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Abstract. Let $\lambda_{(t,\alpha)} = (2a_t(\log(t/a_t) + \alpha \log\log t + (1-\alpha)\log\log a_t))^{-\frac{1}{2}}$ where $0 \le \alpha \le 1$ and $W(t)$ is a standard Wiener process. Suppose that a_t is a nondecreasing function of t such that $0 < a_t \le t$ and a_t/t is nonincreasing. In this paper, we study the almost sure behaviour of $\limsup_{k\to\infty} \sup_{0\le s\le a_{t_k}} \lambda_{(t_k,\alpha)}|W(t_k+s) - W(t_k)|$ where $\{t_k\}$ is an increasing sequence diverging to ∞.

1. Introduction

Let $\{W(t), t \ge 0\}$ be a standard Wiener process. Suppose that a_t is a nondecreasing function of t such that $0 < a_t \le t$ and t/a_t is nonincreasing. In [**1**] Bahram established the following law of the iterated logarithm

$$\limsup_{t\to\infty} \lambda_{(t,\alpha)}|W(t+a_t) - W(t)| = 1 \qquad \text{almost surely (a.s.)}$$

and

$$\limsup_{t\to\infty} \sup_{0\le s\le a_t} \lambda_{(t,\alpha)}|W(t+s) - W(t)| = 1 \quad a.s.,$$

where

$$\lambda_{(t,\alpha)} = (2a_t(\log(t/a_t) + \alpha\log\log t + (1-\alpha)\log\log a_t))^{-\frac{1}{2}} \quad \text{and} \quad 0 \le \alpha \le 1.$$

In the present paper, we look at this problem over a monotonic sequence (t_k), motivated by Gut (1986). Let $Y(t) = W(t+a_t) - W(t)$, $t > 0$. We show that $\limsup_{k\to\infty} \lambda_{(t_k,\alpha)}|Y(t_k)|$ depends on both (t_k) and the function a_t, $t > 0$. The main results of the paper are presented in the next section. In Section 3, similar results are obtained for partial sums of i.i.d. the random variables by appealing to strong approximation theory.

Throughout the paper, ε, c, δ and K (integer), with or without the suffix, stand for positive constants; i.o. means infinitely often; for each $u \ge 0$, we define the functions $\log u = \log(\max(u,1))$, $\log\log u = \log\log(\max(u,3))$, $g(t) = (t\log t)/a_t$ and $g_\alpha(t) = t(\log t)^\alpha(\log a_t)^{1-\alpha}/a_t$ with $0 \le \alpha \le 1$, so that $\lambda_{(t,\alpha)} = (2a_t \log g_\alpha(t))^{-\frac{1}{2}}$.

Received March 4, 2013.
2010 *Mathematics Subject Classification.* Primary 60F15, 60G17.
Key words and phrases. Wiener process; increments of a Wiener process; law of the iterated logarithm; Strong approximations.

2. Main results

Theorem 2.1. *Let a_t, $t > 0$ be a nondecreasing function of t with $0 < a_t < t$ and a_t/t nonincreasing. Let (t_k) be any increasing sequence diverging to ∞ such that*

$$\limsup_{k\to\infty} \frac{t_{k+1} - t_k}{a_{t_k}} < 1.$$

Then

$$\limsup_{k\to\infty} \lambda_{(t_k,\alpha)} |Y(t_k)| = 1 \quad a.s.$$

and

$$\limsup_{k\to\infty} \sup_{0\le s\le a_{t_k}} \lambda_{(t_k,\alpha)} |W(t_k + s) - W(t_k)| = 1 \quad a.s.,$$

where

$$\lambda_{(t,\alpha)} = (2a_t(\log(t/a_t) + \alpha \log\log t + (1-\alpha)\log\log a_t))^{-\frac{1}{2}} \quad and \quad 0 \le \alpha \le 1.$$

Proof of Theorem 2.1. We establish that for any $0 < \varepsilon_1 < 1$,

$$P(\sup_{0\le s\le a_{t_k}} \lambda_{(t_k,\alpha)} |W(t_k + s) - W(t_k)| > 1 + \varepsilon_1 \quad \text{i.o.}) = 0 \tag{1}$$

and

$$P(\lambda_{(t_k,\alpha)} |W(t_k + a_{t_k}) - W(t_k)| > 1 - \varepsilon_1 \quad \text{i.o.}) = 1, \tag{2}$$

which in turn imply the theorem. By Theorem 2.1 of Bahram (see [**1**]) we have

$$\limsup_{t\to\infty} \sup_{0\le s\le a_t} \lambda_{(t_k,\alpha)} |W(t + s) - W(t)| = 1 \;\; a.s.,$$

from which (1) follows.

To prove (2), we proceed as follows. Define a sequence (u_k) by $u_1 = a_{t_1}$ for some $t_1 > 0$ and $u_{k+1} = u_k + a_{u_k}$, $k \ge 1$. Using the well known probability inequality

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right) &\le P(W(1) \ge x) \\ &\le \frac{1}{\sqrt{2\pi}x}\exp\left(-\frac{x^2}{2}\right), \quad \text{for} \quad x \ge 0, \end{aligned} \tag{3}$$

(see, e.g., [**4**, p. 175]), one can find constants c_1, δ_1 and K_1 such that for all $k > K_1$,

$$
\begin{aligned}
P(\lambda_{(t_k,\alpha)}|Y(u_k)| \geq 1-\varepsilon_1) &\geq c_1(\log g_\alpha(u_k))^{\frac{-1}{2}} \exp(-(1-\varepsilon_1)^2 \log g_\alpha(u_k)) \\
&\geq c_1\biggl(g_\alpha(u_k)\biggr)^{-(1-\varepsilon_1)^2-\delta_1} \\
&= c_1\left(\frac{u_k(\log u_k)^\alpha(\log a_{u_k})^{1-\alpha}}{a_{u_k}}\right)^{-(1-\varepsilon_1)^2-\delta_1} \\
&= c_1\left(\frac{a_{u_k}}{u_k}\left(\frac{\log a_{u_k}}{\log u_k}\right)^\alpha \frac{1}{\log a_{u_k}}\right)^{(1-\varepsilon_1)^2+\delta_1} \\
&\geq c_1\left(\frac{a_{u_k}}{u_k}\left(\frac{\log a_{u_k}}{\log u_k}\right) \frac{1}{\log a_{u_k}}\right)^{(1-\varepsilon_1)^2+\delta_1} \\
&= c_1(g(u_k))^{-(1-\varepsilon_1)^2-\delta_1},
\end{aligned}
$$

where δ_1 is chosen such that $(1-\varepsilon_1)^2+\delta_1<1$.

Set

$$S=\sum_{k=K_1}^{\infty}(g(u_k))^{-(1-\varepsilon_2)}, \quad \text{where} \quad 1-\varepsilon_2=(1-\varepsilon_1)^2+\delta_1.$$

From the fact that a_t/t is nonincreasing, we have

$$S \geq \sum_{k=K_1}^{\infty}(g(u_k))^{-1}=\sum_{k=K_1}^{\infty}\frac{u_{k+1}-u_k}{u_k\log u_k}.$$

Observing that

$$\sum_{k=K_1}^{\infty}\frac{u_{k+1}-u_k}{u_k\log u_k} \geq \int_c^\infty \frac{\mathrm{d}t}{t\log t}$$

for some $c>0$ and that

$$\int_c^\infty \frac{\mathrm{d}t}{t\log t}=\infty,$$

one gets $S=\infty$. Let us write

$$S=\sum_{k\geq K_1}(g(u_{2k-1}))^{-(1-\varepsilon_2)}+\sum_{k\geq K_1}(g(u_{2k}))^{-(1-\varepsilon_2)}=S_1+S_2.$$

The fact $S=\infty$ implies that at least one of S_1, S_2 is ∞. Let $S_2=\infty$. By the monotonicity of a_t/t, one can observe that $g(u_{2k}) \leq g(u_{2k-1})$, which in turn implies that $S_1=\infty$. Similarly, with $S_1=\infty$ one can show that $S_2=\infty$. Hence $S=\infty$ implies that both $S_1=\infty$ and $S_2=\infty$.

Let (t'_k) be a subsequence of (t_k) such that $u_k \leq t'_k \leq u_{k+1}$ for all k large. We now claim that such a subsequence exists. Otherwise, it would mean the existence of a subsequence $(u_{k(m)})$ of (u_k) such that no member of (t_k) belongs to the interval $[u_{k(m)}, u_{k(m)+1}]$, $m \geq 1$, i.e., there exists a subsequence $(t_{k(l)})$ of (t_k) such that

$t_{k(l)} < u_{k(m)} < u_{k(m)+1} < t_{k(l)+1}$. In turn we have

$$\frac{t_{k(l)+1} - t_{k(l)}}{a_{t_{k(l)}}} \geq \frac{u_{k(m)+1} - u_{k(m)}}{a_{u_{k(m)}}}$$

which implies that

$$\liminf_{k\to\infty} \frac{t_{k(l)+1} - t_{k(l)}}{a_{t_{k(l)}}} \geq \liminf_{k\to\infty} \frac{u_{k(m)+1} - u_{k(m)}}{a_{u_k(m)}} = 1,$$

contradicting the condition

$$\limsup_{k\to\infty} \frac{t_{k+1} - t_k}{a_{t_k}} < 1$$

of the theorem. Hence such a sequence (t'_k) necessarily exists.

Consider the odd subsequence (t'_{2k-1}) of (t'_k) and define the event

$$A_k = (\lambda_{(t'_{2k-1},\alpha)} |Y(t'_{2k-1})| \geq 1 - \varepsilon_1).$$

By (3), one can find constants c_2, δ_2 and K_2 such that for all $k \geq K_2$,

$$\begin{aligned} P(A_k) &\geq c_2(g_\alpha(t'_{2k-1}))^{-(1-\varepsilon_1)^2-\delta_2} \\ &\geq c_2(g(t'_{2k-1}))^{-(1-\varepsilon_1)^2-\delta_2} \end{aligned}$$

with $(1-\varepsilon_1)^2 + \delta_2 < 1$.

From the fact that $u_{2k-1} < t'_{2k-1} < u_{2k}$ for k large, one can find K_3 such that for all $k \geq K_3$,

$$P(A_k) \geq c_2(g(t'_{2k-1})^{-(1-\varepsilon_2)}$$

where $(1-\varepsilon_1)^2 + \delta_2 = 1 - \varepsilon_2$.

Now $S_2 = \infty$ implies that $\sum_{k=K_3}^{\infty} P(A_k) = \infty$. Also,

$$\begin{aligned} t'_{2k-1} + a_{t'_{2k-1}} &< u_{2k} + a_{u_{2k}} \\ &= u_{2k+1} < t'_{2k+1} \end{aligned}$$

for all $k \geq K_3$, implies that the events $(A_k, \quad k \geq K_3)$ are mutually independent. By appealing to Borel Cantelli Lemma, we have $P(A_k \quad i.o.) = 1$, which in turn implies (2). □

Theorem 2.2. *Let a_t, $t > 0$, be a nondecreasing function of t with $0 < a_t \leq t$ and a_t/t is nonincreasing. Let (t_k) be an increasing sequence diverging to ∞ such that*

$$\liminf_{k\to\infty} \frac{t_{k+1} - t_k}{a_{t_k}} > 1.$$

Then

$$\limsup_{k\to\infty} \lambda_{(t_k,\alpha)} |Y(t_k| = \limsup_{k\to\infty} \sup_{0\leq s\leq a_{t_k}} \lambda_{(t_k,\alpha)} |W(t_k + s) - W(t_k)| = \varepsilon^* \quad a.s.,$$

where

$$\varepsilon^* = \inf\left\{\gamma > 0 : \sum_k (g_\alpha(t_k))^{-\gamma^2} < \infty, \quad 0 \leq \alpha \leq 1\right\}.$$

Proof of Theorem 2.2. Equivalently, we establish that

$$P(\sup_{0\le s\le a_{t_k}} \lambda_{(t_k,\alpha)}|W(t_k+s)-W(t_k)| \ge \varepsilon^*+\varepsilon_1 \quad \text{i.o.})=0 \tag{4}$$

for any $\varepsilon_1>0$ and that

$$P(\lambda_{(t_k,\alpha)}|Y(t_k)| \ge \varepsilon^*-\varepsilon_1 \quad \text{i.o.})=1 \tag{5}$$

for any $0<\varepsilon_1<\varepsilon^*$, when $\varepsilon^*>0$.

We have (see, e.g., [**2**, p. 448])

$$\begin{aligned} &P(\sup_{0\le s\le a_{t_k}} \lambda_{(t_k,\alpha)}|W(t_k+s)-W(t_k)| \ge \varepsilon^*+\varepsilon_1) \\ &\le 2P(\lambda_{(t_k,\alpha)}|Y(t_k)| \ge \varepsilon^*+\varepsilon_1). \end{aligned} \tag{6}$$

By (3), one can find constants c_3 and K_4 such that for all $k\ge K_4$,

$$\begin{aligned} P(\lambda_{(t_k,\alpha)}|Y|(t_k)| \ge \varepsilon^*+\varepsilon_1) &\le c_3(\log g_\alpha(t_k))^{\frac{-1}{2}}\exp\{-(\varepsilon^*+\varepsilon_1)^2\log g_\alpha(t_k)\} \\ &\le c_3(g_\alpha(t_k))^{-(\varepsilon^*+\varepsilon_1)^2}. \end{aligned}$$

From the definition of ε^*, it follows that

$$\sum_{k\ge K_4} g_\alpha(t_k))^{-(\varepsilon^*+\varepsilon_1)^2} < \infty.$$

Now (4) is immediate by appealing to Borel-Cantelli Lemma. This completes the proof of the theorem when $\varepsilon^*=0$.

Consider the case $\varepsilon^*>0$. The condition $\liminf_{k\to\infty}\frac{t_{k+1}-t_k}{a_{t_k}}>1$ implies that there exists K_5 such that $t_{k+1}>t_k+a_{t_k}$ for all $k\ge K_5$. This in turn implies that $(Y(t_k), k\ge K_5)$ is a sequence of mutually independent random variables. By (3), one can find c_4, δ_3 and K_6 such that for all $k\ge K_6$,

$$P(\lambda_{(t_k,\alpha)}|Y(t_k)| \ge \varepsilon^*-\varepsilon_1) \ge c_4(g_\alpha(t_k))^{-(\varepsilon^*-\varepsilon_1)^2-\delta_3} \ge c_4(g(t_k))^{-(\varepsilon^*-\varepsilon_1)^2-\delta_3}$$

where δ_3 can be chosen such that $(\varepsilon^*+\varepsilon_1)^2+\delta_3<\varepsilon^*$. Consequently,

$$\sum_{k=K_6}^{\infty} P(\lambda_{(t_k,\alpha)}|Y(t_k)| \ge \varepsilon^*-\varepsilon_1)=\infty.$$

By appealing to Borel-Cantelli Lemma, (5) follows. □

3. Similar result for partial sums

Let $(X_n, n\ge 1)$ be a sequence of i.i.d. random variables defined on a common probability space $(\Omega,\mathcal{F},P)$ and let $S_n=\sum_{j-1}^{n} S_j$, $n\ge 1$. Assume that $E(X_1)=0$, $E(X_1^2)=1$ and $E(\mathrm{e}^{tX_1})<\infty$ for $|t|\le t_0$, for some $t_0>0$. Let a_n be a nondecreasing function of n, $n\ge 1$, such that (i) $0<a_n\le n$, (ii) $\frac{a_n}{n}$ nonincreasing and (iii) $\lim_{n\to\infty}\left(\frac{a_n}{\log n}\right)=\infty$. Let

$$\lambda_{(n,\alpha)}=\left(2a_n\left(\log\left(\frac{n}{a_n}\right)+\alpha\log\log n+(1-\alpha)\log\log a_n\right)\right)^{-\frac{1}{2}} \quad \text{and } 0\le\alpha\le 1.$$

We have the following theorem.

Theorem 3.1. *Let (n_k) be an increasing sequence of positive integers. Then*

(a) $\limsup_{k\to\infty} \lambda_{(n_k,\alpha)}|S_{n_k+a_{n_k}} - S_{n_k}| = \limsup_{k\to\infty} \sup_{1\le s\le a_{n_k}} \lambda_{(n_k,\alpha)}|S_{n_k+n} - S_{n_k}| = 1$ *a.s.,*

when $\limsup_{k\to\infty} \dfrac{n_{k+1}-n_k}{a_{n_k}} < 1.$

(b) $\limsup_{k\to\infty} \lambda_{(n_k,\alpha)}|S_{n_k+a_{n_k}} - S_{n_k}| = \limsup_{k\to\infty} \sup_{1\le s\le a_{n_k}} \lambda_{(n_k,\alpha)}|S_{n_k+n} - S_{n_k}| = \varepsilon^*$ *a.s.,*

when $\liminf_{k\to\infty} \dfrac{n_{k+1}-n_k}{a_{n_k}} > 1$ *where*

$$\varepsilon^* = \inf\left\{\gamma > 0 : \sum_k \left(\frac{a_{n_k}}{n_k(\log n_k)^\alpha(\log a_{n_k})^{(1-\alpha)}}\right)^{-\gamma^2} < \infty \text{ where } 0\le\alpha\le 1\right\}.$$

Proof of Theorem 3.1. Note that $\frac{a_n}{\log n} \to \infty$ implies that $\lambda_{(n,\alpha)} \log n \to 0$ as $n\to\infty$. In view of the fact that $E(\mathrm{e}^{tX_1}) < \infty$ for $|t| \le t_0$, for some $t_0 > 0$, the result follows from Theorems 2.1 and 2.2 above from Komlös, Major and Tusnädy ([**5, 6**]). The details are omitted. $\square$

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