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SUBTANGENT-LIKE STATISTICAL MANIFOLDS

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Abstract. Subtangent-like statistical manifolds are introduced and characterization theorems for them are given. The special case when the conjugate connections are projectively (or dual-projectively) equivalent is considered.

1. Introduction

A statistical manifold is a Riemannian manifold, whose each point is a probability space. It connects information geometry, affine differential geometry and Hessian geometry. Information geometry is a branch of mathematics that applies the techniques of differential geometry to the field of probability theory. This is done by taking probability distributions for a statistical model as the points of a Riemannian manifold, forming a statistical manifold. The Fisher information metric provides the Riemannian metric. Information geometry can be applied in various areas, where parametrized distributions play a role such as in statistical inference, time series and linear systems, quantum systems, neuronal networks, machine learning, statistical mechanics, biology, mathematical finance, etc. In [**1**], [**2**], [**3**], [**8**], [**9**] the statistical manifolds are studied from the point of view of information geometry and it is given a new description of the statistical distributions by using the obtained geometric structures. The *statistical manifold* was introduced by S. Amari [**1**] as being a triple (M, ∇, g) consisting of a smooth manifold M, a non-degenerate metric g on it and a torsion-free affine connection ∇ with the property that ∇g is symmetric. With it we can associate another torsion-free affine connection ∇^* defined by the relation

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) \tag{1.1}$$

for any X, Y, $Z \in \mathfrak{X}(M)$, called the *conjugate* (or the *dual*) connection of ∇ with respect to g. In this case, $\nabla^* g$ is also symmetric ($\nabla^* g = -\nabla g$), therefore, (M, ∇^*, g) is a statistical manifold, too. It's easy to see that the conjugate of the conjugate connection of ∇ coincides with ∇, i.e., $(\nabla^*)^* = \nabla$.

Starting from this, different notions of generalized connections were also defined. These were contained in the following general formulation, namely, two connections

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are conjugate in a larger sense if there exists a $(0,3)$-tensor field C on M satisfying

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) + C(X,Y,Z) \tag{1.2}$$

for any X, Y, $Z \in \mathfrak{X}(M)$. A simple condition on C is implied by the symmetry of ∇g, that is, $g((\nabla^*)^*_X Y - \nabla_X Y, Z) = C(X,Y,Z) - C(X,Z,Y)$ [**4**]. For a certain tensor field C, some particular cases were stated in [**4**], [**11**], [**12**], [**13**] defining that ∇ and ∇^* are said to be

1. *generalized conjugate* [**11**], [**12**] with respect to g by a 1-form η if
$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) - \eta(X)g(Y,Z);$$
2. *semi-conjugate* [**11**], [**13**] with respect to g by a 1-form η if
$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) + \eta(Z)g(X,Y);$$
3. *dual semi-conjugate* [**4**] with respect to g by a 1-form η if
$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z) - \eta(X)g(Y,Z) - \eta(Y)g(X,Z).$$

The motivation of studying the first types of conjugate connections comes for the first ones from Weyl geometry [**10**]. The second ones naturally appear in affine hypersurface theory [**13**] and the last ones establish the connection between them [**4**]. One important feature of generalized connections is their invariance under gauge transformations [**4**].

In [**16**], K. Takano studied the statistical manifolds with an almost complex structure. In what follows, we shall study the interference of an almost subtangent structure on a statistical manifold. Recall that the almost tangent structures were introduced by R. S. Clark and M. Bruckheimer [**5**], [**6**] and independently, by H. A. Eliopoulos [**14**]. An *almost tangent structure* on a $2n$-dimensional smooth manifold M is an endomorphism J of the tangent bundle TM of constant rank, satisfying

$$\ker J = \operatorname{Im} J. \tag{1.3}$$

The pair (M,J) is called the *almost tangent manifold.* The name is motivated by the fact that (1.3) implies the nilpotence $J^2 = 0$ exactly as the natural tangent structure of tangent bundles. It is known that the most important G-structures of the first type are those defined by linear operators satisfying certain algebraic relations. Note that the almost tangent structures define a class of conjugate G-structures on M, a group G for a representative structure consisting of all matrices of the form $\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$, where A, B are matrices of order $n \times n$ and A is non-singular.

In addition, if we assume that J is integrable, i.e.,

$$N_J(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X,Y] = 0 \tag{1.4}$$

then J is called the *tangent structure* and (M,J) is called *tangent manifold.*

Basic facts following directly from the definition are stated in [**17**]:

(i) the distribution $\operatorname{Im} J (= \ker J)$ defines a foliation denoted by $V(M)$ and called the the *vertical distribution.*

Example 1.1. [7] $M = \mathbb{R}^2$, $J_e(x,y) = (0,x)$ is a tangent structure with $\ker J_e$ the Y-axis, hence the name. The subscript e comes from "Euclidean".

(ii) there exists an atlas on M with local coordinates $(x,y) = (x^i, y^i)_{1\leq i\leq n}$ such that $J = \frac{\partial}{\partial y^i} \otimes \mathrm{d}x^i$, i.e.,

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \qquad J\left(\frac{\partial}{\partial y^i}\right) = 0. \tag{1.5}$$

We call (x,y) *canonical coordinates* and the change of canonical coordinates $(x,y) \to (\widetilde{x}, \widetilde{y})$ is given by

$$\begin{cases} \widetilde{x}^i = \widetilde{x}^i\,(x) \\ \widetilde{y}^i = \dfrac{\partial \widetilde{x}^i}{\partial x^a} y^a + B^i(x). \end{cases} \tag{1.6}$$

It results the description in terms of G-structures. Namely, a tangent structure is a G-structure with

$$G = \left\{ C = \begin{pmatrix} A & O_n \\ B & A \end{pmatrix} \in GL(2n,\mathbb{R})\ ;\ A \in GL(n,\mathbb{R}),\ B \in gl(n,\mathbb{R}) \right\} \tag{1.7}$$

and G is the invariance group of matrix $J = \begin{pmatrix} O_n & O_n \\ I_n & O_n \end{pmatrix}$, i.e., $C \in G$ if and only if $C \cdot J = J \cdot C$.

The natural almost tangent structure J of $M = TN$ is an example of tangent structure having exactly the expression (1.5) if (x^i) are the coordinates on N and (y^i) are the coordinates in the fibers of $TN \to N$. Also, J_e in Example 1.1 has the above expression (1.5) with $n = 1$, whence it is integrable.

If the condition 1.3 is weakened, requiring that only J squares to 0, we call J an *almost subtangent structure*. In this case, $\operatorname{Im} J \subset \ker J$ and for a non-degenerate metric g on M, $\ker J$ is the Lagrangian distribution for the almost symplectic structure $\omega_{JJ}(X,Y) := g(X,JY) - g(JX,Y)$, X, $Y \in \mathfrak{X}(M)$. In this context we introduce the analogue notion of a holomorphic statistical manifold, namely, the special statistical manifold and give a construction of strong special statistical manifolds.

2. Subtangent-like statistical manifolds

Let M be a smooth manifold, g a non-degenerate metric and J an almost subtangent structure on M.

Definition 2.1. We say that (M,g,J) is an almost subtangent-like manifold if there exists an endomorphism of the tangent bundle J^* satisfying

$$g(JX,Y) + g(X,J^*Y) = 0 \tag{2.1}$$

for any X, $Y \in \mathfrak{X}(M)$.

In this case, $(J^*)^2 = 0$, $(J^*)^* = J$ and J^* is called an *conjugate* (or the *dual*) almost subtangent structure of J.

If J and J^* are two conjugate almost subtangent structures, then $J - J^*$ and $J + J^*$, respectively, are symmetric and skew-symmetric with respect to g. Also, $JJ^* + J^*J$ is symmetric with respect to g.

Example 2.1. Regarding the Example 1.1 (i), we get the metric $g = \mathrm{diag}(1, -1)$ and then $(\mathbb{R}^2, g, J_e)$ is an almost subtangent-like statistical manifold with

$$J_e^*(x, y) = (y, 0)$$

or equivalently

$$J_e^*\left(\frac{\partial}{\partial x}\right) = 0, \qquad J_e^*\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}.$$

Definition 2.2. We say that (M, ∇, g, J) is an almost subtangent-like statistical manifold if (M, ∇, g) is statistical manifold and (M, g, J) is an almost subtangent-like manifold. Moreover, if $\nabla J = 0$, we drop the adjective almost.

From (2.1), we get that $\mathrm{Im}\, J^* \subset \ker J^* \perp_g \mathrm{Im}\, J \subset \ker J$.

Note that on a subtangent-like statistical manifold (M, ∇, g, J), the linear connection ∇ restricts to the distribution $\ker J$, which means that for $Y \in \ker J$, it follows $\nabla_X Y \in \ker J$ for any $X \in \mathfrak{X}(M)$.

Concerning the conjugate structures of an almost subtangent-like statistical manifold, we can state the following result

Proposition 2.3. *Let (M, ∇, g, J) be an almost subtangent-like statistical manifold, ∇^* the conjugate connection of ∇ and J^* a conjugate almost subtangent structure of J. Then*

1. *(M, ∇^*, g, J^*) is an almost subtangent-like statistical manifold;*
2. *$g((\nabla_X J)Y, Z) + g(Y, (\nabla^*_X J^*)Z) = 0$ for any X, Y, $Z \in \mathfrak{X}(M)$;*
3. *if (M, ∇, g, J) is a subtangent-like statistical manifold, then (M, ∇^*, g, J^*) is a subtangent-like statistical manifold, too.*

Proof. 1. From the previous considerations. 2. From a direct computation. 3. From 1 and 2. □

3. J-conjugate connections and the statistical structure

Let g be a non-degenerate metric on M, ∇ an affine connection, ∇^* its conjugate w.r.t. g, J an almost subtangent structure and J^* its conjugate w.r.t. g.

Define the J-conjugate connections of ∇ and ∇^* by

$$\nabla^{(J)} := \nabla - J \circ \nabla J, \qquad \nabla^{*(J)} := \nabla^* - J \circ \nabla^* J$$

and the J^*-conjugate connections of ∇ and ∇^* by

$$\nabla^{(J^*)} := \nabla - J^* \circ \nabla J^*, \qquad \nabla^{*(J^*)} := \nabla^* - J^* \circ \nabla^* J^*.$$

W.r.t. g, the conjugate connection of $\nabla^{(J)}$ is $\nabla^{*(J^*)}$ and the conjugate connection of $\nabla^{(J^*)}$ is $\nabla^{*(J)}$, i.e.,

$$(\nabla^{(J)})^* = \nabla^{*(J^*)}, \qquad (\nabla^{(J^*)})^* = \nabla^{*(J)}.$$

The following propositions establish properties of the J-(and J^*-)conjugate connections of an affine connection and its dual and provide necessary and sufficient conditions for these connections to give rise to statistical or almost subtangent-like statistical manifolds.

Proposition 3.1. *Let (M, ∇, g) be a statistical manifold and J an almost subtangent structure. Then $(M, \nabla^{(J)}, g)$ is a statistical manifold if and only if $\nabla_X JY - \nabla_Y JX \in \ker J$ and $g(X, (\nabla_Z J)Y) = g(Z, (\nabla_X J)Y)$ for any X, Y, $Z \in \mathfrak{X}(M)$.*

Proof. Notice that

$$(\nabla^{(J)}_X g)(Y, Z) = (\nabla_X g)(Y, Z) + g(J(\nabla_X JY), Z) + g(Y, J(\nabla_X JZ))$$

and the relation between the torsion of $\nabla^{(J)}$ and the torsion of ∇ is

$$T_{\nabla^{(J)}}(X, Y) = T_\nabla(X, Y) - J(\nabla_X JY - \nabla_Y JX)$$

for any X, Y, $Z \in \mathfrak{X}(M)$. □

In particular, if J is a subtangent structure, then $(M, \nabla^{(J)}, g)$ is a statistical manifold.

Proposition 3.2. *Let (M, ∇, g, J) be an almost subtangent-like statistical manifold, ∇^* the conjugate connection of ∇ and J^* a conjugate almost subtangent structure of J.*

1. (a) *$(M, \nabla^{(J)}, g, J)$ is an almost subtangent-like statistical manifold if and only if $\nabla_X JY - \nabla_Y JX \in \ker J$ and $g(X, (\nabla_Z J)Y) = g(Z, (\nabla_X J)Y)$ for any X, Y, $Z \in \mathfrak{X}(M)$.*
   (b) *If $(M, \nabla^{(J)}, g, J)$ is an almost subtangent-like statistical manifold, then $(M, (\nabla^{(J)})^*, g, J^*)$ is an almost subtangent-like statistical manifold or equivalent, $(M, \nabla^{*(J^*)}, g, J^*)$ is almost subtangent-like statistical manifold.*
2. (a) *$(M, \nabla^{(J^*)}, g, J^*)$ is an almost subtangent-like statistical manifold if and only if $\nabla_X J^*Y - \nabla_Y J^*X \in \ker J^*$ and $g(X, (\nabla_Z J^*)Y) = g(Z, (\nabla_X J^*)Y)$ for any X, Y, $Z \in \mathfrak{X}(M)$.*
   (b) *If $(M, \nabla^{(J^*)}, g, J^*)$ is an almost subtangent-like statistical manifold, then $(M, (\nabla^{(J^*)})^*, g, J)$ is an almost subtangent-like statistical manifold or equivalent, $(M, \nabla^{*(J)}, g, J)$ is an almost subtangent-like statistical manifold.*

Proof. (a) follows from Proposition 2.3 and (b) from Proposition 2.3 and the observation that $(\nabla^{(J^*)})^* = \nabla^{*(J)}$. □

4. Special statistical manifolds

An analogue of the notion of *holomorphic statistical manifold* defined in [**15**] can be here considered, namely, the *special statistical manifold* (s.s.m.).

4.1. Weak special statistical manifold

Definition 4.1. We say that the subtangent-like statistical manifold (M,∇,g,J) is weak s.s.m. if the 2-form $\omega_J(X,Y) := g(X,JY)$ is ∇-parallel.

A necessary and sufficient condition for a subtangent-like statistical manifold to be weak s.s.m. is given in the following proposition

Proposition 4.2. *The subtangent-like statistical manifold (M,∇,g,J) is weak s.s.m. if and only if*

$$(\nabla_X g)\circ(J\times I) = -g\circ(\nabla_X J\times I)$$

for any $X\in\mathfrak{X}(M)$.

Proof. Computing $(\nabla_X\omega_J)(Y,Z) = -(\nabla_X g)(JY,Z) - g((\nabla_X J)Y,Z)$ for any X, Y, $Z\in\mathfrak{X}(M)$ and from the condition $\nabla\omega_J = 0$, we get the required relation. □

Proposition 4.3. *Under the hypothesis above,*
1. *(M,∇,g,J) is weak s.s.m. if and only if (M,∇,g,J^*) is weak s.s.m..*
2. *(M,∇^*,g,J) is weak s.s.m. if and only if (M,∇^*,g,J^*) is weak s.s.m..*

Proof. It is a consequence of the previous relations. □

4.2. Strong special statistical manifold

Definition 4.4. We say that the subtangent-like statistical manifold (M,∇,g,J) is strong s.s.m. if the 2-form $\omega_{JJ}(X,Y) := g(X,JY) - g(JX,Y)$ is ∇-parallel.

A necessary and sufficient condition for a subtangent-like statistical manifold to be strong s.s.m. is given in the following proposition

Proposition 4.5. *The subtangent-like statistical manifold (M,∇,g,J) is strong s.s.m. if and only if*

$$(\nabla_X g)\circ(I\times J) - (\nabla_X g)\circ(J\times I) = g\circ(\nabla_X J\times I) - g\circ(I\times\nabla_X J),$$

Proof. Follows from Proposition 4.2. □

Consider (M,∇,g,J) an almost subtangent-like statistical manifold, ∇^* the conjugate connection of ∇ and J^* a conjugate almost subtangent structure of J and define

$$\omega_{JJ^*}(X,Y) := g(X,JY) - g(J^*X,Y),$$
$$\omega_{J^*J}(X,Y) := g(X,J^*Y) - g(JX,Y),$$
$$\omega_{J^*J^*}(X,Y) := g(X,J^*Y) - g(J^*X,Y).$$

A simple computation shows that on a subtangent-like statistical manifold (M,∇,g,J), the relation between ω_J, ω_{J^*} and the 2-forms are defined above

$$\omega_{JJ}(X,Y) = \omega_J(X,Y) - \omega_J(Y,X),$$

$$\omega_{JJ^*} = 2\omega_J,\qquad \omega_{J^*J} = 2\omega_{J^*},\qquad \omega_{J^*J^*} = -\omega_{JJ}$$

and

$$(\nabla_X \omega_{JJ^*})(Y, Z) = -(\nabla_X \omega_{J^*J})(Z, Y),$$
$$(\nabla^*_X \omega_{JJ^*})(Y, Z) = -(\nabla^*_X \omega_{J^*J})(Z, Y),$$
$$(\nabla_X \omega_{JJ^*})(Y, Z) + (\nabla^*_X \omega_{JJ^*})(Y, Z) = -g((\nabla_X J^*)Y + (\nabla^*_X J^*)Y, Z)$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Proposition 4.6. *Under the hypothesis above,*
1. (M, ∇, g, J) *is strong s.s.m. if and only if* (M, ∇, g, J^*) *is strong s.s.m..*
2. (M, ∇^*, g, J) *is strong s.s.m. if and only if* (M, ∇^*, g, J^*) *is strong s.s.m..*

Proof. It is a consequence of the fact that $\omega_{JJ} = -\omega_{J^*J^*}$. □

Remark that the notion of weak s.s.m. implies the strong s.s.m., but conversely it's not always true. The notions are equivalent if and only if

$$(\nabla_X \omega_J)(Y, Z) = (\nabla_X \omega_J)(Z, Y),$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Now using the 2-form ω_{JJ}, we can associate to any affine connection ∇ another affine connection ∇^* such that ω_{JJ} is ∇^*-parallel

$$\omega_{JJ}(Y, \nabla^*_X Z) = \omega_{JJ}(Y, \nabla_X Z) + \frac{1}{2}(\nabla_X \omega_{JJ})(Y, Z), \tag{4.1}$$

$X, Y, Z \in \mathfrak{X}(M)$. Indeed, from the skew-symmetry of ω_{JJ}, we get

$$(\nabla^*_X \omega_{JJ})(Y, Z) = -\frac{1}{2}[(\nabla_X \omega_{JJ})(Y, Z) + (\nabla_X \omega_{JJ})(Z, Y)] = 0.$$

Therefore, this procedure gives a way to construct strong special statistical manifolds.

Theorem 4.7. *Let* (M, ∇, g, J) *be an almost subtangent-like statistical manifold and* J^* *a conjugate almost subtangent structure of* J. *If* $(\nabla_{\bar{J}X} g)(Y, Z) = g((\nabla_Y \bar{J})X, Z)$ *for any* $X, Y, Z \in \mathfrak{X}(M)$, *where* $\bar{J} := J + J^*$, *then the affine connection* ∇^* *defined by relation* (4.1) *is the conjugate connection of* ∇. *In this case,* (M, ∇^*, g, J) *is a strong special statistical manifold.*

Proof. Replacing ω_{JJ} in (4.1) and considering the symmetry of ∇g, we obtain the required relation. □

5. Projectively equivalent statistical manifolds

Geometrically, two torsion-free affine connections are projectively equivalent if they have the same geodesics as unparameterized curves. Thus, they determine a class of equivalence on a given manifold called the *projective structure*.

We say that two affine connections ∇ and ∇^* on M are

1. *projectively equivalent* if there exists a 1-form η on M such that

$$\nabla^*_X Y = \nabla_X Y + \eta(X)Y + \eta(Y)X \tag{5.1}$$

for any $X, Y \in \mathfrak{X}(M)$.

2. *dual-projectively equivalent* if there exists a 1-form η on M such that

$$\nabla^*_X Y = \nabla_X Y - g(X,Y)\eta^{\sharp_g}, \tag{5.2}$$

for any X, $Y \in \mathfrak{X}(M)$, where $g(\eta^{\sharp_g}, X) = \eta(X)$, $X \in \mathfrak{X}(M)$.

Note that if two connections are projectively equivalent or dual-projectively equivalent, their conjugate connections may not be projectively or dual-projectively equivalent, respectively.

Proposition 5.1. *If (M, ∇, g) is a statistical manifold and ∇^* is the conjugate connection of ∇, then ∇ and ∇^* are η-projective equivalent or η-dual-projective equivalent if and only if $\eta \otimes I = I \otimes \eta$.*

Proof. Replacing the expression of ∇^* in 1.1 and taking into account that ∇g is symmetric, we obtain the required relation. □

Remark that if $\eta \otimes I = I \otimes \eta$, then for any endomorphism J of the tangent bundle, $\eta \otimes J = (\eta \circ J) \otimes I$ and $J \otimes \eta = I \otimes (\eta \circ J)$.

Proposition 5.2. *Let (M, ∇, g) be a statistical manifold and ∇^* the conjugate connection of ∇. Consider ω a 2-form on M.*

1. *If ∇ and ∇^* are η-projective equivalent, then $\nabla\omega = \nabla^*\omega + 4\eta \otimes \omega$.*
2. *If ∇ and ∇^* are η-dual-projective equivalent, then*

$$(\nabla_X \omega)(Y, Z) = (\nabla^*_X \omega)(Y, Z) - g(X, Y)\omega(\eta^{\sharp_g}, Z) - g(X, Z)\omega(Y, \eta^{\sharp_g})$$

for any X, Y, $Z \in \mathfrak{X}(M)$.

Proof. From the previous proposition. □

Corollary 5.3. *Let (M, ∇, g, J) be an almost subtangent-like statistical manifold and ∇^* the conjugate connection of ∇.*

1. *If ∇ and ∇^* are η-projective equivalent, then $\nabla J = \nabla^* J$ and $\nabla\eta = \nabla^*\eta + 2\eta \otimes \eta$.*
2. *If ∇ and ∇^* are η-dual-projective equivalent, then*

$$(\nabla_X J)Y = (\nabla^*_X J)Y - g(X, Y)J(\eta^{\sharp_g}) + g(X, JY)\eta^{\sharp_g}$$

and

$$(\nabla_X \eta)Y = (\nabla^*_X \eta)Y - g(X, Y)\eta(\eta^{\sharp_g})$$

for any X, $Y \in \mathfrak{X}(M)$.

Proof. From the previous propositions. □

Corollary 5.4. *Let (M, ∇, g, J) be a special almost subtangent-like statistical manifold, ∇^* the conjugate connection of ∇ and ω_J the 2-form defined by (g, J).*

1. *If ∇ and ∇^* are η-projective equivalent, then $\nabla J = \nabla^* J$, $\nabla\eta = \nabla^*\eta + 2\eta \otimes \eta$ and $0 = \nabla^*\omega_J + 4\eta \otimes \omega$.*

2. *If ∇ and ∇^* are η-dual-projective equivalent, then*

$$(\nabla_X J)Y = (\nabla^*_X J)Y - g(X,Y)J(\eta^{\sharp_g}) + g(X,JY)\eta^{\sharp_g},$$
$$(\nabla_X \eta)Y = (\nabla^*_X \eta)Y - g(X,Y)\eta(\eta^{\sharp_g}),$$
$$0 = (\nabla^*_X \omega_J)(Y,Z) - g(X,JY)\eta(Z) - g(JX,Z)\eta(Y)$$

for any X, Y, $Z \in \mathfrak{X}(M)$.

Proof. From the previous propositions and from the fact that $\eta\otimes J^* = (\eta\circ J^*)\otimes I$ and $J\otimes\eta = I\otimes(\eta\circ J)$. □

Remark 5.1. To the class of pairs (∇, ∇^*) which are solutions of the nonlinear system

$$\begin{aligned} \nabla J &= \nabla^* J \\ \nabla\eta &= \nabla^*\eta + 2\eta\otimes\eta \\ \nabla\omega &= \nabla^*\omega + 4\eta\otimes\omega \end{aligned} \tag{5.3}$$

for J, η and ω apriori given, for $\omega_J(X,Y) = g(X,JY)$, the η-projective equivalent conjugate connections on the special almost subtangent-like statistical manifold (M,∇,g,J) belong.

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