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COMPARISON RESULTS FOR NONLINEAR ELLIPTIC EQUATIONS INVOLVING A FINSLER-LAPLACIAN

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Abstract. Picone identity for a Finsler-Lapace operator is established and comparison theorems of the Leighton type for a pair of nonlinear elliptic equations involving such operators are obtained with the help of this new formula.

1. Introduction

The purpose of this paper is to present an identity of the Picone type for the operator of the form

$$\Delta_{H,A} v := \operatorname{div}\big(A(x)H(\nabla v)\nabla_\xi H(\nabla v)\big) \tag{1.1}$$

where $A \in C^1(\overline{\Omega})$ with $A(x) > 0$ on $\overline{\Omega}$ for some bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a piecewise smooth boundary $\partial\Omega$, $H\colon \mathbb{R}^n \to [0, +\infty)$ is a convex function of the class $C^1(\mathbb{R}^n \setminus \{0\})$ which is positively homogeneous of degree 1, and ∇ and ∇_ξ stand for usual gradient operators with respect to the variables x and ξ, respectively. We refer to the operator $\Delta_{H,A}$ as the (weighted) Finsler-Laplacian. An example of H satisfying the above conditions is the l_r-norm

$$H(\xi) = \|\xi\|_r = \left(\sum_{i=1}^{n} |\xi_i|^r\right)^{1/r}, \qquad r > 1, \tag{1.2}$$

for which the operator $\Delta_{H,A}$ has the form

$$\Delta_{H,A} v = \operatorname{div}\left(A(x)\|\nabla v\|_r^{2-r}\nabla^r v\right)$$

where

$$\nabla^r v := \left(\left|\frac{\partial v}{\partial x_1}\right|^{r-2}\frac{\partial v}{\partial x_1}, \dots, \left|\frac{\partial v}{\partial x_n}\right|^{r-2}\frac{\partial v}{\partial x_n}\right). \tag{1.3}$$

Note that $\Delta_{H,A}$ is a nonlinear operator unless $r = 2$ when it reduces to the usual weighted Laplacian $\operatorname{div}(A\nabla v)$. Various elliptic problems involving the Finsler-Laplacian $\Delta_{H,A}$ with $A \equiv 1$ have been recently studied by several authors including [**3**]–[**5**], [**8**]–[**9**], [**12**], [**17**]–[**19**].

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In the case of the Euclidean norm $H(\xi) = \|\xi\|_2, \ \xi \in \mathbb{R}^n$, the following simple formula (that became known as Picone's identity) holds true (see [14]).

Lemma 1.1. *If u, v and $A\nabla v$ are differentiable in a given domain $\Omega \subset \mathbb{R}^n$ and $v(x) \neq 0$ in Ω, then*

$$\begin{aligned} &\operatorname{div}\left(\frac{u^2}{v}A(x)\nabla v\right) \\ &= \frac{u^2}{v}\operatorname{div}\,(A(x)\nabla v) + A(x)\|\nabla u\|_2^2 - A(x)\|\nabla u - \frac{u}{v}\nabla v\|_2^2. \end{aligned} \tag{1.4}$$

Because of its simplicity and wide applicability, the identity (1.4) has become one of the most popular tools of the qualitative and comparison theory of linear differential equations and continues to be the topic of various extensions and generalizations; see, for example, [**1**]–[**2**], [**6**]–[**7**], [**10**]–[**11**], [**13**], [**16**], [**20**].

One of typical results that can easily be obtained by integrating (1.4) over Ω and using the divergence theorem asserts that if the equation $\operatorname{div}(A\nabla v) + Cv = 0$ where $C \in C(\overline{\Omega})$ has solutions satisfying $v(x) \neq 0$ in $\overline{\Omega}$, then

$$J[u;\Omega] := \int_\Omega \big[A(x)\|\nabla u\|_2^2 - C(x)u^2\big]\mathrm{d}x > 0 \tag{1.5}$$

for all $u \in W_0^{1,2}(\Omega) \setminus \{0\}$.

An extended version of formula (1.4) which is sometimes called the "second Picone's identity" says that if a satisfies the same conditions as A and $u, v, a\nabla u$ and $A\nabla v$ are differentiable in Ω with $v(x) \neq 0$, then

$$\begin{aligned} \operatorname{div}\left(ua(x)\nabla u - \frac{u^2}{v}A(x)\nabla v\right) &= u\operatorname{div}(a(x)\nabla u) - \frac{u^2}{v}\operatorname{div}(A(x)\nabla v) \\ &\quad + \big(a(x) - A(x)\big)\,\|\nabla u\|_2^2 \\ &\quad + A(x)\|\nabla u - \frac{u}{v}\nabla v\|_2^2. \end{aligned} \tag{1.6}$$

Formula (1.6) provides a tool for simple proofs of comparison theorems concerning a pair of elliptic equations involving the weighted Laplacians $\operatorname{div}(a\nabla u)$ and $\operatorname{div}(A\nabla v)$, respectively. An example of such results is the Leighton-type integral comparison theorem which asserts that if

$$V[u;\Omega] := \int_\Omega \big[\big(a(x) - A(x)\big)\|\nabla u\|_2^2 + \big((C(x) - c(x)\big)u^2\big]\mathrm{d}x \geq 0 \tag{1.7}$$

for some nontrivial solution u of $\operatorname{div}(a\nabla u) + cu = 0$ satisfying $u = 0$ on $\partial\Omega$, then any solution v of the equation $\operatorname{div}\,(A\nabla v) + Cv = 0$ either has a zero in Ω or it is a constant multiple of u. In particular, if $a(x) \geq A(x)$ and $C(x) \geq c(x)$ for all $x \in \Omega$, then the condition (1.7) is clearly satisfied and from the above result we get the classical Sturm-Picone comparison theorem.

For a survey of other applications of identities (1.4) and (1.6), see [16].

The purpose of this paper is to generalize identity (1.4) to the case where the Euclidean norm $\|.\|_2$ is replaced by an arbitrary norm $H(.)$ in $\mathbb{R}^n$ and to obtain the

Leighton-type comparison result concerning a pair of nonlinear degenerate elliptic equations of the form

$$\operatorname{div}\big(a(x)H(\nabla u)\nabla_\xi H(\nabla u)\big) + c(x)u = 0 \tag{1.8}$$

and

$$\operatorname{div}\big(A(x)H(\nabla v)\nabla_\xi H(\nabla v)\big) + C(x)v = 0 \tag{1.9}$$

where a, c, A, C and H are as above.

The paper is organized as follows. In Section 2 we survey basic properties of general norms in $\mathbb{R}^n$. Section 3 contains an extension of Picone's identity to the Finsler-Laplace operator and comparison results for nonlinear elliptic equations obtained with the help of this new identity. In Section 4 we show how the comparison principle developed in the preceding section yields the nonexistence of positive solutions in exterior domains for a class of equations of the form (1.9).

2. Preliminaries

In this section we recall some of elementary properties of general norms in $\mathbb{R}^n$ which are needed in the sequel. For the proofs see, for instance, [**3**] or [**8**].

Let H be an arbitrary norm in $\mathbb{R}^n$, i.e., a convex function $H\colon \mathbb{R}^n \to [0,\infty)$ satisfying $H(\xi) > 0$ for all $\xi \neq 0$ which is positively homogeneous of degree 1, so that

$$H(t\xi) = |t|H(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } t \in \mathbb{R}. \tag{2.1}$$

Since all norms in $\mathbb{R}^n$ are equivalent, for H there exist positive constants α and β such that

$$\alpha\|\xi\|_2 \le H(\xi) \le \beta\|\xi\|_2$$

for all $\xi \in \mathbb{R}^n$. Let $\langle , \rangle$ denote the usual inner product in $\mathbb{R}^n$ and define the dual norm H_0 of H by

$$H_0(x) = \sup_{\xi\neq 0} \frac{\langle x, \xi\rangle}{H(\xi)} \qquad \text{for } x \in \mathbb{R}^n. \tag{2.2}$$

The set $\mathcal{W}_H := \{x \in \mathbb{R}^n : H_0(x) < 1\}$ is sometimes called the *Wulff shape* (or *equilibrium crystal shape*) of H.

If we assume that $H \in C^1(\mathbb{R}^n \setminus \{0\})$, then from (2.1) it follows that

$$\nabla_\xi H(t\xi) = \operatorname{sgn} t \; \nabla_\xi H(\xi) \qquad \text{for all } \xi \neq 0 \text{ and } t \neq 0 \tag{2.3}$$

and

$$\langle \xi, \nabla_\xi H(\xi)\rangle = H(\xi) \qquad \text{for all } \xi \in \mathbb{R}^n \tag{2.4}$$

where the left-hand side is defined to be 0 if $\xi = 0$. Moreover,

$$H_0(\nabla_\xi H(\xi)) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.5}$$

Similarly, if H_0 is of class C^1 for $x \neq 0$, then

$$H(\nabla H_0(x)) = 1 \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{2.6}$$

Also, the identities

$$H\big[H_0(x)\nabla H_0(x)\big]\nabla_\xi H\big[H_0(x)\nabla H_0(x)\big] = x, \tag{2.7}$$

and

$$H_0\big[H(\xi)\nabla_\xi H(\xi)\big]\nabla H_0\big[H(\xi)\nabla_\xi H(\xi)\big] = \xi, \tag{2.8}$$

hold for all $x, \xi \in \mathbb{R}^n$, where $H(0)\nabla_\xi H(0)$ and $H_0(0)\nabla H_0(0)$ are defined to be 0.

From the definition (2.2) we easily obtain the Hölder-type inequality

$$\langle x, \xi\rangle \le H(\xi)H_0(x) \qquad \text{for all } x, \xi \in \mathbb{R}^n \tag{2.9}$$

with equality holding if and only if

$$x = H(\xi)\nabla_\xi H(\xi). \tag{2.10}$$

In the proof of our main result we will need the following simple lemma which is a consequence of the well-know result from the convex analysis asserting that a continuously differentiable function F defined in an open convex subset of $\mathbb{R}^n$ is strictly convex there if and only if

$$F(y) - F(x) - \big\langle \nabla F(x), y - x\big\rangle > 0 \tag{2.11}$$

for all $x \ne y$.

Lemma 2.1. *Let H be a norm in $\mathbb{R}^n$ such that $H \in C^1(\mathbb{R}^n \setminus \{0\})$ and H^2 is strictly convex. If*

$$H(x)^2 - 2\big\langle x, H(y)\nabla H(y))\big\rangle + H(y)^2 = 0 \tag{2.12}$$

for some $x, y \in \mathbb{R}^n$, $y \ne 0$, and $H(x) = H(y)$, then $x = y$.

Proof. Let $x, y \in \mathbb{R}^n$ with $y \ne 0$ satisfy $H(x) = H(y)$ and (2.12). Adding and subtracting $2\langle y, H(y)\nabla H(y)\rangle$ in (2.12) and using (2.4), we obtain

$$\begin{aligned} 0 &= 2H(y)^2 - 2\big\langle y, H(y)\nabla H(y)\big\rangle + 2\big\langle y - x, H(y)\nabla H(y)\big\rangle \\ &= 2H(y)^2 - 2H(y)\langle y, \nabla H(y)\rangle + 2\big\langle y - x, H(y)\nabla H(y)\big\rangle \\ &= 2\big\langle y - x, H(y)\nabla H(y)\big\rangle. \end{aligned} \tag{2.13}$$

Notice that $2H(y)\nabla H(y) = \nabla\big(H(y)\big)^2 \ne 0$. Indeed, if $\nabla\big(H(y)\big)^2$ were the zero vector for some $y \in \mathbb{R}^n$, i.e., even strictly convex function $H(y)^2$ attained its global minimum at y, then y would necessarily be equal to 0, a contradiction. Therefore, by strict convexity of H^2, $x = y$, and the proof is complete. □

Another elementary inequality that will be needed in the sequel is an immediate consequence of the property $|H(y) - H(x)| \le H(y - x)$ which holds for each $x, y \in \mathbb{R}^n$ and any norm H.

Lemma 2.2. *If H is an arbitrary norm in $\mathbb{R}^n$, then*

$$|H(y)^2 - H(x)^2| \le \big[H(x) + H(y)\big]H(y - x) \tag{2.14}$$

for any $x, y \in \mathbb{R}^n$.

3. Basic identity and comparison theorems

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a piecewise boundary. The following is an extension of Picone's identity (1.4) to the Finsler-Laplace operator $\Delta_{H,A}$ given by (1.1).

Theorem 3.1 (Finsler-Picone identity)**.** *Let H be an arbitrary norm in $\mathbb{R}^n$ which is of class C^1 for $x \neq 0$. If u, v and $AH(\nabla v)\nabla_\xi H(\nabla v)$ are differentiable in a given domain Ω and $v(x) \neq 0$ in Ω, then*

$$\begin{aligned} \operatorname{div}\left(\frac{u^2}{v}A(x)H(\nabla v)\nabla_\xi H(\nabla v)\right) &= \frac{u^2}{v}\Delta_{H,A}v + A(x)H(\nabla u)^2 \\ &- A(x)\left\{H(\nabla u)^2 - 2\frac{u}{v}\langle \nabla u, H(\nabla v)\nabla_\xi H(\nabla v)\rangle + \frac{u^2}{v^2}H(\nabla v)^2\right\}. \end{aligned} \tag{3.1}$$

Moreover, the bracketed expression in (3.1)*, denoted by $\Phi(u,v)$, is nonnegative in Ω. If, in addition, H^2 is strictly convex in $\mathbb{R}^n$, then $\Phi(u,v) = 0$ in Ω if and only if u is a constant multiple of v in each component of Ω.*

Proof. The relation (3.1) can be easily verified by a direct computation. To prove that $\Phi(u,v) \geq 0$, notice that it can be rewritten as $\Phi(u,v) = \Phi_1(u,v) + \Phi_2(u,v)$, where

$$\Phi_1(u,v) = H(\nabla u)^2 - 2H(\nabla u)H\left(\frac{u}{v}\nabla v\right) + H\left(\frac{u}{v}\nabla v\right)^2 = \left[H(\nabla u) - H\left(\frac{u}{v}\nabla v\right)\right]^2$$

and

$$\Phi_2(u,v) = 2\left\{H(\nabla u)H\left(\frac{u}{v}\nabla v\right) - \left\langle \nabla u, H\left(\frac{u}{v}\nabla v\right)\nabla_\xi H\left(\frac{u}{v}\nabla v\right)\right\rangle\right\}.$$

Clearly, $\Phi_1(u,v) \geq 0$ in Ω. The nonnegativity of $\Phi_2(u,v)$ follows from the Hölder inequality (2.9).

Finally, the equality case in $\Phi(u,v) \geq 0$ can occur only if both $\Phi_1(u,v) = 0$ and $\Phi_2(u,v) = 0$ in Ω. The first condition is satisfied if and only if

$$H(\nabla u) = H\left(\frac{u}{v}\nabla v\right) \qquad \text{in } \Omega. \tag{3.2}$$

If $\big(u\nabla v/v\big)(x_0) \neq 0$ for some $x_0 \in S$, then by Lemma 2.1, we have $\nabla u = u\nabla v/v$ at x_0, or equivalently, $\nabla(u/v)(x_0) = 0$. On the other hand, if $u\nabla v/v = 0$, then from (3.2), we get $\nabla u = 0$ which again implies $\nabla(u/v) = 0$. Summarizing the above observations we get $\nabla(u/v) = 0$ in Ω which forces u/v to be constant in each component of Ω. □

In the special case when $H(\xi)$ is an r-norm (1.2), the identity (3.1) becomes

$$\begin{aligned} \operatorname{div}\left(\frac{u^2}{v}\|\nabla v\|_r^{2-r}\nabla^r v\right) &= \frac{u^2}{v}\operatorname{div}\left(\|\nabla v\|_r^{2-r}\nabla^r v\right) + \|\nabla u\|_r^2 \\ &- \left\{\|\nabla u\|_2^r - 2\frac{u}{v}\left\langle \|\nabla v\|_r^{2-r}\nabla^r v, \nabla u\right\rangle + \frac{u^2}{v^2}\|\nabla v\|_r^2\right\}, \end{aligned} \tag{3.3}$$

where $\nabla^r v$ is defined by (1.3).

In what follows, we always assume that the norm $H(\xi)$ is continuously differentiable for $\xi \neq 0$ and that $H(\xi)^2$ is strictly convex in $\mathbb{R}^n$.

As an immediate consequence of the Finsler-Picone identity (3.1) we get the following necessary condition for the existence of zero-free solutions (in $\overline{\Omega}$) of the equation (1.9).

Theorem 3.2. *If* (1.9) *possesses a solution v which satisfies $v(x) \neq 0$ in $\overline{\Omega}$, then*

$$J_H[u;\Omega] := \int_\Omega \big[A(x)H(\nabla u)^2 - C(x)u^2\big]\mathrm{d}x > 0, \tag{3.4}$$

for all $0 \not\equiv u \in D(\Omega) := \{\phi \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$.

Proof. Integrating (3.1) over Ω and making use of the divergence theorem yields

$$J_H[u;\Omega] = \int_\Omega A(x)\Phi(u,v)\mathrm{d}x \geq 0.$$

Since $A(x) > 0, v(x) \neq 0$ in $\overline{\Omega}$ and $u = 0$ on $\partial\Omega$, equality $J_H[u;\Omega] = 0$ cannot occur and the proof is complete. □

The above theorem can be reformulated as a criterion for the existence of zeros of solutions of (1.9) in $\overline{\Omega}$. Such a result belongs to "weaker" Sturmian conlusions in the sense that it establishes the existence of a zero in $\Omega \cup \partial\Omega$ rather than in Ω. Under the additional assumption that the boundary of a domain Ω is smooth, we can prove the following stronger result.

Theorem 3.3. *Let $\partial\Omega \in C^1$. Assume that there exists a nontrivial function $u \in C^1(\overline{\Omega})$ vanishing on $\partial\Omega$ and satisfying*

$$J_H[u;\Omega] := \int_\Omega \big[A(x)H(\nabla u)^2 - C(x)u^2\big]\mathrm{d}x \leq 0. \tag{3.5}$$

Then every solution v of (1.9) *must have a zero in Ω unless v is a constant multiple of u.*

Proof. Suppose that there exists a solution v of (1.9) such that $v(x) \neq 0$ in Ω. Let $\{u_k\}$ denote a sequence of $C_0^\infty(\Omega)$ functions converging to u in the norm

$$\|w\| := \left(\int_\Omega \big[H(\nabla w)^2 + w^2\big]\mathrm{d}x\right)^{\frac{1}{2}}.$$

First, an integration of the identity (3.1) with $u = u_k$ over Ω yields

$$\begin{aligned} J_H[u_k;\Omega] = \int_\Omega A(x)\Big[H(\nabla u_k)^2 &+ H\left(\frac{u_k}{v}\nabla v\right)^2 \\ &- 2\frac{u_k^2}{v^2}\big\langle H(\nabla v)\nabla_\xi H(\nabla v), \nabla u_k\big\rangle\Big]\mathrm{d}x \geq 0. \end{aligned} \tag{3.6}$$

Next, we show that $\lim_{k\to\infty} J_H[u_k;\Omega] = J_H[u;\Omega] = 0$. Since A and C are uniformly bounded, there is a constant $K_1 > 0$ such that

$$\big|J_H[u_k;\Omega] - J_H[u;\Omega]\big| \le K_1 \int_\Omega |H(\nabla u_k)^2 - H(\nabla u)^2|\mathrm{d}x + K_1 \int_\Omega \big|u_k^2 - u^2\big|\mathrm{d}x. \tag{3.7}$$

Observing that

$$\big|H(\nabla u_k)^2 - H(\nabla u)^2\big| \le \big[H(\nabla u_k) + H(\nabla u)\big] H(\nabla(u_k - u)) \tag{3.8}$$

(cf. (2.14)) and using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}&\int_\Omega \big|H(\nabla u_k)^2 - H(\nabla u)^2\big|\mathrm{d}x\\ &\le \Big(\int_\Omega \big[H(\nabla u_k) + H(\nabla u)\big]^2\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_\Omega H(\nabla(u_k - u))^2\mathrm{d}x\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.9}$$

Similarly,

$$\int_\Omega \big|u_k^2 - u^2\big|\mathrm{d}x \le \Big(\int_\Omega (|u_k| + |u|)^2\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_\Omega (u_k - u)^2\mathrm{d}x\Big)^{\frac{1}{2}}. \tag{3.10}$$

Collecting (3.7), (3.9) and (3.10) yields

$$\big|J_H[u_k;\Omega] - J_H[u;\Omega]\big| \le K_2\big(\|u_k\| + \|u\|\big)\|u_k - u\|$$

for some positive constant K_2 which does not depend on k. It follows that $\lim_{k\to\infty} J_H[u_k;\Omega] = J_H[u;\Omega]$. From (3.6) we have $J_H[u;\Omega] \ge 0$, which together with (3.5) implies that $J_H[u;\Omega] = 0$.

Let S be an arbitrary domain with $\bar{S} \subset \Omega$. Then for sufficiently large k, the support of u_k contains $\bar{S}$, so that

$$0 \le \int_S A(x)\Phi(u_k, v)\mathrm{d}x \le \int_\Omega A(x)\Phi(u_k, v)\mathrm{d}x = J_H[u_k;\Omega] \tag{3.11}$$

for all such k. Applying (3.8) and Hölder inequality, we can show analogously as in the first part of the proof that

$$\int_S A(x)\Phi(u_k, v)\mathrm{d}x \to \int_S A(x)\Phi(u, v)\mathrm{d}x \qquad \text{as } k \to \infty.$$

Letting $k \to \infty$ in (3.11), we obtain that

$$\int_S A(x)\Phi(u, v)\mathrm{d}x = 0.$$

Since $A(x) > 0$ in Ω, it follows that $\Phi(u, v) \equiv 0$ identically in S. By the last assertion in Theorem 3.1, v must be a constant multiple of u in S and thus in Ω. This completes the proof. □

Our next result is the Leighton-type integral comparison theorem.

Theorem 3.4. *Let $\partial\Omega \in C^1$. Assume that there exists a nontrivial solution u of* (1.8) *vanishing on $\partial\Omega$ and satisfying*

$$V_H[u;\Omega] := \int_\Omega \big[\big(a(x)-A(x)\big)H(\nabla u)^2 + \big(C(x)-c(x)\big)u^2\big]\mathrm{d}x \geq 0. \tag{3.12}$$

Then every solution v of (1.9) *must have a zero in Ω unless v is a constant multiple of u.*

Proof. If the function u is a nontrivial solution of Eq. (1.8) which satisfies $u = 0$ on $\partial\Omega$, it follows from the divergence theorem that

$$F_H[u;\Omega] := \int_\Omega \big[a(x)H(\nabla u)^2 - c(x)u^2\big]\mathrm{d}x = 0. \tag{3.13}$$

Thus, the condition (3.12) implies

$$J_H[u;\Omega] = F_H[u;\Omega] - V_H[u;\Omega] \leq 0,$$

and the assertion follows from Theorem 3.3. □

The pointwise comparison principle of the Sturm-Picone type for the pair of nonlinear elliptic equations (1.8) and (1.9) with general norms in the principal differential operators is an immediate consequence of Theorem 3.4.

Corollary 1. *Assume that $a(x) \geq A(x)$ and $C(x) \geq c(x)$ in Ω and* (1.8) *has a nontrivial solution u such that $u = 0$ on $\partial\Omega$. Then any solution v of* (1.9) *is either zero at some point in Ω or else $v = ku$ for a nonzero constant k.*

4. Nonexistence of positive solutions in exterior domains

Let $\Omega_r := \{x \in \mathbb{R}^n : H_0(x) > r\}$, $r \geq r_0 > 0$, be the exterior of the H_0-ball with radius r centered at the origin. We apply Theorem 3.4 to demonstrate that the equation

$$\operatorname{div}\big(A(x)H(\nabla v)\nabla_\xi H(\nabla v)\big) + C(x)v = 0, \qquad x \in \Omega_{r_0}, \tag{4.1}$$

may have no positive solutions in Ω_r for any $r > r_0$. This is done by comparing (4.1) with another equation of the same form which is H_0-radially symmetric in the sense that its coefficients $\widetilde{a}$ and $\widetilde{c}$ depend only on $H_0(x)$:

$$\operatorname{div}\big(\widetilde{a}(H_0(x))H(\nabla u)\nabla_\xi H(\nabla u)\big) + \widetilde{c}(H_0(x))u = 0, \qquad x \in \Omega_{r_0}. \tag{4.2}$$

If $u = y(H_0(x))$ is an H_0-radially symmetric solution of (4.2), then $y(r)$ is easily seen to satisfy the linear ODE

$$\big(r^{n-1}\widetilde{a}(r)y'\big)' + r^{n-1}\widetilde{c}(r)y = 0, \qquad r \geq r_0, \tag{4.3}$$

where $' = \mathrm{d}/\mathrm{d}r$.

Theorem 4.1. *Assume that there exist continuous real-valued functions $\widetilde{a}$ and $\widetilde{c}$ defined on $[r_0,\infty)$ with $\widetilde{a}(r) > 0$ in $[r_0,\infty)$ such that* (4.3) *is oscillatory in the sense that any of its solutions has a sequence of zeros clustering at infinity. Let*

$$\max_{H_0(x)=r} A(x) \leq \widetilde{a}(r) \quad \text{and} \quad \min_{H_0(x)=r} C(x) \geq \widetilde{c}(r), \qquad r \geq r_0 > 0. \tag{4.4}$$

Then (4.2) *cannot have solutions v such that $v(x) \neq 0$ in Ω_r for any $r \geq r_0$.*

Proof. Let $y(r)$ be an oscillatory solution of (4.3) on $[r_0, \infty)$ and $\{r_i\}$ be the sequence of its consecutive zeros satisfying $r_0 \leq r_1 < \ldots < r_i < \ldots$, $\lim_{i\to\infty} r_i = \infty$. Then the function u defined by $u(x) := y(H_0(x))$ is an H_0-radially symmetric solution of (4.2) in Ω_{r_0} such that $u(x) = 0$ on $S_{r_i} := \{x \in \mathbb{R}^n : H_0(x) = r_i\}$, $i = 1, 2, \ldots$. Define

$$\Omega_{r_i,r_{i+1}} := \{x \in \mathbb{R}^n : r_i < H_0(x) < r_{i+1}\}, \qquad i = 1, 2, \ldots.$$

Let v be a solution of (4.2) in Ω_r for some $r \geq r_0$. Then $\Omega_{r_i,r_{i+1}} \subset \Omega_r$ for sufficiently large i and

$$\begin{aligned} V_H[u; \Omega_{r_i,r_{i+1}}] = \int_{\Omega_{r_i,r_{i+1}}} & \big[\big(A(x) - \widetilde{a}(H_0(x)\big) H(\nabla u)^2 \\ & - \big(C(x) - \widetilde{c}(H_0(x))\big) u^2\big] \mathrm{d}x \leq 0 \end{aligned} \tag{4.5}$$

because of (4.4). Theorem 3.4 now implies that v must vanish at some points of $\overline{\Omega}_{r_i,r_{i+1}}$ for all i large enough, and the proof is complete. □

An alternative way how to reduce the problem of the existence (nonexistence) of positive solutions of the PDE (4.1) in exterior domains to the one-dimensional oscillation problem is to replace $\widetilde{a}(r)$ and $\widetilde{c}(r)$ in (4.3) by the spherical means $\bar{a}(r)$ and $\bar{c}(r)$ of the coefficients $A(x)$ and $C(x)$ over the Wulff sphere $S_r := \{x \in \mathbb{R}^n : H_0(x) = r\}$, respectively, defined by

$$\bar{a}(r) := \frac{1}{\alpha_n r^{n-1}} \int_{S_r} A(x) \mathrm{d}\sigma, \qquad \bar{c}(r) := \frac{1}{\alpha_n r^{n-1}} \int_{S_r} C(x) \mathrm{d}\sigma, \tag{4.6}$$

where α_n is the surface area of the unit H_0-sphere S_1.

Theorem 4.2. *If the linear ODE*

$$\big(r^{n-1} \bar{a}(r) y'\big)' + r^{n-1} \bar{c}(r) y = 0, \qquad r \geq r_0 > 0, \tag{4.7}$$

with $\bar{a}$ and $\bar{c}$ given by (4.6) *is oscillatory, then the equation* (4.2) *cannot have positive (or negative) solutions in Ω_r for any $r > r_0$.*

Proof. Let $y(r)$ be an oscillatory solution of (4.7) and $(r_0 \leq)$ $r_1 < r_2 < \ldots r_i < \ldots$ be its consecutive zeros with $r_i \to \infty$ as $t \to \infty$. Integrating (4.7) from r_i to r_{i+1} by parts, we have

$$\int_{r_i}^{r_{i+1}} r^{n-1} \big[\bar{a}(r) y'(r)^2 - \bar{c}(r) y^2\big] \mathrm{d}r = 0, \quad i = 1, 2, \ldots.$$

Define the function u by $u(x) := y(H_0(x))$. Then

$$\begin{aligned} J_H[u; \Omega_{r_i,r_{i+1}}] &= \int_{\Omega_{r_i,r_{i+1}}} \big[A(x) H(\nabla u)^2 - C(x) u^2\big] \mathrm{d}x \\ &= \int_{r_i}^{r_{i+1}} \big[y'(r)^2 \int_{S_r} A(x) dS_r - y(r)^2 \int_{S_r} C(x) dS_r\big] \mathrm{d}r \\ &= \alpha_n \int_{r_i}^{r_{i+1}} r^{n-1} \big[\bar{a}(r) y'(r)^2 - \bar{c}(r) y(r)^2\big] \mathrm{d}r = 0. \end{aligned}$$

Thus, the condition (3.5) of Theorem 3.3 is satisfied and consequently, any solution v of (4.1) must have zero in $\Omega_{r_i,r_{i+1}}$ which means that it cannot be positive (or negative) through Ω_r for any $r \geq r_0$. This completes the proof. □

There is a large body of literature on oscillation of the linear Sturm-Liouville equation

$$(p(t)y')' + q(t)y = 0 \tag{4.8}$$

where p and q are continuous functions on $[t_0, \infty)$ with $p(t) > 0$ for $t \geq t_0$ (see, for instance, [**15**] and references therein). Any of the available oscillation criteria for (4.8) when applied to (4.3) or (4.7) yield the corresponding nonexistence result for the original PDE (4.1). For example, the application of the well-known Leighton-Wintner criterion which says that the satisfaction of the conditions

$$\int_{t_0}^{\infty} [1/p(t)]\mathrm{d}t = +\infty, \quad \int_{t_0}^{\infty} q(t)\mathrm{d}t = +\infty \tag{4.9}$$

implies oscillation of Eq.(4.8) gives the following result.

Corollary 2. *Suppose that the continuous functions* $\widetilde{a}(r)$ *and* $\widetilde{c}(r)$ *defined on* $[r_0, \infty)$ *with* $\tilde{a}(r) > 0$ *in* $[r_0, \infty)$ *satisfy* (4.4),

$$\int_{r_0}^{\infty} r^{1-n}\widetilde{a}(r)^{-1}\mathrm{d}r = \infty, \tag{4.10}$$

and

$$\int_{r_0}^{\infty} r^{n-1}\widetilde{c}(r)\mathrm{d}r = \infty. \tag{4.11}$$

Then (4.1) *has no positive solutions in the domain* Ω_r *for any* $r \geq r_0$.

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