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DEGENERATE MONGE-TYPE HYPERSURFACES

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Abstract. In this note, we extend the notion of a Monge hypersurface from its roots in a semi-Euclidean space to more general spaces. For the degenerate case, the geometry of these structures is studied using the Bejancu-Duggal method of screen distributions.

1. Introduction

Semi-Riemannian geometry is a well established branch of mathematics. By comparison, the theory of lightlike manifolds is still relatively new and less developed. If $(\overline{M},\overline{g})$ is a semi-Riemannian manifold and (M,g) is a semi-Riemannian submanifold of $(\overline{M},\overline{g})$, the key to relating the geometry on M with that of $\overline{M}$ is the fact that the tangent bundle of $\overline{M}$ splits as

$$T\overline{M}|_M = TM \oplus NM,$$

where TM and NM, respectively, are the tangent bundle and normal bundle of M. In lightlike geometry, this decomposition is no longer possible since a degenerate metric produces a non-empty intersection between TM and NM.

Lightlike submanifolds arise naturally in semi-Riemannian geometry as well as physics. In semi-Riemannian geometry, the metric tensor is indefinite. Consequently, there is no assurance that the induced metric on any given submanifold will remain non-degenerate. In general relativity, lightlike submanifolds model various types of horizons [**1**], [**14**], [**15**].

To deal with the problems posed by lightlike submanifolds[1], Bejancu and Duggal introduced the notion of *screen distributions* in [**6**], which provides a direct sum decomposition of $T\overline{M}$ with certain nice properties. With a choice of a screen distribution, one can induce geometric objects on a lightlike submanifold in a manner which is analogous to what is done in the classical theory of submanifolds.

For any developing theory of mathematics, examples clearly play an important role in testing ideas, developing concepts, and shaping the overall theory. For the field of lightlike geometry, a number of instructive examples have come in the form of a Monge hypersurface. In addition to being a source of examples for the field,

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[1]For an alternate approach to lightlike geometry, see [**12**], [**13**].

Monge hypersurfaces are also interesting geometric objects in their own right [**4**], [**5**], [**10**], [**6**]. As defined in [**6**], [**10**], a Monge hypersurface lives in semi-Euclidean space, which places limitations on their geometry. In this note, we extend the notion of a Monge hypersurface from its roots in semi-Euclidean space to more general spaces. These new structures, which we call *Monge-type hypersurfaces*, allow for more general geometries, and could, in time, be a source of new and interesting examples of lightlike hypersurfaces.

The rest of the paper is organized as follows. In Section 2, we review the method of screen distributions introduced by Bejancu and Duggal in [**6**]. In Section 3, we develop the basic theory of Monge-type hypersurfaces as it pertains to lightlike geometry. Lastly, in Section 4, we conclude the paper with some basic examples.

2. Preliminaries

In this section, we review the Bejancu-Duggal approach to lightlike geometry [**10**], [**6**]. We begin with the following definition

Definition 2.1. An r-lightlike (or degenerate) manifold (M, g) is a smooth manifold M with a degenerate metric g, which satisfies the following conditions:

1. The radical space

$$\operatorname{Rad} T_pM := \{u \in T_pM \mid g(u, v) = 0 \ \forall v \in T_pM\} \tag{2.1}$$

   has dimension $r > 0$ for all $p \in M$;
2. The distribution defined via $p \mapsto \operatorname{Rad} T_pM$ is smooth.

Definition 2.2. Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold. A submanifold M of $\overline{M}$ is a *lightlike submanifold* if (M, g) is a lightlike manifold, where g is the induced metric on M.

Hence, the fibers of $\operatorname{Rad} TM$ are

$$\operatorname{Rad} T_pM = T_pM \cap T_pM^{\perp}, \tag{2.2}$$

where

$$T_pM^{\perp} := \{u \in T_p\overline{M} \mid \overline{g}(u, v) = 0 \ \forall v \in T_pM\}. \tag{2.3}$$

Since $\dim \operatorname{Rad}\ T_pM > 0$, $T\overline{M}|_M$ does not decompose as the direct sum of TM and $TM^{\perp}$. Consequently, the classical Gauss-Weingarten formulas breakdown for lightlike submanifolds. As a way to remedy this problem, Bejancu and Duggal [**6**] introduced the notion of *screen distributions*. We now review this approach for the special case when (M, g) is a lightlike hypersurface, that is, a lightlike submanifold of codimension 1 in $(\overline{M}, \overline{g})$. Notice that this implies that $TM^{\perp}$ is a line bundle and $TM^{\perp} \subset TM$. By (2.2), we have

$$\operatorname{Rad} TM = TM^{\perp}. \tag{2.4}$$

A *screen distribution* $S(TM)$ is any smooth vector bundle for which

$$TM = S(TM) \oplus TM^{\perp}. \tag{2.5}$$

Notice that (2.5) implies that $\overline{g}$ is non-degenerate on $S(TM)$. The fundamental result of [**6**] for the case of lightlike hypersurfaces can be stated as follows

Theorem 2.3. *Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and let (M, g) be a lightlike hypersurface of $\overline{M}$. For each screen distribution $S(TM)$, there exists a unique line bundle* $\mathrm{tr}(TM)$ *which satisfies the following conditions:*

(i) $T\overline{M}|_M = TM \oplus \mathrm{tr}(TM)$
(ii) *given a non-vanishing local section ξ of $TM^\perp$ which is defined on a neighborhood U of $p \in M$, there exists a unique, non-vanishing local section N_ξ of* $\mathrm{tr}(TM)$ *defined on a neighborhood $U' \subset U$ of p such that*
  (a) $\overline{g}(\xi, N_\xi) = 1$
  (b) $\overline{g}(N_\xi, N_\xi) = 0$
  (c) $\overline{g}(W, N_\xi) = 0$ *for all* $W \in \Gamma(S(TM)|_{U'})$.

The line bundle $\mathrm{tr}(TM)$ appearing in Theorem 2.3 is called the *lightlike transversal bundle*. Explicitly, $\mathrm{tr}(TM)$ is constructed as follows. Let F be any vector bundle for which

$$F \oplus TM^\perp = S(TM)^\perp. \tag{2.6}$$

Notice that F is necessarily a line bundle. For all $p \in M$, choose a non-vanishing local section ξ of $TM^\perp$ and a non-vanishing local section V of F which are both defined on a neighborhood U of p. Since $\overline{g}$ is non-degenerate on $S(TM)$, it follows that

$$T\overline{M}|_M = S(TM) \oplus S(TM)^\perp. \tag{2.7}$$

This implies

$$\overline{g}(\xi, V) \neq 0$$

on U. The local section N_ξ in Theorem 2.3 is then given by

$$N_\xi := \frac{1}{\overline{g}(\xi, V)} \left(V - \frac{\overline{g}(V, V)}{2\overline{g}(\xi, V)} \xi \right). \tag{2.8}$$

It can be shown that the 1-dimensional space spanned by N_ξ is independent of the choice of ξ or the bundle F. Hence, N_ξ determines a rank 1 distribution, which in turn defines the line bundle $\mathrm{tr}(TM)$.

Using the decomposition of Theorem 2.3, one obtains a modified version of the Gauss-Weingarten formulas for lightlike hypersurfaces:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.9}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V \tag{2.10}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(\mathrm{tr}(TM))$, where

(i) $\overline{\nabla}$ is the Levi-Civita connection on $(\overline{M}, \overline{g})$
(ii) $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$
(iii) $h(X, Y)$ and $\nabla^t_X V$ belong to $\Gamma(\mathrm{tr}(TM))$.

It follows from (2.9) that ∇ is a connection on M and h is a $\Gamma(\mathrm{tr}(TM))$-valued $C^\infty(M)$-bilinear form. In addition, a direct verification shows that ∇ is torsion-free and h is symmetric. In (2.10), A_V is a $C^\infty(M)$-linear operator on $\Gamma(TM)$ and ∇^t is a connection on $\mathrm{tr}(TM)$. As in the classical submanifold theory, h is called the *second fundamental tensor*, and A_V is the *shape operator* of M in $\overline{M}$. Lastly, the second fundamental form B_ξ associated with a local section ξ of $TM^\perp$ is defined by

$$B_\xi(X,Y) := \overline{g}(\overline{\nabla}_X Y, \xi). \tag{2.11}$$

It follows from this definition that

$$h(X,Y) = B_\xi(X,Y)N_\xi. \tag{2.12}$$

The primary shortcoming of this approach is that some (not all) of the induced geometric objects on (M,g) are dependent on the choice of $S(TM)$. Hence, the search for spaces with canonical or unique screen distributions has been an area of research for this approach [**7**], [**8**], [**9**], [**2**], [**3**]. Fortunately, this framework does contain objects which are independent of the choice of a screen distribution. Consequently, these objects provide the aforementioned theory with well-defined invariants. We conclude this section by recalling some of these invariants.

Definition 2.4. Let (M,g) be a lightlike hypersurface with the screen distribution $S(TM)$ and let B_ξ denote the second fundamental form associated with a local section ξ of $TM^\perp$. Then (M,g) is

(i) totally geodesic if $B_\xi \equiv 0$
(ii) totally umbilical if $B_\xi = \rho g$ for a smooth function ρ
(iii) minimal if
$$\sum_{i=1}^n \varepsilon_i B_\xi(E_i,E_i) = 0,$$
where E_i, $i = 1,\dots,n$ is any orthonormal local frame of $S(TM)$, and $\varepsilon_i := g(E_i,E_i) \in \{-1,1\}$.

Remark 2.5. Since B_ξ is given by (2.11), statements (i) and (ii) of the above definition are clearly independent of the choice of a screen distribution. Although not quite as apparent, statement (iii) of the above definition is independent of both the choice of orthonormal frame and the choice of a screen distribution. In addition, notice that if ξ' is another local section of $TM^\perp$ (defined on the same open set as ξ), then

$$B_{\xi'} = \lambda B_\xi, \tag{2.13}$$

for a smooth non-vanishing function λ. Hence, (i)–(iii) are also independent of the choice of local section ξ.

3. Monge-Type Hypersurfaces

We begin with the following definition.

Definition 3.1. A *Monge-type hypersurface* (M, g) and its ambient space $(\overline{M}, \overline{g})$ are generated by a triple $(\widehat{M}, \widehat{g}, F)$, where

(i) $(\widehat{M}, \widehat{g})$ is a semi-Riemannian manifold, and
(ii) $F\colon \widehat{M} \to \mathbb{R}$ is a smooth function.

The ambient space $(\overline{M}, \overline{g})$ is the semi-Riemannian manifold defined by

$$\begin{aligned} \overline{M} &:= \mathbb{R} \times \widehat{M} \\ \overline{g} &:= -\mathrm{d}x^0 \otimes \mathrm{d}x^0 + \pi^* \widehat{g}, \end{aligned}$$

where $\pi\colon \overline{M} \to \widehat{M}$ is the natural projection map and x^0 is the natural coordinate associated with the $\mathbb{R}$-component of $\overline{M}$. (M, g) is the hypersuface in $(\overline{M}, \overline{g})$ defined by

$$\begin{aligned} M &:= \{(t, p) \in \overline{M} \mid t = F(p)\} \\ g &:= i^* \overline{g}, \end{aligned}$$

where $i\colon M \hookrightarrow \overline{M}$ is the inclusion map. The triple $(\widehat{M}, \widehat{g}, F)$ is a *Monge-type generator.*

Remark 3.2. Let $\mathbb{R}^{n+1}_{k-1}$ denote an $(n+1)$-dimensional semi-Euclidean space with index $k-1$, that is, the space $\mathbb{R}^{n+1}$ with the metric

$$\eta := -\sum_{i=1}^{k-1} \mathrm{d}x^i \otimes \mathrm{d}x^i + \sum_{j=k}^{n+1} \mathrm{d}x^j \otimes \mathrm{d}x^j.$$

If $\widehat{M}$ in Definition 3.1 is an open submanifold of $\mathbb{R}^{n+1}_{k-1}$ (with the induced metric), then (M, g) coincides with the definition given in [**6**] for a Monge hypersurface.

Definition 3.3. A Monge-type generator is degenerate if its associated Monge-type hypersurface is degenerate.

Theorem 3.4. *Let $(\widehat{M}, \widehat{g}, F)$ be a Monge-type generator. Then the associated Monge-type hypersurface is lightlike iff $\widehat{g}(\widehat{\xi}, \widehat{\xi}) = 1$, where $\widehat{\xi}$ is the gradient of F with respect to $\widehat{g}$.*

Proof. Let $(\overline{M}, \overline{g})$ and (M, g) be defined as in Definition 3.1. Set

$$G := F \circ \pi - x^0 \tag{3.1}$$

$$\xi := \operatorname{grad}_{\overline{g}} G, \tag{3.2}$$

where $\pi\colon \overline{M} \to \widehat{M}$ is the projection map and $\operatorname{grad}_{\overline{g}} G$ is the gradient of G with respect to $\overline{g}$. Since $M = G^{-1}(0)$, it follows that ξ is normal to M. Hence, M is lightlike iff

$$\overline{g}(\xi, \xi) = 0. \tag{3.3}$$

Let $\widehat{\xi}_L$ denote the unique lift of $\widehat{\xi}$ to $\overline{M}$. Then

$$\xi = \frac{\partial}{\partial x^0} + \widehat{\xi}_L. \tag{3.4}$$

Theorem 3.4 then follows from the fact that

$$\overline{g}(\xi, \xi) = -1 + \overline{g}(\widehat{\xi}_L, \widehat{\xi}_L) = -1 + \widehat{g}(\widehat{\xi}, \widehat{\xi}). \tag{3.5}$$

□

Remark 3.5. Let $(\widehat{M}, \widehat{g}, F)$ be a Monge-type generator and $(\overline{M}, \overline{g})$ its associated ambient space. If X is any vector field on $\widehat{M}$, we will denote its unique lift to $\overline{M}$ by X_L, that is, if

$$\pi_1 : \overline{M} \to \mathbb{R}$$
$$\pi_2 : \overline{M} \to \widehat{M}$$

are the natural projection maps, then

$$\pi_{1*} X_L = 0$$
$$\pi_{2*} X_L = X.$$

To simplify notation in some places, we will not distinguish between X_L and X.

From the proof of Theorem 3.4, we have the following corollaries

Corollary 3.6. *Let $(\widehat{M}, \widehat{g}, F)$ be a Monge-type generator and let $(\overline{M}, \overline{g})$ and (M, g) be defined as in Definition 3.1. Then*

$$\xi := \frac{\partial}{\partial x^0} + \widehat{\xi}_L,$$

is normal to M, where $\widehat{\xi} := \mathrm{grad}_{\widehat{g}} F$.

Corollary 3.7. *Let $(\widehat{M}, \widehat{g}, F)$ be a Monge-type generator. If the associated Monge-type hypersurface is degenerate in the sense of Definition 2.1, then $\widehat{M}$ cannot be compact.*

Proof. Let $\widehat{\xi} := \mathrm{grad}_{\widehat{g}} F$. If $\widehat{M}$ is compact, then F must have a critical point somewhere on $\widehat{M}$. Hence, $\widehat{\xi}$ must vanish at some point. The statement of the corollary now follows from Theorem 3.4. □

Corollary 3.8. *Let (M, g) be a Monge hypersurface with generator $(U, \widehat{\eta}, F)$, that is, U is an open submanifold of $\mathbb{R}^{n+1}_{k-1}$ and $\widehat{\eta}$ is the induced metric. Then (M, g) is lightlike iff*

$$-\sum_{i=1}^{k-1} \left(\frac{\partial F}{\partial x^i} \right)^2 + \sum_{j=k}^{n+1} \left(\frac{\partial F}{\partial x^j} \right)^2 = 1.$$

Proof. Let $\widehat{\xi}$ denote the gradient of F with respect to $\widehat{\eta}$. Then

$$\widehat{\xi} = -\sum_{i=1}^{k-1} \frac{\partial F}{\partial x^i} \frac{\partial}{\partial x^i} + \sum_{j=k}^{n+1} \frac{\partial F}{\partial x^j} \frac{\partial}{\partial x^j}. \tag{3.6}$$

For $v \in T_x\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$,

$$\widehat{\eta}(v,v) := -\sum_{i=1}^{k-1} (v^i)^2 + \sum_{j=k}^{n+1} (v^j)^2. \tag{3.7}$$

The corollary now follows from Theorem 3.4. □

The following is an immediate consequence of Definition 3.1.

Lemma 3.9. *Let (M,g) be an $(n+1)$-dimensional Monge-type hypersurface with generator $(\widehat{M}, \widehat{g}, F)$. For $p \in \widehat{M}$ and (U, x^i) a coordinate neighborhood of p, the vector fields*

$$e_i := \frac{\partial F}{\partial x^i} \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^i},\ i = 1, \dots, n+1$$

make up a local frame on M in a neighborhood of $(F(p), p)$.

Theorem 3.10. *Let $(\widehat{M}, \widehat{g}, F)$ be a generator with the ambient space $(\overline{M}, \overline{g})$ and Monge-type hypersurface (M,g). In addition, let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$ and let ξ be defined as in Corollary 3.6. Then the second fundamental form B_ξ of M satisfies*

$$B_\xi(X,Y) = -\operatorname{Hess}(F)(\pi_* X, \pi_* Y), \tag{3.8}$$

for all $X, Y \in \Gamma(TM)$, where $\operatorname{Hess}(F)$ is the Hessian of F in $(\widehat{M}, \widehat{g})$.

Proof. By definition,

$$B_\xi(X,Y) := \overline{g}(\overline{\nabla}_X Y, \xi) = -\overline{g}(Y, \overline{\nabla}_X \xi). \tag{3.9}$$

Let $q \in M \subset \overline{M}$ and let (U, x^i) be a coordinate neighborhood of $\pi(q)$ in $\widehat{M}$. Since B_ξ is $C^\infty(M)$-bilinear, it suffices to show that

$$B_\xi(e_i, e_j) = -\operatorname{Hess}(F)(\pi_* e_i, \pi_* e_j), \tag{3.10}$$

where $\{e_i\}$ is the local frame on M in Lemma 3.9 associated with (U, x^i). Consider the coordinate system $(\mathbb{R} \times U, (x^0, x^i))$ of q in $\overline{M}$. In this coordinate system, the coefficients of $\overline{g}$ and $\widehat{g}$ are related via

$$\overline{g}_{ij} = \widehat{g}_{ij},\ i,j > 0 \tag{3.11}$$

$$\overline{g}_{0i} = 0,\ i > 0 \tag{3.12}$$

$$\overline{g}_{00} = -1. \tag{3.13}$$

Let $\widehat{\nabla}$ be the Levi-Civita connection on $(\widehat{M}, \widehat{g})$. Then (3.11)–(3.13) implies

$$\overline{\nabla}_{e_i} e_j = \frac{\partial^2 F}{\partial x^i \partial x^j} \frac{\partial}{\partial x^0} + \left(\widehat{\nabla}_{\partial_i} \partial_j\right)_L, \tag{3.14}$$

where

$$\partial_i := \frac{\partial}{\partial x^i}.$$

Hence,

$$\overline{g}\left(\overline{\nabla}_{e_i} e_j, \xi\right) = \frac{\partial^2 F}{\partial x^i \partial x^j}\overline{g}(\partial_0, \xi) + \overline{g}\left(\left(\widehat{\nabla}_{\partial_i}\partial_j\right)_L, \xi\right) \tag{3.15}$$

$$= -\frac{\partial^2 F}{\partial x^i \partial x^j} + \widehat{g}\left(\widehat{\nabla}_{\partial_i}\partial_j, \widehat{\xi}\right) \tag{3.16}$$

$$= -\frac{\partial^2 F}{\partial x^i \partial x^j} + \mathrm{d}F\left(\widehat{\nabla}_{\partial_i}\partial_j\right) \tag{3.17}$$

$$= -\operatorname{Hess}(F)(\partial_i, \partial_j) \tag{3.18}$$

$$= -\operatorname{Hess}(F)(\pi_* e_i, \pi_* e_j) \tag{3.19}$$

where the third to last equality follows from the definition of $\widehat{\xi}$ in Theorem 3.4. □

Corollary 3.11. *Let (M, g) be a Monge-type hypersurface with generator $(\widehat{M}, \widehat{g}, F)$ and ambient space $(\overline{M}, \overline{g})$. Then M is totally geodesic iff* $\operatorname{Hess}(F) \equiv 0$ *on $\widehat{M}$.*

In terms of its generator, the following gives a necessary and sufficient condition for a Monge-type hypersurface to be totally umbilical.

Theorem 3.12. *Let (M, g) be a Monge-type hypersurface with generator $(\widehat{M}, \widehat{g}, F)$ and ambient space $(\overline{M}, \overline{g})$. Then M is totally umbilical in $\overline{M}$ iff for all $p \in \widehat{M}$, there exists a neighborhood $\widehat{U}$ of p and a smooth function $\widehat{\rho} \in C^\infty(\widehat{U})$ such that*

$$\operatorname{Hess}(F) = \widehat{\rho}\,(dF \otimes dF - \widehat{g})$$

on $\widehat{U}$.

Proof. Let ξ be defined as in Corollary 3.6, let $q \in M$ be any point, let (U, x^i) be a coordinate neighborhood of $\pi(q)$ in $\widehat{M}$, where π is the projection map from $\overline{M}$ to $\widehat{M}$, and let $\{e_i\}$ be the local frame on M from Lemma 3.9 associated with (U, x^i). From Theorem 3.10, the condition that M is totally umbilical is equivalent to

$$\operatorname{Hess}(F)(\pi_* e_i, \pi_* e_j) = -\rho g(e_i, e_j) \tag{3.20}$$

for some smooth function ρ defined on a neighborhood

$$V \subset \pi^{-1}(U) \cap M \tag{3.21}$$

of q in M. Let $\widehat{\rho}$ be the smooth function on the open set $\pi(V) \subset \widehat{M}$ defined by

$$\rho = \widehat{\rho} \circ \pi. \tag{3.22}$$

Expanding the right side of (3.20) gives

$$-\rho g(e_i, e_j) = -\rho\overline{g}(e_i, e_j) \tag{3.23}$$

$$-\rho g(e_i, e_j) = -\widehat{\rho}\left(-\frac{\partial F}{\partial x^i}\frac{\partial F}{\partial x^j} + \widehat{g}(\partial_i, \partial_j)\right) \tag{3.24}$$

$$-\rho g(e_i, e_j) = \widehat{\rho}\left(\frac{\partial F}{\partial x^i}\frac{\partial F}{\partial x^j} - \widehat{g}(\partial_i, \partial_j)\right) \tag{3.25}$$

where it is understood that if the left side of (3.25) is evaluated at $p \in V$, the right side is evaluated at $\pi(p)$. Since $\pi_* e_i = \partial_i$ and

$$\mathrm{d}F(\partial_i) = \frac{\partial F}{\partial x^i}, \tag{3.26}$$

we have

$$\mathrm{Hess}(F) = \widehat{\rho}\,(\mathrm{d}F \otimes \mathrm{d}F - \widehat{g}) \tag{3.27}$$

on $\pi(V)$. This completes the proof. □

Corollary 3.13. *Let M be a Monge hypersurface of $\mathbb{R}^{n+2}_k$ with generator $(\widehat{U}, \widehat{\eta}, F)$. Then M is totally umbilical iff there exists a smooth function $\widehat{\rho}$ on $\widehat{U}$ such that*

$$\frac{\partial^2 F}{\partial x^i \partial x^j} = \widehat{\rho}\left(\frac{\partial F}{\partial x^i}\frac{\partial F}{\partial x^j} - \widehat{\eta}_{ij}\right), \qquad 1 \le i, j \le n+1. \tag{3.28}$$

Theorem 3.14. *Every degenerate Monge-type hypersurface has a canonical screen distribution which is integrable. If $(\widehat{M}, \widehat{g}, F)$ is a lightlike generator with ambient space $(\overline{M}, \overline{g})$, the lightlike transversal line bundle associated with the canonical screen is spanned by*

$$N_\xi = -\frac{1}{2}\left(\frac{\partial}{\partial x^0} - \widehat{\xi}_L\right),$$

where $\widehat{\xi} := \mathrm{grad}_{\widehat{g}} F$. In addition, the vector field N_ξ satisfies $\overline{g}(\xi, N_\xi) = 1$.

Proof. Let (M, g) be a degenerate Monge-type hypersurface with generator $(\widehat{M}, \widehat{g}, F)$ and ambient space $(\overline{M}, \overline{g})$. Let

$$V := -\frac{\partial}{\partial x^0} \in \Gamma(T\overline{M}) \tag{3.29}$$

and let ξ be defined as in Corollary 3.6. The canonical screen distribution is then defined by setting

$$S(TM) = (\mathcal{L}_\xi \oplus \mathcal{L}_V)^\perp, \tag{3.30}$$

where $\mathcal{L}_\xi$ and $\mathcal{L}_V$ are the line bundles over M with sections $\xi|_M$ and $V|_M$, respectively. Since

$$TM = \mathcal{L}_\xi^\perp, \tag{3.31}$$

it follows that $S(TM) \subset TM$. In addition, since

$$\overline{g}(\xi, V) = 1, \tag{3.32}$$

it follows that $\overline{g}$ is non-degenerate on $S(TM)$. Hence, $\xi_p \notin S(TM)_p$ for all $p \in M$. This implies

$$TM = S(TM) \oplus TM^{\perp}. \tag{3.33}$$

From (2.8), the lightlike transversal line bundle $\mathrm{tr}(TM)$ associated with $S(TM)$ is spanned by the vector field

$$N_\xi = -\frac{1}{2}\left(\frac{\partial}{\partial x^0} - \widehat{\xi}_L\right). \tag{3.34}$$

The fact that $\overline{g}(\xi, N_\xi) = 1$ is a consequence of Theorem 2.3.

To show that $S(TM)$ is integrable, notice from the definition of V and $\overline{g}$ that

$$\overline{g}(W, V) = 0 \Longleftrightarrow Wx^0 = 0 \tag{3.35}$$

for all $W \in \Gamma(T\overline{M})$. Let $X, Y \in \Gamma(S(TM))$. From the definition of $S(TM)$, it follows that

$$[X, Y]x^0 = X(Yx^0) - Y(Xx^0) = X(0) - Y(0) = 0. \tag{3.36}$$

Hence $[X, Y] \in \mathcal{L}_V^{\perp}$. Lastly, since X, Y are vector fields on M, so is $[X, Y]$. Hence, $[X, Y] \in \mathcal{L}_\xi^{\perp}$. This completes the proof. □

Regarding minimal Monge-type hypersurfaces, we have the following result.

Theorem 3.15. *Let $(\widehat{M}, \widehat{g}, F)$ be a degenerate Monge-type generator. The associated Monge-type hypersurface (M, g) is minimal iff for all $\widehat{p} \in \widehat{M}$, there exists a neighborhood $\widehat{U}$ of $\widehat{p}$ and an orthonormal frame $\{\widehat{E}_i\}$ of the kernel of $\mathrm{d}F|_{\widehat{U}}$ such that*

$$\sum_{i=1}^{n} \varepsilon_i \operatorname{Hess}(F)(\widehat{E}_i, \widehat{E}_i) = 0, \tag{3.37}$$

where $\varepsilon_i = \widehat{g}(\widehat{E}_i, \widehat{E}_i) \in \{-1, 1\}$ and $n := \dim \widehat{M} - 1$.

Proof. Let $S(TM)$ be the canonical screen distribution on (M, g) and let $(\overline{M}, \overline{g})$ be the ambient space associated with $(\widehat{M}, \widehat{g}, F)$. In addition, let ξ be the null vector field tangent to M given by Corollary 3.6.

Suppose that (M, g) is minimal. By definition, this means that for all $p \in M$, there exists a neighborhood U of p and an orthonormal frame $\{E_i\}_{i=1}^n$ of $S(TM)|_U$ such that

$$\sum_{i=1}^{n} \varepsilon_i B_\xi(E_i, E_i) = 0, \tag{3.38}$$

where $\varepsilon_i = g(E_i, E_i)$. Let $\pi \colon \overline{M} \to \widehat{M}$ be the projection map. If necessary, shrink U so that the open set $\widehat{U} := \pi(U)$ is covered by a coordinate system $(x^j)_{j=1}^{n+1}$. Let $\{e_j\}_{j=1}^{n+1}$ be the local frame on M associated with $(\widehat{U}, x^j)$ (see Lemma 3.9). Then

$$E_i = \sum_{j=1}^{n+1} \alpha_i^j e_j = \left(\sum_{j=1}^{n+1} \alpha_i^j \frac{\partial F}{\partial x^j}\right) \frac{\partial}{\partial x^0} + \sum_{j=1}^{n+1} \alpha_i^j \frac{\partial}{\partial x^j}, \quad i = 1, \ldots, n \tag{3.39}$$

for some smooth functions α_i^j, $j = 1, \ldots, n+1$, $i = 1, \ldots, n$. Since E_i is a section of $S(TM)|_U$, we have $\overline{g}(E_i, \partial_0) = 0$, which is equivalent to

$$\sum_{j=1}^{n+1} \alpha_i^j \frac{\partial F}{\partial x^j} = 0. \tag{3.40}$$

Set

$$\widehat{E}_i = \sum_{j=1}^{n+1} \alpha_i^j \frac{\partial}{\partial x^j}, \qquad i = 1, \ldots, n \tag{3.41}$$

where $\widehat{E}_i$ is regarded as a vector field on $\widehat{U}$. As a consequence of (3.40), we have

$$E_i = (\widehat{E}_i)_L, \tag{3.42}$$

where $(\widehat{E}_i)_L$ is the unique lift of $\widehat{E}_i$ to U. In addition, notice that (3.40) is equivalent to

$$\mathrm{d}F(\widehat{E}_i) = 0, \qquad i = 1, \ldots, n. \tag{3.43}$$

Moreover,

$$g(E_i, E_j) = \widehat{g}(\widehat{E}_i, \widehat{E}_j) = \varepsilon_j \delta_{ij}, \tag{3.44}$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise. Equation (3.43) shows that $\widehat{E}_i$ belongs to the kernel of $\mathrm{d}F$ on $\widehat{U}$. By Theorem 3.10, we have

$$\begin{aligned}
\sum_{i=1}^{n} \varepsilon_i \operatorname{Hess}(F)(\widehat{E}_i, \widehat{E}_i) &= \sum_{i=1}^{n} \varepsilon_i \operatorname{Hess}(F)(\pi_*(\widehat{E}_i)_L, \pi_*(\widehat{E}_i)_L) \\
&= -\sum_{i=1}^{n} \varepsilon_i B_\xi((\widehat{E}_i)_L, (\widehat{E}_i)_L) \\
&= -\sum_{i=1}^{n} \varepsilon_i B_\xi(E_i, E_i) \\
&= 0.
\end{aligned}$$

For the converse, we use (3.42) to define E_i in terms of $\widehat{E_i}$. This immediately guarantees that the E_i are orthonormal and satisfy

$$\begin{aligned}
\overline{g}(E_i, \partial_0) &= 0 \\
\overline{g}(E_i, \xi) &= 0,
\end{aligned}$$

where the last equality follows from the fact that the $\widehat{E_i}$ belong to the kernel of $\mathrm{d}F$. Hence, $\{E_i\}_{i=1}^n$ is a local orthonormal frame on $S(TM)$. Running the above calculation in reverse shows that (M, g) is minimal. This completes the proof. $\square$

4. Examples

In this section, two basic examples of degenerate Monge-type hypersurfaces are presented; both examples are totally umbilical.

Example 4.1. Let $(\widehat{M}, \widehat{g}, F)$ be the generator defined by

$$\begin{aligned}\widehat{M} &:= \{x \in \mathbb{R}^{n+1} \mid x^{n+1} > 0\}\\ \widehat{g} &:= \frac{1}{(x^{n+1})^2}(\mathrm{d}x^1 \otimes \mathrm{d}x^1 + \cdots + \mathrm{d}x^{n+1} \otimes \mathrm{d}x^{n+1})\\ F &:= \ln(x^{n+1}).\end{aligned}$$

In other words, $(\widehat{M}, \widehat{g})$ is an $(n+1)$-dimensional hyperbolic space. Let

$$\widehat{\xi} := \mathrm{grad}_{\widehat{g}}\, F = x^{n+1}\frac{\partial}{\partial x^{n+1}}. \tag{4.1}$$

Since $\widehat{g}(\widehat{\xi}, \widehat{\xi}) = 1$, it follows from Theorem 3.4 that the associated Monge-type hypersurface is lightlike. A direct verification shows that

$$\mathrm{Hess}(F)(\partial_i, \partial_j) = \begin{cases} -\dfrac{1}{(x^{n+1})^2} & i = j < n+1\\ 0 & i = j = n+1\\ 0 & i \neq j\end{cases}$$

From this it follows easily that

$$\mathrm{Hess}(F) = \mathrm{d}F \otimes \mathrm{d}F - \widehat{g}.$$

Hence, by Theorem 3.12, the associated Monge-type hypersurface (M, g) is totally umbilical. The null vector field ξ tangent to M (see Corollary 3.6) is

$$\xi = \frac{\partial}{\partial x^0} + x^{n+1}\frac{\partial}{\partial x^{n+1}}. \tag{4.2}$$

Let $\mathrm{tr}(TM)$ denote the lightlike transversal bundle associated with the canonical screen (see Theorem 3.14), and let N_ξ denote the unique section of $\mathrm{tr}(TM)$ associated with ξ . The induced linear connection ∇ on (M, g) is then given by

$$\overline{\nabla}_X Y = \nabla_X Y + g(X, Y)N_\xi \tag{4.3}$$

for all $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ is the Levi-Civita connection on the ambient space $(\overline{M}, \overline{g})$. (Note that the second fundamental form B_ξ is precisely g in this example.)

Example 4.2. Let $(\widehat{M}, \widehat{g}, F)$ be the generator[2] defined by

$$\begin{aligned}\widehat{M} &:= \{(t, r) \mid -\infty < t < \infty, \quad r > R\}\\ \widehat{g} &:= -\left(1 - \frac{R}{r}\right)\mathrm{d}t \otimes \mathrm{d}t + \left(1 - \frac{R}{r}\right)^{-1}\mathrm{d}r \otimes \mathrm{d}r\\ F &:= \sqrt{r}\sqrt{r-R} + R\ln\left(\sqrt{r} + \sqrt{r-R}\right),\end{aligned}$$

[2]The Lorentz manifold $(\widehat{M}, \widehat{g})$ is actually a 2-dimensional submanifold of the Schwartzchild spacetime (see [**11**, pp. 149]).

where $R > 0$ is a constant. A direct verification shows that

$$\widehat{\xi} := \operatorname{grad}_{\widehat{g}} F = \sqrt{\frac{r-R}{r}}\frac{\partial}{\partial r}. \tag{4.4}$$

This shows that $\widehat{g}(\widehat{\xi}, \widehat{\xi}) = 1$. By Theorem 3.4, the associated Monge-type hypersurface (M, g) is lightlike. For the Hessian of F, we have

$$\operatorname{Hess}(F)(\partial_i, \partial_j) = \begin{cases} -\dfrac{R\sqrt{r-R}}{2r^{5/2}} & i = j = t \\ 0 & \text{otherwise} \end{cases}$$

From this, we have

$$\operatorname{Hess}(F) = \rho(\mathrm{d}F \otimes \mathrm{d}F - \widehat{g}),$$

where

$$\rho = -\frac{R}{2r^{3/2}\sqrt{r-R}}. \tag{4.5}$$

Hence, (M, g) is totally umbilical (in the ambient space $(\overline{M}, \overline{g})$) by Theorem 3.12.

The null vector field ξ tangent to M (see Corollary 3.6) is

$$\xi := \frac{\partial}{\partial x^0} + \sqrt{\frac{r-R}{r}}\frac{\partial}{\partial r}.$$

Let $\operatorname{tr}(TM)$ be the lightlike transversal bundle associated with the canonical screen on M (see Theorem 3.14). The unique section of $\operatorname{tr}(TM)$ associated with ξ is then

$$N_\xi = -\frac{1}{2}\left(\frac{\partial}{\partial x^0} - \sqrt{\frac{r-R}{r}}\frac{\partial}{\partial r}\right).$$

Using the above information, the induced linear connection on (M, g) is then given by

$$\overline{\nabla}_X Y = \nabla_X Y + \rho g(X, Y)N_\xi, \quad \forall\ X, Y \in \Gamma(TM). \tag{4.6}$$

References

1. Ashtekar A. and Krishnan B., *Dynamical horizons and their properties*, Physical Review D **68** (2003), 10430–10455.

2. Bejancu A., *A canonical screen distribution on a degenerate hypersurface*, Scientific Bulletin, Series A, Applied Math. and Physics **55** (1993), 55–61.

3. Bejancu A., Ferrandez A. and Lucas P., *A new viewpoint on the geometry of a lightlike hypersurface in a semi-Euclidean space*, Saitama Math. J. **16** (1998), 31–38.

4. Calin C., *Totally umbilical degenerate Monge hypersurfaces of* $\mathbb{R}^4_1$, Publications de L'Institut Mathematique, Novelle serie **60(74)** 1996, 137–142.

5. Cheshkova M. A., *Monge hypersurfaces in Euclidean space*, Russian Mathematics **45(3)** (2001), 69–70.

6. Duggal K., Bejancu A., *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic, 364, 1996.

7. Duggal K., *A report on canonical null curves and screen distribution for lightlike geometry*, Acta. Appl. Math. **95** (2007), 135–149.

8. ———, *On canonical screen for lightlike submanifolds of codimension two*, Central European Journal of Math. **5** (2007), 710–719.

9. ______, *On existence of canonical screens for coisotropic submanifolds*, Int. Electronic J. Geom. **1(1)** (2008), 25–32.

10. Duggal K. and Sahin B., *Differential Geometry of Lightlike Submanifolds*, Birkauser Verlag AG, Bassel 2010.

11. Hawking S. W. and Ellis G. F. R., *The Large Scale Structure of Spacetime*, Cambridge University Press, 1973.

12. Kupeli D., *Singualr Semi-Riemannian Geometry*, Kluwer Academic, 366, 1996.

13. ______, *On Null Submanifolds in Spacetimes*, Geomertiae Dedicata. **23(1)** (1987), 33–51.

14. Sultana J. and Dyer C. C., *Conformal Killing horizons*, J. Math. Phys. **45(12)** (2004), 4764–4776.

15. ______, *Cosmological black holes: A black hole in the Einstein-de Sitter universe*, Gen. Rel. Grav. **37(8)** (2005), 1349–1370.

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