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SOME RESULTS OF F-BIHARMONIC MAPS

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ABSTRACT. In this paper, we give the notion of F-biharmonic maps, which is a generalization of biharmonic maps. We derive the first variation formula which yields F-biharmonic maps. Then we investigate the harmonicity of F-biharmonic maps under the curvature conditions on the target manifold (N, h). We also introduce the stress F-bienergy tensor $S_{F,2}$. Then, by using the stress F-bienergy tensor $S_{F,2}$, we obtain some nonexistence results of proper F-biharmonic maps under the assumption that $S_{F,2} = 0$. Moreover, we derive some monotonicity formulas for the special case of the biharmonic map, i.e., where F-biharmonic map with $F(t) = t$. Then, by using these monotonicity formulas, we obtain new results on the non existence of proper biharmonic isometric immersions from complete manifolds.

1. INTRODUCTION

Harmonic maps play a central roll in variational problems for smooth maps between manifolds $u\colon (M, g) \to (N, h)$ as the critical points of the energy functional $E(u) = \frac{1}{2}\int_M \|du\|^2 dv_g$. On the other hand, in 1981, J. Eells and L. Lemaire [**7**] proposed the problem to consider the k-harmonic maps which are critical maps of the functional

$$E_k(u) = \int_M \frac{\|(d+\delta)^k u\|^2}{2} dv_g$$

for smooth maps $u : M \to N$. G. Y. Jiang [**9**] studied the first and second variation formulas of the bienergy E_2 where critical maps of E_2 are called biharmonic maps. There have been extensive studies on biharmonic maps (for instance, see [**9, 13, 14, 15, 16, 18, 19**]).

Let $F\colon [0, \infty) \to [0, \infty)$ be a C^3 function such that $F' > 0$ on $(0, \infty)$. For a smooth map $u\colon (M, g) \to (N, h)$ between Riemannian manifolds (M, g) and (N, h), we define the F-k-energy $E_{F,k}(u)$ of u by

$$E_{F,k}(u) = \int_M F\Big(\frac{\|(d+\delta)^k u\|^2}{2}\Big) dv_g,$$

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which is $E_k(u)$ if $F(t) = t$. When $k = 1$, we have

$$E_{F,1}(u) = \int_M F\left(\frac{\|du\|^2}{2}\right) dv_g = E_F(u),$$

which was introduced by M. Ara in [**1**]. The critical maps of $E_F(u)$ are called F-harmonic maps which are the generalization of harmonic maps, p-harmonic maps or exponentially harmonic maps. There have been extensive studies in this area (for instance, [**4, 5, 11, 12**]). When $k = 2$, we have

$$E_{F,2}(u) = \int_M F\left(\frac{\|\tau(u)\|^2}{2}\right) dv_g,$$

where $\tau(u) = -\delta du = \operatorname{trace} \widetilde{\nabla}(du)$. It is the bienergy of G.Y. Jiang [**9**], the p-bienergy of P. Hornung and R. Moser [**6**] or exponentially bienergy when $F(t) = t$, $F(t) = (2t)^{\frac{p}{2}}$ or $F(t) = \mathrm{e}^t$. We say that u is an F-biharmonic map if

$$\frac{d}{dt} E_{F,2}(u_t)|_{t=0} = 0$$

for any compactly supported variation $u_t \colon M \to N$ with $u_0 = u$. In this note, we derive the first variation formula which yields F-biharmonic maps. Then we investigate the harmonicity of F-biharmonic maps under the curvature conditions on the target manifold (N, h). We also introduce the stress F-bienergy tensor $S_{F,2}$. Then, by using the stress F-bienergy tensor $S_{F,2}$, we obtain some non existence results of proper F-biharmonic maps under the assumption $S_{F,2} = 0$. Also, we derive some monotonicity formulas for the special case of a biharmonic map, i.e., an F-biharmonic map with $F(t) = t$. Then, by using these monotonicity formulas, we investigate the harmonicity of biharmonic isometric maps from complete manifolds.

Remark 1.1. In [**17**], the authors introduced f-biharmonic maps which are critical points of the bi-f-energy functional

$$E_f^2(u) = \frac{1}{2} \int_M \|\tau_f(u)\|^2 dv_g,$$

where $\tau_f(u) = f\tau(u) + du(\operatorname{grad} f)$ and $f \in C^\infty(M)$. We think that it is more reasonable to call them "bi-f-harmonic maps" as parallel to "biharmonic maps".

2. The first variation formula

Let ∇ and ${}^N\nabla$ always denote the Levi-Civita connections of M and N, respectively. Let $\widetilde{\nabla}$ be the induced connection on $u^{-1}TN$ defined by $\widetilde{\nabla}_X W = {}^N\nabla_{du(X)} W$, where X is a tangent vector of M and W is a section of $u^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}$ on M. We define the F-bitension field $\tau_{F,2}(u)$

of u by

$$\begin{aligned}\tau_{F,2}(u) &= -J(F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u))\\ &= -\widetilde{\triangle}(F'\left(\frac{\|\tau(u)\|^2}{2})\tau(u)\right) - \sum_i R^N(du(e_i), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u))du(e_i),\end{aligned}$$

where J is the Jacobi operator of the second variation for the energy $E(u) = \frac{1}{2}\int_M \|du\|^2 dv_g$, $\widetilde{\triangle} = -\sum_i(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{\nabla_{e_i}e_i})$ is the rough Laplacian on the section of $u^{-1}TN$ and $R^N(X,Y) = [^N\nabla_X, ^N\nabla_Y] - {}^N\nabla_{[X,Y]}$ is the curvature operator on N.

Under the notation above we have the following theorem

Theorem 2.1 (The first variation formula)**.** *Let $u\colon M \to N$ be a smooth map. Then*

$$\frac{d}{dt}E_{F,2}(u_t)\,|_{t=0} = \int_M h(\tau_{F,2}(u), V)dv_g, \tag{1}$$

where $V = \frac{d}{dt}u_t|_{t=0}$.

Proof. Let $\Psi\colon (-\varepsilon,\varepsilon)\times M \to N$ be defined by $\Psi(t,x) = u_t(x)$, where $(-\varepsilon,\varepsilon)\times M$ is equipped with the product metric. We extend the vector fields $\frac{\partial}{\partial t}$ on $(-\varepsilon,\varepsilon)$, X on M naturally on $(-\varepsilon,\varepsilon)\times M$, and denote those also by $\frac{\partial}{\partial t}$, X. Then

$$d\Psi\left(\frac{\partial}{\partial t}\right) = \frac{d}{dt}u_t|_{t=0} = V.$$

We shall use the same notations ∇ and $\widetilde{\nabla}$ for the Levi-Civita connection on $(-\varepsilon,\varepsilon)\times M$ and the induced connection on $\Psi^{-1}TN$, respectively.

We compute

$$\begin{aligned}&\frac{\partial}{\partial t}F\left(\frac{\|\tau(u_t)\|^2}{2}\right)\\ &= F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\frac{1}{2}\frac{\partial}{\partial t}\|\tau(u_t)\|^2\\ &= F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)h\left(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\tau(u_t), \tau(u_t)\right)\\ &= \sum_i F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)h\left(\widetilde{\nabla}_{\frac{\partial}{\partial t}}[(\widetilde{\nabla}_{e_i}d\Psi)(e_i)], \tau(u_t)\right) \qquad (2)\\ &= \sum_i h\left(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\widetilde{\nabla}_{e_i}d\Psi(e_i) - \widetilde{\nabla}_{\frac{\partial}{\partial t}}d\Psi(\nabla_{e_i}e_i)\right., F'\left(\frac{\|\tau(u_t)\|^2}{2})\tau(u_t)\right)\\ &= \sum_i h(R^N\left(d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_i))d\Psi(e_i), F'\left(\frac{\|\tau u_t\|^2}{2}\right)\tau(u_t)\right)\\ &\quad + \sum_i h\left(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}d\Psi\left(\frac{\partial}{\partial t}\right) - \widetilde{\nabla}_{\nabla_{e_i}e_i}d\Psi(\frac{\partial}{\partial t}\right), F'\left(\frac{\|\tau(u_t)\|^2}{2})\tau(u_t)\right),\end{aligned}$$

where we use

$$\widetilde{\nabla}_{\frac{\partial}{\partial t}} d\Psi(e_i) - \widetilde{\nabla}_{e_i} d\Psi\left(\frac{\partial}{\partial t}\right) = d\Psi\left[\frac{\partial}{\partial t}, e_i\right] = 0$$

and

$$\widetilde{\nabla}_{\frac{\partial}{\partial t}} d\Psi(\nabla_{e_i} e_i) - \widetilde{\nabla}_{\nabla_{e_i} e_i} d\Psi\left(\frac{\partial}{\partial t}\right) = d\Psi\left[\frac{\partial}{\partial t}, \nabla_{e_i} e_i\right] = 0$$

for the fifth equality.

Let X_t and Y_t be two compactly supported vector fields on M such that $g(X_t, Z) = h(\widetilde{\nabla}_Z d\Psi\left(\frac{\partial}{\partial t}\right), F'\left(\frac{\|\tau u_t\|^2}{2}\right)\tau(u_t)\Big)$ and $g(Y_t, Z) = h\left(d\Psi\left(\frac{\partial}{\partial t}\right),\right.$ $\left.\widetilde{\nabla}_Z\left(F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t)\right)\right)$ for any vector field Z on M. Then the divergence of X_t and Y_t are given by the following:

$$\begin{aligned}
\operatorname{div}(X_t) = & \sum_k g(\nabla_{e_k} X_t, e_k) = \sum_k e_k g(X_t, e_k) - \sum_k g(X_t, \nabla_{e_k} e_k) \\
= & \sum_k e_k h\left(\widetilde{\nabla}_{e_k} d\Psi\left(\frac{\partial}{\partial t}\right), F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t)\right) \\
& - \sum_k h\left(\widetilde{\nabla}_{\nabla_{e_k} e_k} d\Psi\left(\frac{\partial}{\partial t}\right), F'(\frac{\|\tau(u_t)\|^2}{2})\tau(u_t)\right) \\
= & \sum_k h\left(\widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_k} d\Psi\left(\frac{\partial}{\partial t}\right)\right. \\
& \left. - \widetilde{\nabla}_{\nabla_{e_k} e_k} d\Psi\left(\frac{\partial}{\partial t}\right), F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t)\right) \\
& + \sum_k h\left(\widetilde{\nabla}_{e_k} d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{e_k}\left[F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t)\right]\right)
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
\operatorname{div}(Y_t) = & \sum_k g(\nabla_{e_k} Y_t, e_k) = \sum_k e_k g(Y_t, e_k) - \sum_k g(Y_t, \nabla_{e_k} e_k) \\
= & \sum_k e_k h\left(d\Psi(\frac{\partial}{\partial t}), \widetilde{\nabla}_{e_k}\left[F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t)\right]\right) \\
& - \sum_k h(d\Psi(\frac{\partial}{\partial t}), \widetilde{\nabla}_{\nabla_{e_k} e_k}(F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t))\Big) \\
= & \sum_k h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_k}\left[F'\left(\frac{\|\tau u_t\|^2}{2}\right)\tau(u_t)\right]\right. \\
& \left. - \widetilde{\nabla}_{\nabla_{e_k} e_k}\left[F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t)\right]\right) \\
& + \sum_k h\left(\widetilde{\nabla}_{e_k} d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{e_k}\left[F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t)\right]\right).
\end{aligned} \tag{4}$$

From (2), (3) and (4), we have

$$\begin{aligned}
&\frac{\partial}{\partial t} F\left(\frac{\|\tau(u_t)\|^2}{2}\right)\\
&= \sum_i h\left(R^N\left(d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_i)\right) d\Psi(e_i), F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t)\right)\\
&\quad + \sum_i h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}\left[F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t)\right]\right.\\
&\qquad\qquad \left. - \widetilde{\nabla}_{\nabla_{e_i} e_i}\left[F'\left(\frac{\|\tau(u_t)\|^2}{2}\right)\tau(u_t)\right]\right)\\
&\quad + \operatorname{div}(X_t) - \operatorname{div}(Y_t).
\end{aligned} \tag{5}$$

By (5) and Green's theorem, we have

$$\begin{aligned}
&\frac{d}{dt} E_{F,2}(u_t)|_{t=0}\\
&= \int_M \frac{\partial}{\partial t} F\left(\frac{\|\tau(u_t)\|^2}{2}\right)\bigg|_{t=0} dv_g\\
&= \int_M h\left(-\widetilde{\triangle}\left[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right]\right.\\
&\qquad\qquad \left. - \sum_i R^N\left(du(e_i), \left[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right]\right) du(e_i), V\right) dv_g\\
&= \int_M h(\tau_{F,2}(u), V) dv_g.
\end{aligned}$$

This proves Theorem 2.1. □

The first variation formula allows us to define the notion of an F-biharmonic map for the functional $E_{F,2}(u)$.

Definition 2.2. A smooth map u is called an F-biharmonic map for the functional $E_{F,2}(u)$ if it is a solution of the Euler-Lagrange equation $\tau_{F,2}(u) = 0$.

Remark 2.3. By Definition 2.2, we know that any harmonic map is an F-biharmonic map.

Proposition 2.4. *Let $u\colon M \to N$ be a smooth map. If $\|\tau(u)\|^2$ is constant, then u is F-biharmonic if and only if it is biharmonic.*

Proof. Since $\|\tau(u)\|^2$ is constant, we have

$$\begin{aligned}
\tau_{F,2}(u) &= F'\left(\frac{\|\tau(u)\|^2}{2}\right)\left[-\widetilde{\triangle}(\tau(u)) - \sum_i R^N(du(e_i), \tau(u))du(e_i)\right]\\
&= F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau_2(u),
\end{aligned}$$

so we know that u is F-biharmonic if and only if it is biharmonic. □

Remark 2.5. When $\|\tau(u)\|^2$ is non-constant, we have

$$\begin{aligned}\tau_{F,2}(u) &= F'\left(\frac{\|\tau(u)\|^2}{2}\right)\left[-\widetilde{\triangle}(\tau(u)) - \sum_i R^N(du(e_i),\tau(u))du(e_i)\right]\\ &\quad - \left[\widetilde{\triangle}F'\left(\frac{\|\tau(u)\|^2}{2}\right)\right]\tau(u) + \widetilde{\nabla}_{\operatorname{grad} F'\left(\frac{\|\tau(u)\|^2}{2}\right)}\tau(u)\\ &= F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau_2(u) - \left[\widetilde{\triangle}F'\left(\frac{\|\tau(u)\|^2}{2}\right)\right]\tau(u) + \widetilde{\nabla}_{\operatorname{grad} F'\left(\frac{\|\tau(u)\|^2}{2}\right)}\tau(u).\end{aligned}$$

From this equation, we know that there are many differences between F-biharmonic maps and biharmonic maps when $F(t) = (2t)^{\frac{p}{2}}$, $(p > 2)$ or $F(t) = \mathrm{e}^t$.

3. Non-existence results for F-biharmonic maps

From the definition of an F-biharmonic map, we know that a harmonic map is F-biharmonic map, so a basic question in theory is to understand under what conditions the converse is true. A first general answer to this problem for $F(t) = t$, proved by G. Y. Jiang [**9**], is the following theorem

Theorem 3.1 ([**9**])**.** *Let $u\colon (M,g) \to (N,h)$ be a smooth map. If M is compact, orientable and the sectional curvature of (N,h) is non-positive, i.e.,* $\mathrm{Riem}^N \le 0$*, then u is a biharmonic map if and only if it is harmonic.*

In this section, we will obtain the following results

Theorem 3.2. *Let $u\colon (M,g) \to (N,h)$ be a smooth map. If M is compact, orientable and the sectional curvature of (N,h) is non-positive, i.e.,* $\mathrm{Riem}^N \le 0$*, then u is an F-biharmonic map if and only if it is harmonic.*

Proof. Computing the Laplacian of the function $\|F'(\frac{\|\tau(u)\|^2}{2})\tau(u)\|^2$, we have

$$\begin{aligned}&\triangle\left\|F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right\|^2\\ &= 2\sum_k h\left(\widetilde{\nabla}_{e_k}\left[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right], \widetilde{\nabla}_{e_k}\left[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right]\right)\\ &\quad + 2h\left(-\widetilde{\triangle}\left[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right], F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right).\end{aligned} \tag{6}$$

Since u is an F-biharmonic map, we have

$$\begin{aligned}\tau_{F,2}(u) = &-\widetilde{\triangle}\left(F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right)\\ &- \sum_i R^N\left(du(e_i), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right)du(e_i) = 0.\end{aligned} \tag{7}$$

From (6) and (7), we have

$$\begin{aligned}
&\triangle \| F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\Big\|^2 \\
&= 2\sum_k h(\widetilde{\nabla}_{e_k}\left[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right], \widetilde{\nabla}_{e_k}\left[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right]\Big) \\
&+ 2\sum_i h(R^N\left(du\left(e_i\right), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right) du(e_i), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\Big). \qquad (8)
\end{aligned}$$

Since the section curvature of N is non-positive, i.e., $\mathrm{Riem}^N \leq 0$ and by (8), we have

$$\triangle \|F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\|^2 \geq 0 \qquad (9)$$

By the Green's theorem $\int_M \triangle\|F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\|^2 dv_g = 0$ and (9), we have

$$\triangle\|F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\|^2 = 0,$$

so then $\|F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\|^2$ is constant. From (8), we have

$$\widetilde{\nabla}_{e_k}\left[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right] = 0, \qquad \text{for } k = 1, \ldots, m. \qquad (10)$$

Setting $X = \sum_i h\left(du(e_i), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right)e_i$, we have

$$\begin{aligned}
\mathrm{div}(X) = &\sum_k g(\nabla_{e_k} X, e_k) \\
&= h\left(\tau(u), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right) \\
&\quad + \sum_i h\left(du(e_i), \widetilde{\nabla}_{e_i}\left[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right]\right) \qquad (11) \\
&= h\left(\tau(u), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right) \\
&= F'\left(\frac{\|\tau(u)\|^2}{2}\right)\|\tau(u)\|^2.
\end{aligned}$$

Integrating (11) over M, we have

$$0 = \int_M \mathrm{div}(X) dv_g = \int_M F'\left(\frac{\|\tau(u)\|^2}{2}\right)\|\tau(u)\|^2 dv_g. \qquad (12)$$

From $F'(t) > 0$ on $(0, \infty)$ and (12), we have $\tau(u) = 0$.

□

When u is a Riemannian immersion and dim$M$=dim$N-1$, we can replace the hypothesis $\mathrm{Riem}^N \leq 0$ with the hypothesis $\mathrm{Ricci}^N \leq 0$, and we obtain the following theorem.

Theorem 3.3. *Let $u\colon (M,g) \to (N,h)$ be a Rimannian immersion. If M is compact, orientable, $\mathrm{Ricci}^N \leq 0$ and $\dim M = \dim N - 1$, then u is an F-biharmonic map if and only if it is harmonic.*

Proof. Since u is a Riemannian immersion and $\dim M = \dim N - 1$, we have

$$\begin{aligned}&\sum_i h\left(R^N\left(du(e_i), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right)du(e_i), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right)\\ &= -\,\mathrm{Ricci}^N\left(F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right).\end{aligned} \tag{13}$$

From (8), (13) and $\mathrm{Ricci}^N \leq 0$, we have

$$\triangle\left\|F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right\|^2 \geq 0.$$

Applying the same argument as in the proof of Theorem 3.2, we get the result. □

Theorem 3.4. *Let (M,g) be an m-dimensional complete manifold with $\mathrm{Vol}(M,g) = \infty$. If $u\colon (M,g) \to (N,h)$ is an F-biharmonic map, the sectional curvature of (N,h) is non-positive, i.e., $\mathrm{Riem}^N \leq 0$ and $\int_M \left\|F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right\|^2 dv_g < \infty$, then u is harmonic.*

Proof. Since u is an F-biharmonic map, we have

$$\begin{aligned}\tau_{F,2}(u) =& -\widetilde{\triangle}\left(F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right)\\ &-\sum_i R^N\left(du(e_i), F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right)du(e_i) = 0.\end{aligned} \tag{14}$$

Take any point $x_0 \in M$ and for every $r > 0$, let us consider the following cut off function $\lambda(x)$ on M:

$$\begin{cases} 0 \leq \lambda(x) \leq 1, & x \in M,\\ \lambda(x) = 1, & x \in B_r(x_0),\\ \lambda(x) = 0, & x \in M - B_{2r}(x_0),\\ |\nabla\lambda| \leq \frac{2}{r}, & x \in M, \end{cases} \tag{15}$$

where $B_r(x_0) = \{x \in M : \ d(x,x_0) < r\}$ and d is the distance of (M,g).

Let X be a compactly supported vector field on M such that

$$g(X,Y) = h\left(\widetilde{\nabla}_Y\left[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)], \lambda^2[F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right]\right).$$

Then the divergence of X is given by the following expression

$$
\begin{aligned}
&\operatorname{div}(X)\\
&=\sum_k g(\nabla_{e_k}X,e_k)=\sum_k e_k g(X,e_k)-\sum_k g(X,\nabla_{e_k}e_k)\\
&=\sum_k e_k h\left(\widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big],\lambda^2\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\\
&\quad-\sum_k h\left(\widetilde{\nabla}_{\nabla_{e_k}e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big],\lambda^2\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\\
&=h\left(-\widetilde{\triangle}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big],\lambda^2\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\\
&\quad+\sum_k h\left(\widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big],\widetilde{\nabla}_{e_k}\left(\lambda^2\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\right).
\end{aligned}
\tag{16}
$$

From (14) and (16), we have

$$
\begin{aligned}
&\operatorname{div}(X)\\
&=\sum_k h\left(R^N\left(du(e_k),F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right)du(e_k),\lambda^2\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\\
&\quad+\sum_k h\left(\widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big],\widetilde{\nabla}_{e_k}\left(\lambda^2\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\right).
\end{aligned}
\tag{17}
$$

Integrating (17) over M and $\mathrm{Riem}^N\leq 0$, we get

$$
\begin{aligned}
&\sum_k\int_M h\left(\widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big],\widetilde{\nabla}_{e_k}\left(\lambda^2\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\right)dv_g\\
&=-\sum_k\int_M h\left(R^N\left(du(e_k),F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right)du(e_k),\lambda^2\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)dv_g\\
&=\sum_k\int_M h\left(R^N\left(F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u),du(e_k)\right)du(e_k),\lambda^2\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)dv_g\\
&\leq 0.
\end{aligned}
\tag{18}
$$

From (18), we have

$$
\begin{aligned}
0&\geq\sum_k\int_M h\left(\widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big],\widetilde{\nabla}_{e_k}\left(\lambda^2\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\right)dv_g\\
&=\sum_k\int_M\lambda^2\left\|\widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right\|^2dv_g\\
&\quad+2\sum_k\int_M\lambda e_k(\lambda)h\left(\widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big],\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)dv_g.
\end{aligned}
\tag{19}
$$

Therefore, we have

$$
\begin{aligned}
&\sum_k \int_M \lambda^2 \left\| \widetilde{\nabla}_{e_k} \left[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right] \right\|^2 dv_g \\
&\leq -2\sum_k \int_M \lambda e_k(\lambda) h \left(\widetilde{\nabla}_{e_k} \left[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right], \left[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right] \right) dv_g \\
&= -\sum_k \int_M 2h \left(\lambda\widetilde{\nabla}_{e_k} \left[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right], e_k(\lambda)\left[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right] \right) dv_g \qquad (20) \\
&\leq \sum_k \int_M \left\{ \frac{1}{2}\lambda^2 \left\| \widetilde{\nabla}_{e_k} \left[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right] \right\|^2 \right. \\
&\qquad\qquad \left. + 2[e_k(\lambda)]^2 \left\| F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right\|^2 \right\} dv_g,
\end{aligned}
$$

where we use the following Cauchy-Schwarz inequality

$$
\pm 2h(V,W) \leq \varepsilon\|V\|^2 + \frac{1}{\varepsilon}\|W\|^2
$$

for the second inequality and $\varepsilon = \frac{1}{2}$.

From (20), we have

$$
\begin{aligned}
&\sum_k \int_M \lambda^2 \left\| \widetilde{\nabla}_{e_k} \left[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right] \right\|^2 dv_g \\
&\leq 4\int_M \sum_k [e_k(\lambda)]^2 \left\| F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right\|^2 dv_g, \qquad (21) \\
&\leq \frac{16}{r^2}\int_M \left\| F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right\|^2 dv_g.
\end{aligned}
$$

Since $\int_M \|F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\|^2 dv_g < \infty$ and (M,g) is complete, then we have $(r\to\infty)$

$$
\int_M \sum_k \left\| \widetilde{\nabla}_{e_k} \left[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right] \right\|^2 dv_g = 0.
$$

For every vector field X on M, we have

$$
\widetilde{\nabla}_X \left[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right] = 0.
$$

So we know that $\left\| F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right\|^2$ is constant, say C. Therefore, if $\mathrm{Vol}(M,g)=\infty$ and $C\neq 0$, then

$$
\int_M \left\| F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\right\|^2 dv_g = C^2\,\mathrm{Vol}(M,g) = \infty,
$$

which yields a contradiction. Thus, we have $\left\|F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right\|^2 = C = 0$. From $F'(t) > 0$ on $(0,\infty)$ and $\left\|F'\left(\frac{\|\tau(u)\|^2}{2}\right)\tau(u)\right\|^2 = 0$, we know that $\tau(u) = 0$, i.e. u is harmonic. $\square$

From Theorem 3.4, we have the following corollaries:

Corollary 3.5. *Let (M, g) be an m-dimensional complete manifold with $\mathrm{Vol}(M, g) = \infty$. If $u\colon (M, g) \to (N, h)$ is an exponentially biharmonic map, the sectional curvature of (N, h) is non-positive, i.e.,*

$$\mathrm{Riem}^N \leq 0 \quad \textit{and} \quad \int_M \|\tau(u)\|^2\, \mathrm{e}^{\|\tau(u)\|^2}\, dv_g < \infty,$$

then u is harmonic.

Corollary 3.6. *Let (M, g) be an m-dimensional complete manifold with $\mathrm{Vol}(M, g) = \infty$. If $u\colon (M, g) \to (N, h)$ is a p-biharmonic map, the sectional curvature of (N, h) is non-positive, i.e., $\mathrm{Riem}^N \leq 0$ and $\int_M \|\tau(u)\|^{2p-2} dv_g < \infty$, then u is harmonic.*

Corollary 3.7 ([**15**])**.** *Let (M, g) be an m-dimensional complete manifold with $\mathrm{Vol}(M, g) = \infty$. If $u\colon (M, g) \to (N, h)$ is a biharmonic map, the sectional curvature of (N, h) is non-positive, i.e., $\mathrm{Riem}^N \leq 0$ and $\int_M \|\tau(u)\|^2 dv_g < \infty$, then u is harmonic.*

4. Stress F-bienergy tensor

The stress bienergy tensor and the conservation law of a biharmonic map between Riemannian manifolds were first studied by G.Y. Jiang in [**10**]. Following Jiang's notion, we define the stress F-bienergy tensor of a smooth map as follows.

Definition 4.1. Let $u\colon (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds. The stress F-bienergy tensor of u is defined by

$$\begin{aligned} &S_{F,2}(X,Y) \\ &= F\Big(\frac{\|\tau(u)\|^2}{2}\Big)g(X,Y) + \sum_k h\Big(du(e_k), \widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\Big)g(X,Y) \\ &\quad - h\Big(du(X), \widetilde{\nabla}_Y\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\Big) - h\Big(du(Y), \widetilde{\nabla}_X\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\Big) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$.

Remark 4.2. When $F(t) = t$, we have $S_{F,2}(X,Y) = S_2(X,Y)$, where S_2 is stress bienergy tensor in [**10**].

Theorem 4.3. *For any smooth map $u\colon (M, g) \to (N, h)$*

$$(\mathrm{div}\, S_{F,2})(X) = -h(\tau_{F,2}(u), du(X)) - F''\Big(\frac{\|\tau(u)\|^2}{2}\Big)X\Big(\frac{\|\tau(u)\|^4}{4}\Big)$$

for any vector field $X \in \Gamma(TM)$.

Proof. We choose a local orthonormal frame field $\{e_i\}$ on M with $\nabla_{e_i} e_i|_x = 0$ at a point $x \in M$. Let X be a vector field on M. At x, we compute

$$
\begin{aligned}
&(\operatorname{div} S_{F,2})(X)\\
=\ & \sum_i (\nabla_{e_i} S_{F,2})(e_i, X)\\
=\ & \sum_i e_i S_{F,2}(e_i, X) - S_{F,2}(e_i, \nabla_{e_i} X)\\
=\ & \sum_i e_i \left[F\left(\frac{\|\tau(u)\|^2}{2}\right) g(e_i, X) + \sum_k h\left(du(e_k), \widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right) g(e_i, X)\right.\\
&\left. - h\left(du(e_i), \widetilde{\nabla}_X \Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right) - h\left(du(X), \widetilde{\nabla}_{e_i}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\right]\\
& - \sum_i \left[F\left(\frac{\|\tau(u)\|^2}{2}\right) g(e_i, \nabla_{e_i} X)\right.\\
& + \sum_k h\left(du(e_k), \widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right) g(e_i, \nabla_{e_i} X)\\
& \left. - h\left(du(e_i), \widetilde{\nabla}_{\nabla_{e_i} X}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right) - h\left(du(\nabla_{e_i} X), \widetilde{\nabla}_{e_i}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\right]\\
=\ & X\left(F\left(\frac{\|\tau(u)\|^2}{2}\right)\right) + \sum_k h\left((\widetilde{\nabla} du)(X, e_k), \widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\\
& + \sum_k h\left(du(e_k), \widetilde{\nabla}_X \widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right) - h\left(\tau(u), \widetilde{\nabla}_X\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\\
& - \sum_i h\left(du(e_i), \widetilde{\nabla}_{e_i}\widetilde{\nabla}_X\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\\
& - \sum_k h\left((\widetilde{\nabla} du)(X, e_k), \widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\\
& + \sum_i h\left(du(e_i), \widetilde{\nabla}_{\nabla_{e_i} X}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)\\
& - \sum_i h\left(du(X), \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right)]\\
=\ & - F''\Big(\frac{\|\tau(u)\|^2}{2}\Big) X\Big(\frac{\|\tau(u)\|^4}{4}\Big)\\
& + h\left(\widetilde{\triangle}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big] + \sum_i R^N\left(du(e_i), \Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\right) du(e_i), du(X)\right)\\
=\ & - h(\tau_{F,2}(u), du(X)) - F''\Big(\frac{\|\tau(u)\|^2}{2}\Big) X\Big(\frac{\|\tau(u)\|^4}{4}\Big).
\end{aligned}
$$

□

From Theorem 3.1, we know that if $u\colon M \to N$ is an F-biharmonic map, then

$$
(\operatorname{div} S_{F,2})(X) = -F''\Big(\frac{\|\tau(u)\|^2}{2}\Big) X\Big(\frac{\|\tau(u)\|^4}{4}\Big). \tag{22}
$$

Proposition 4.4. *Let $c\colon I \subset R \to (N, h)$ be a curve parametrized by arc-length. Assume that $S_{F,2} = 0$ and $l_F = \inf_{t\geq 0} \frac{tF'(t)}{F(t)} > 0$. Then c is geodesic.*

Proof. A direct computation shows that

$$\begin{aligned}
0 = S_{F,2}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) &= F\Big(\frac{\|\tau(c)\|^2}{2}\Big) - h\Big(dc\Big(\frac{\partial}{\partial t}\Big), \widetilde{\nabla}_{\frac{\partial}{\partial t}}\Big[F'\Big(\frac{\|\tau(c)\|^2}{2})\tau(c)\Big]\Big)\\
&= F\Big(\frac{\|\tau(c)\|^2}{2}\Big) + h\Big(\tau(c), \Big[F'\Big(\frac{\|\tau(c)\|^2}{2}\Big)\tau(c)\Big]\Big),\\
&> (1 + 2l_F)F\Big(\frac{\|\tau(c)\|^2}{2}\Big).
\end{aligned}$$

If $F(\frac{\|\tau(c)\|^2}{2}) = 0$, then $\tau(c) = 0$. □

Proposition 4.5. *Let $u\colon (M^2, g) \to (N, h)$ be a map from a surface. Then $S_{F,2} = 0$ implies u is harmonic.*

Proof. The trace of $S_{F,2}$ gives the equality

$$\begin{aligned}
0 = \operatorname{trace} S_{F,2} &= F\Big(\frac{\|\tau(u)\|^2}{2}\Big) + 2\Big\langle du, \widetilde{\nabla}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\Big\rangle\\
&\quad - 2\Big\langle du, \widetilde{\nabla}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\Big\rangle\\
&= F\Big(\frac{\|\tau(u)\|^2}{2}\Big),
\end{aligned}$$

so we have $\tau(u) = 0$. □

Proposition 4.6. *Let $u\colon (M^m, g) \to (N, h)$, $m \neq 2$. Then $S_{F,2} = 0$ if and only if*

$$\begin{aligned}
&\frac{2}{m-2}F\Big(\frac{\|\tau(u)\|^2}{2}\Big)g(X,Y) + h\Big(du(X), \widetilde{\nabla}_Y[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)]\Big)\\
&+ h\Big(du(Y), \widetilde{\nabla}_X\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\Big) = 0
\end{aligned} \tag{23}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Since $S_{F,2} = 0$, we have trace$S_{F,2} = 0$. Therefore,

$$\sum_k h\Big(du(e_k), \widetilde{\nabla}_{e_k}\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\Big) = -\frac{m}{m-2}F\Big(\frac{\|\tau(u)\|^2}{2}\Big).$$

Substituting it into the definition of $S_{F,2}$, we obtain

$$\begin{aligned}
0 = S_{F,2}(X,Y) &= -\frac{2}{m-2}F\Big(\frac{\|\tau(u)\|^2}{2}\Big)g(X,Y)\\
&- h\Big(du(X), \widetilde{\nabla}_Y\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\Big) - h\Big(du(Y), \widetilde{\nabla}_X\Big[F'\Big(\frac{\|\tau(u)\|^2}{2}\Big)\tau(u)\Big]\Big).
\end{aligned}$$

□

Proposition 4.7. *A map* $u\colon (M^m, g) \to (N, h)$*,* $m > 2$*, with* $S_{F,2} = 0$ *and* $\operatorname{rank} u \leq m-1$ *is harmonic.*

Proof. Take $p \in M$. Since $\operatorname{rank} u(p) \leq m-1$, there exists a unit vector $X_p \in \operatorname{Ker} du_p$ and for $X = Y = X_p$, (23) becomes $F\Big(\frac{\|\tau(u)\|^2}{2}\Big) = 0$, so $\tau(u) = 0$. □

Corollary 4.8. *Let* $u\colon (M^m, g) \to (N^n, h)$ *be a submersion* ($m > n$)*, if* $S_{F,2} = 0$*, then* u *is harmonic.*

Recall that for two 2-tensors T_1, $T_2 \in \Gamma(T^*M \otimes T^*M)$, their inner product is defined as follows:

$$\langle T_1, T_2 \rangle = \sum_{ij} T(e_i, e_j) T_2(e_i, e_j), \tag{24}$$

where $\{e_i\}$ is an orthonormal basis of M with respect to g. For a vector field $X \in \Gamma(TM)$, by θ_X we denote its dual one form, i.e., $\theta_X(Y) = g(X, Y)$. The covariant derivative of θ_X gives a 2-tensor field $\nabla\theta_X$

$$(\nabla\theta_X)(Y, Z) = (\nabla_Z \theta_X)(Y) = g(\nabla_Z X, Y). \tag{25}$$

If $X = \nabla\varphi$ is the gradient of some function φ on M, then $\theta_X = d\varphi$ and $\nabla\theta_X = \operatorname{Hess}\varphi$.

Lemma 4.9 (cf. [**2, 4**]). *Let* T *be a symmetric* $(0, 2)$*-type tensor field and let* X *be a vector field. Then*

$$\operatorname{div}(i_X T) = (\operatorname{div} T)(X) + \langle T, \nabla\theta_X \rangle = (\operatorname{div} T)(X) + \frac{1}{2}\langle T, L_X g \rangle. \tag{26}$$

Let D be any bounded domain of M with C^1 boundary. By using the Stokes' theorem, we immediately have the following integral formula

$$\int_{\partial D} T(X, \nu) ds_g = \int_D [\langle T, \frac{1}{2} L_X g \rangle + \operatorname{div}(T)(X)] dv_g \tag{27}$$

where ν is the unit outward normal vector field along ∂D.

By (22) and (3), we have

$$\begin{aligned} &\int_{\partial D} S_{F,2}(X, \nu) ds_g \\ &= \int_D \Big[\langle S_{F,2}, \frac{1}{2} L_X g \rangle - F''\Big(\frac{\|\tau(u)\|^2}{2}\Big) X\Big(\frac{\|\tau(u)\|^4}{4}\Big)\Big] dv_g. \end{aligned} \tag{28}$$

When $F(t) = t$, the equation (28) turns into the following equation

$$\int_{\partial D} S_2(X, \nu) ds_g = \int_D \langle S_2, \frac{1}{2} L_X g \rangle dv_g. \tag{29}$$

5. Monotonicity formulas for biharmonic maps

In this section, we investigate the special case of F-biharmonic maps, i.e., biharmonic maps.

Let (M^m, g) be a complete Riemannian manifold with pole x_0. By $r(x)$ denote the g-distance function relative to the pole x_0, that is, $r(x) = \mathrm{dist}_g(x, x_0)$. Set $B(r) = \{x \in M^m : r(x) \le r\}$. By $\lambda_{\max}$ (resp. $\lambda_{\min}$) denote the maximum (resp. minimal) eigenvalues of $\mathrm{Hess}(r^2) - dr \otimes dr$ at each point of $M - \{x_0\}$.

Theorem 5.1. *Let $u\colon (M, g) \to (N, h)$ be an isometric immersion. Assume that there is a constant $\sigma > 0$ such that*

$$\frac{m-1}{2}\lambda_{\min} + 1 - 2\max\{2, \lambda_{\max}\} \ge \sigma. \tag{30}$$

If u is a biharmonic map and $h(\tau(u), \widetilde{\nabla}_{\frac{\partial}{\partial r}} du(\frac{\partial}{\partial r})) \ge 0$, then we have

$$\frac{\int_{B(\rho_1)} \frac{\|\tau(u)\|^2}{2} dv_g}{\rho_1^\sigma} \le \frac{\int_{B(\rho_2)} \frac{\|\tau(u)\|^2}{2} dv_g}{\rho_2^\sigma} \tag{31}$$

for any $0 < \rho_1 \le \rho_2$.

Proof. Since $u\colon M^m \to N$ is an isometric immersion, we have $\tau(u) = mH$, where H is the mean curvature vector field of M in N, so we know that

$$h(\tau(u), du(X)) = h(mH, du(X)) = 0 \tag{32}$$

for any tangent vector field X on M.

Taking $D = B(r)$ and $X = r\frac{\partial}{\partial r}$ in (29), we have

$$\begin{aligned} \int_{\partial B(r)} S_2\Big(r\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big) ds_g &= \int_{B(r)} \langle S_2, \frac{1}{2} L_{r\frac{\partial}{\partial r}} g\rangle dv_g \\ &= \frac{1}{2}\int_{B(r)} \langle S_2, \mathrm{Hess}(r^2)\rangle dv_g. \end{aligned} \tag{33}$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal basis on M and $e_m = \frac{\partial}{\partial r}$. We may assume that $\mathrm{Hess}(r^2)$ becomes a diagonal matrix with respect to $\{e_i\}$.

$$\begin{aligned} -\frac{1}{2}\langle S_2, \mathrm{Hess}(r^2)\rangle &= -\frac{1}{2}\sum_{i,j} S_2(e_i, e_j)\,\mathrm{Hess}(r^2)(e_i, e_j) \\ &= -\frac{1}{2}\{\sum_i \frac{\|\tau(u)\|^2}{2}\,\mathrm{Hess}(r^2)(e_i, e_i) \\ &\quad + \sum_k h(\widetilde{\nabla}_{e_k}\tau(u), du(e_k)) \sum_i \mathrm{Hess}(r^2)(e_i, e_i) \\ &\quad - 2\sum_{i,j} h(du(e_i), \widetilde{\nabla}_{e_j}\tau(u))\,\mathrm{Hess}(r^2)(e_i, e_j)\} \end{aligned} \tag{34}$$

$$
\begin{aligned}
&= -\frac{1}{2}\Big\{-\frac{\|\tau(u)\|^2}{2}\sum_i \mathrm{Hess}(r^2)(e_i,e_i)\\
&\quad + 2\sum_i h(\tau(u),\widetilde{\nabla}_{e_i}du(e_i))\,\mathrm{Hess}(r^2)(e_i,e_i)\Big\}\\
&\geq \frac{\|\tau(u)\|^2}{2}\Big[\frac{m-1}{2}\lambda_{\min}+1-2\max\{2,\lambda_{\max}\}\Big]\\
&\geq \sigma\frac{\|\tau(u)\|^2}{2},
\end{aligned} \tag{34}
$$

where the equation (32) is used for the third equality and the equation (30) for the last inequality.

On the other hand, by the coarea formula, we have

$$
\begin{aligned}
-\int_{\partial B(r)} S_2\Big(r\frac{\partial}{\partial r},\frac{\partial}{\partial r}\Big)ds_g &= -\int_{\partial B(r)}\Big\{\Big[\frac{\|\tau(u)\|^2}{2}+\sum_k h(du(e_k),\widetilde{\nabla}_{e_k}\tau(u))\Big]g\Big(r\frac{\partial}{\partial r},\frac{\partial}{\partial r}\Big)\\
&\quad - 2rh(du\Big(\frac{\partial}{\partial r}\Big),\widetilde{\nabla}_{\frac{\partial}{\partial r}}\tau(u)\Big)\Big\}ds_g\\
&= \int_{\partial B(r)}\Big\{r\frac{\|\tau(u)\|^2}{2} - rh\Big(\tau(u),\widetilde{\nabla}_{\frac{\partial}{\partial r}}du\Big(\frac{\partial}{\partial r}\Big)\Big)\Big\}ds_g\\
&\leq \int_{\partial B(r)} r\frac{\|\tau(u)\|^2}{2}ds_g\\
&= r\frac{d}{dr}\int_{B(r)}\frac{\|\tau(u)\|^2}{2}dv_g,
\end{aligned} \tag{35}
$$

where the condition $h(\tau(u),\widetilde{\nabla}_{\frac{\partial}{\partial r}}du(\frac{\partial}{\partial r}))\geq 0$ is used for the inequality.

From (33), (34) and (35), we have

$$
\sigma\int_{B(r)}\frac{\|\tau(u)\|^2}{2}dv_g \leq r\frac{d}{dr}\int_{B(r)}\frac{\|\tau(u)\|^2}{2}dv_g \tag{36}
$$

i.e.

$$
\frac{d}{dr}\frac{\int_{B(r)}\frac{\|\tau(u)\|^2}{2}dv_g}{r^\sigma}\geq 0. \tag{37}
$$

Therefore,

$$
\frac{\int_{B(\rho_1)}\frac{\|\tau(u)\|^2}{2}dv_g}{\rho_1^\sigma}\leq\frac{\int_{B(\rho_2)}\frac{\|\tau(u)\|^2}{2}dv_g}{\rho_2^\sigma}
$$

for any $0<\rho_1\leq\rho_2$. □

Lemma 5.2 ([**4, 8**])**.** *Let (M^m,g) be a complete Riemannian manifold with a pole x_0. By K_r denote the radial curvature of M as follows*

(i) *if $-\alpha^2\leq K_r\leq-\beta^2$ with $\alpha\geq\beta>0$, then*

$$
\beta\coth(\beta r)[g-dr\otimes dr]\leq\mathrm{Hess}(r)\leq\alpha coth(\alpha r)[g-dr\otimes dr],
$$

(ii) *if* $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ *with* $\varepsilon > 0$, $A \geq 0$ *and* $0 \leq B < 2\varepsilon$, *then*

$$\frac{1 - B/2\varepsilon}{r}[g - dr \otimes dr] \leq \mathrm{Hess}(r) \leq \frac{\mathrm{e}^{A/2\varepsilon}}{r}[g - dr \otimes dr],$$

(iii) *if* $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ *with* $a \geq 0$ *and* $b^2 \in [0, \frac{1}{4}]$, *then*

$$\frac{1 + \sqrt{1 - 4b^2}}{2r}[g - dr \otimes dr] \leq \mathrm{Hess}(r) \leq \frac{1 + \sqrt{1 + 4a^2}}{2r}[g - dr \otimes dr].$$

Lemma 5.3. *Let* (M^m, g) *be a complete Riemannian manifold with a pole* x_0. *By* K_r *denote the radial curvature of* M *as follows*

(i) *if* $-\alpha^2 \leq K_r \leq -\beta^2$ *with* $\alpha \geq \beta > 0$ *and* $(m-1)\beta - 4\alpha \geq 0$, *then*

$$\frac{(m-1)}{2}\lambda_{\min} + 1 - 2\max\{2, \lambda_{\max}\} \geq m - \frac{4\alpha}{\beta}.$$

(ii) *if* $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ *with* $\varepsilon > 0$, $A \geq 0$ *and* $0 \leq B < 2\varepsilon$, *then*

$$\frac{(m-1)}{2}\lambda_{\min} + 1 - 2\max\{2, \lambda_{\max}\} \geq 1 + (m-1)(1 - \frac{B}{2\varepsilon}) - 4e^{\frac{A}{2\varepsilon}}.$$

(iii) *if* $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ *with* $a \geq 0$ *and* $b^2 \in [0, \frac{1}{4}]$, *then*

$$\begin{aligned} &\frac{(m-1)}{2}\lambda_{\min} + 1 - 2\max\{2, \lambda_{\max}\} \\ &\geq [1 + (m-1)\frac{1 + \sqrt{1 - 4b^2}}{2} - 4\frac{1 + \sqrt{1 + 4a^2}}{2}. \end{aligned}$$

Proof. If K_r satisfies (i), then by Lemma 5.2, for every $r > 0$, we have on $B(r) - \{x_0\}$,

$$\begin{aligned} &\frac{1}{2}[(m-1)\lambda_{\min} + 2 - 4\max\{2, \lambda_{\max}\}] \\ &\geq \frac{1}{2}[(m-1)2\beta r\coth(\beta r) + 2 - 4 \times 2\alpha r\coth(\alpha r)] \\ &= 1 + \beta r\coth(\beta r)\Big(m - 1 - \frac{4\alpha}{\beta}\frac{\coth(\alpha r)}{\coth(\beta r)}\Big) \\ &\geq 1 + 1.\Big(m-1) - \frac{4\alpha}{\beta}\Big) \\ &= m - \frac{4\alpha}{\beta}. \end{aligned}$$

where the second inequality is valid the increasing function $\beta r\coth(\beta r) \to 1$ as $r \to 0$, and $\frac{\coth(\alpha r)}{\coth(\beta r)} < 1$ for $0 < \beta < \alpha$. Similarly, from Lemma 5.2, the above inequality holds for the cases (ii) and (iii) on $B(r)$. □

Theorem 5.4. *Let* (M, g) *be an* m*-dimensional complete manifold with a pole* x_0. *Assume that the radial curvature* K_r *of* M *satisfies one of the following three conditions:*

(i) *if if* $-\alpha^2 \leq K_r \leq -\beta^2$ *with* $\alpha \geq \beta > 0$ *and* $(m-1)\beta - 4\alpha \geq 0$,

(ii) *if* $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ *with* $\varepsilon > 0$, $A \geq 0$, $0 \geq B < 2\varepsilon$ *and* $1 + (m-1)(1 - \frac{B}{2\varepsilon}) - 4\,\mathrm{e}^{\frac{A}{2\varepsilon}} > 0$,

(iii) *if* $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ *with* $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ *and* $1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\frac{1+\sqrt{1+4a^2}}{2} > 0$.

If $u\colon (M,g) \to (N,h)$ *is a biharmonic isometric immersion and* $h(\tau(u), \widetilde{\nabla}_{\frac{\partial}{\partial r}} du(\frac{\partial}{\partial r})) \geq 0$, *then*

$$\frac{\int_{B(\rho_1)} \frac{\|\tau(u)\|^2}{2} dv_g}{\rho_1^\Lambda} \leq \frac{\int_{B(\rho_2)} \frac{\|\tau(u)\|^2}{2} dv_g}{\rho_2^\Lambda} \tag{38}$$

for any $0 < \rho_1 \leq \rho_2$, *where*

$$\Lambda = \begin{cases} m - \dfrac{4\alpha}{\beta}, & \text{if } K_r \text{ satisfies (i)} \\ 1 + (m-1)\Big(1 - \dfrac{B}{2\varepsilon}\Big) - 4\,\mathrm{e}^{\frac{A}{2\varepsilon}}, & \text{if } K_r \text{ satisfies (ii)} \\ 1 + (m-1)\dfrac{1+\sqrt{1-4b^2}}{2} - 4\dfrac{1+\sqrt{1+4a^2}}{2}, & \text{if } K_r \text{ satisfies (iii)} \end{cases} \tag{39}$$

Proof. From the proof of Theorem 5.1 and Lemma 5.3, we have

$$\frac{d}{dr}\frac{\int_{B(r)} \frac{\|\tau(u)\|^2}{2} dv_g}{r^\Lambda} \geq 0.$$

Therefore, we get the monotonicity formula

$$\frac{\int_{B(\rho_1)} \frac{\|\tau(u)\|^2}{2} dv_g}{\rho_1^\Lambda} \leq \frac{\int_{B(\rho_2)} \frac{\|\tau(u)\|^2}{2} dv_g}{\rho_2^\Lambda}$$

for any $0 < \rho_1 \leq \rho_2$. □

Corollary 5.5. *Let* M, K_r *and* Λ *be as in Theorem 5.4. Assume that* $u\colon (M,g) \to (N,h)$ *is a biharmonic isometric immersion and* $h(\tau(u), \widetilde{\nabla}_{\frac{\partial}{\partial r}} du(\frac{\partial}{\partial r})) \geq 0$. *If*

$$\int_{B(R)} \frac{\|\tau(u)\|^2}{2} dv_g = o(R^\Lambda),$$

then u *is harmonic.*

We say the bienergy $E_2(u)$ of u is slowly divergent if there exists a positive function $\psi(r)$ with $\int_{R_0}^\infty \frac{dr}{r\psi(r)} = +\infty$ ($R_0 > 0$) such that

$$\lim_{R\to\infty} \int_{B(R)} \frac{\frac{\|\tau(u)\|^2}{2}}{\psi(r(x))} dv_g < \infty. \tag{40}$$

Theorem 5.6. *Let $u\colon (M,g) \to (N,h)$ be a biharmonic isometric immersion. Assume that there is a constant $\sigma > 0$ such that*

$$\frac{m-1}{2}\lambda_{\min} + 1 - 2\max\{2, \lambda_{\max}\} \geq \sigma.$$

If $E_2(u)$ is slowly divergent and $h(\tau(u), \widetilde{\nabla}_{\frac{\partial}{\partial r}} du(\frac{\partial}{\partial r})) \geq 0$, then u is harmonic, i.e., $\tau(u) = 0$.

Proof. From the proof of Theorem 5.1, we have

$$\sigma \int_{B(r)} \frac{\|\tau(u)\|^2}{2} dv_g \leq r \int_{\partial B(r)} \frac{\|\tau(u)\|^2}{2} ds_g. \tag{41}$$

Now suppose that u is not harmonic, so there exists $R_0 > 0$ such that for $R \geq R_0$,

$$\sigma \int_{B(R)} \frac{\|\tau(u)\|^2}{2} dv_g \geq c_1, \tag{42}$$

where c_1 is a positive constant. From (41) and (42), we have

$$c_1\sigma \leq R \int_{\partial B(R)} \frac{\|\tau(u)\|^2}{2} ds_g. \tag{43}$$

for $R \geq R_0$ and

$$\begin{aligned}
\lim_{R\to\infty} \int_{B(R)} \frac{\frac{\|\tau(u)\|^2}{2}}{\psi(r(x))} dv_g &= \int_0^\infty \frac{dR}{\psi(R)} \int_{\partial B(R)} \frac{\|\tau(u)\|^2}{2} ds_g \\
&\geq \int_{R_0}^\infty \frac{dR}{\psi(R)} \int_{\partial B(R)} \frac{\|\tau(u)\|^2}{2} ds_g \\
&\geq c_1\sigma \int_{R_0}^\infty \frac{dR}{R\psi(R)} = \infty,
\end{aligned}$$

which contradicts (40), therefore, u is harmonic. $\square$

From the proof of Theorem 5.6, we immediately get the following theorem.

Theorem 5.7. *Let M, K_r and Λ be as in Theorem 5.4. If $u\colon (M,g) \to (N,h)$ is a biharmonic isometric immersion, the bienergy $E_2(u)$ is slowly divergent and $h(\tau(u), \widetilde{\nabla}_{\frac{\partial}{\partial r}} du(\frac{\partial}{\partial r})) \geq 0$, then u is harmonic.*

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