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ON THE TIME GROWTH OF THE ERROR OF THE DISCONTINUOUS GALERKIN METHOD FOR ADVECTION-REACTION PROBLEMS *

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Abstract. This contribution presents an overview of the results of the paper [6] on the time growth of the error of the discontinuous Galerkin (DG) method. When estimating quantities of interest in differential equations, the application of Gronwall's lemma gives estimates which grow exponentially in time even for problems where such behavior is unnatural. In the case of a nonstationary advection-diffusion equation we can circumvent this problem by considering a general space-time exponential scaling argument. Thus we obtain error estimates for DG which grow exponentially not in time, but in the time particles carried by the flow field spend in the spatial domain. If this is uniformly bounded, one obtains an error estimate of the form $C(h^{p+1/2})$, where p is the degree of polynomials used in the DG method and C is independent of time. We discuss the time growth of the exact solution and the exponential scaling argument and give an overview of results from [6] and the tools necessary for the analysis.

Key words. discontinuous Galerkin method, error estimates, exponential scaling

AMS subject classifications. 35L03, 65M15, 65M60

Introduction. We give an overview of the paper [6] dealing with estimates of the time growth of the error of the discontinuous Galerkin method applied to a nonstationary advection-reaction equation. To put the results of paper [6] into appropriate context, in Section 1, we first comment on the role of ellipticity in producing estimates that are uniform with respect to time – this includes the heat equation and limitations of attempts to apply similar techniques straightforwardly to advection-reaction problems. In Section 2, we introduce the continuous problem and in Section 2.1 we give estimates of the time growth of the solution using streamlines, which gives further justification to the results of [6]. In Section 3, we introduce exponential scaling, a key tool used in the analysis. To give further insight into the results of [6], we give an overview of the main ideas of the proof of Theorem 3.2 from [2]. Finally, in Section 4 we introduce the DG method and give an overview of the results in [6] and their proofs. We note that the full analysis is rather technical, however the main ideas upon which the theory is built are rather straightforward.

1. Estimation and ellipticity in differential equations. When dealing with differential equations, one typically estimates quantities of interest to prove the desired results, e.g. regularity of the solution, etc. Let us consider the simple heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0 \tag{1.1}$$

in a spatial domain Ω with homogeneous Dirichlet boundary conditions, for simplicity. The weak formulation reads

$$\Big(\frac{\partial u(t)}{\partial t}, v\Big) + \big(\nabla u(t), \nabla v\big) = 0 \tag{1.2}$$

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for all $v \in V = H_0^1(\Omega)$ and all t, where $(\cdot\,,\cdot)$ is the standard $L^2(\Omega)$-inner product. Let's say we wish to estimate the norm of the solution. To this end we test (1.2) with $v := u(t)$ to obtain

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + |u(t)|_1^2 = 0, \tag{1.3}$$

where $\|\cdot\|$ is the $L^2(\Omega)$-norm and $|\cdot|_1$ is the $H_0^1(\Omega)$-seminorm. Neglecting the seminorm in (1.3) gives us

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 \le 0, \tag{1.4}$$

i.e. the $L^2(\Omega)$-norm of the solution is non-increasing in time.

Compare the consideration above with the advection-reaction problem

$$\frac{\partial u}{\partial t} + a\cdot\nabla u + cu = 0, \tag{1.5}$$

where a is a given advection vector field and c is a given reaction coefficient. If we test the weak form of (1.5) with $v := u(t)$ and apply Green's theorem in the advection terms, we obtain

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \int_\Omega (c - \tfrac{1}{2}\mathrm{div}\, a)u^2\mathrm{d}x + \int_{\partial\Omega} a\cdot\mathbf{n}\, u^2\mathrm{d}S = 0. \tag{1.6}$$

Assuming for simplicity homogeneous Dirichlet boundary conditions on the inlet boundary and neglecting the terms over the outflow boundary, where $a\cdot\mathbf{n} > 0$, we get

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \int_\Omega (c - \tfrac{1}{2}\mathrm{div}\, a)u^2\mathrm{d}x \le 0. \tag{1.7}$$

Now we distinguish two cases. First, we consider the case when $(c - \frac{1}{2}\mathrm{div}\, a) \ge 0$. In this case, similarly to the heat equation, we get (1.4), hence again the $L^2(\Omega)$-norm of the solution is non-increasing in time. On the other hand, if $(c - \frac{1}{2}\mathrm{div}\, a) \le 0$, the best we can do is estimate

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 \le 2\sup|c - \tfrac{1}{2}\mathrm{div}\, a|\, \|u\|^2. \tag{1.8}$$

It is natural to apply Gronwall's lemma to (1.8), which results in the estimate

$$\|u(t)\|^2 \le \|u(0)\|^2 \exp\bigl(2t\sup|c - \tfrac{1}{2}\mathrm{div}\, a|\bigr), \tag{1.9}$$

which grows exponentially w.r.t. time. However, as we shall see in the following section, the distinction based on the sign of $c - \frac{1}{2}\mathrm{div}\, a$ is completely artificial, as is the exponential growth of the estimate (1.9). The problem arised due to the crude use of Gronwall's lemma, as well as relying on ellipticity of the advection-reaction terms. In the following section we will present how to circumvent these obstacles in the case of estimation of the error of the DG method using more refined estimation techniques based on the so-called exponential scaling trick.

2. Continuous problem. Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ be a bounded polygonal (polyhedral) domain with Lipschitz boundary $\partial\Omega$. Let $0 < T \le +\infty$ and let $Q_T = \Omega \times (0, T)$ be the space-time domain. We consider the following nonstationary advection-reaction equation: We seek $u : Q_T \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + a\cdot\nabla u + cu = 0 \quad \text{in } Q_T, \tag{2.1}$$

along with the initial condition $u(x,0) = u^0(x)$ and boundary condition $u = u_D$ on $\partial\Omega^- \times (0,T)$. Here $a : \overline{Q_T} \to \mathbb{R}^d$ and $c : \overline{Q_T} \to \mathbb{R}$ are the given advective field and reaction coefficient, respectively. By $\partial\Omega^-$ we denote the *inflow* boundary $\{x \in \partial\Omega; a(x,t)\cdot \mathbf{n}(x) < 0, \forall t \in (0,T)\}$, where $\mathbf{n}(x)$ is the unit outer normal to $\partial\Omega$ at x. We assume that $c \in C([0,T); L^\infty(\Omega)) \cap L^\infty(Q_T)$ and $a \in C([0,T); W^{1,\infty}(\Omega))$ with $a, \nabla a$ uniformly bounded a.e. in Q_T.

2.1. Time growth of the exact solution. In order to put the results on the time growth of error of the DG method into perspective, it is useful to first gain insight into the time growth of the exact solution itself in a more refined manner than in Section 1. For this purpose, we define *pathlines* of the flow, i.e. the family of curves $S(t; x_0, t_0)$, each originating at (x_0, t_0), by

$$S(t_0; x_0, t_0) = x_0 \in \overline{\Omega}, \quad \frac{\mathrm{d}S(t; x_0, t_0)}{\mathrm{d}t} = a(S(t; x_0, t_0), t).$$

This means that $S(\cdot; t_0, x_0)$ is the trajectory of a massless particle in the nonstationary flow field a passing through point x_0 at time t_0. Equation (2.1) can then be rewritten along each pathline:

$$\frac{\mathrm{d}u(S(t; x_0, t_0), t)}{\mathrm{d}t} + (cu)(S(t; x_0, t_0), t) = \Big(\frac{\partial u}{\partial t} + a\cdot\nabla u + cu\Big)(S(t; x_0, t_0), t) = 0,$$

which is an ordinary differential equation with the solution

$$u(S(t; x_0, t_0), t) = u(x_0, t_0) \exp\Big(- \int_{t_0}^{t} c(S(s; x_0, t_0), s)\,\mathrm{d}s\Big). \tag{2.2}$$

For simplicity, if we take $c(x,t) \equiv c_0 \in \mathbb{R}$, i.e. a constant, this reduces to

$$u(S(t; x_0, t_0), t) = u(x_0, t_0) e^{-c_0(t-t_0)}. \tag{2.3}$$

From this we can see that along pathlines, the exact solution of (2.1) exponentially grows ($c_0 < 0$) or decays ($c_0 > 0$) with the rate $-c$. In the special case of pure advection ($c_0 = 0$) the function u is constant along each pathline.

In this short note and in the paper [6] we are concerned with the case of uniformly bounded solutions and errors. One case when this can occur is when the maximal particle 'life-time' $\widehat{T}$ is finite. By this we mean that the maximal time any massless particle carried by the flow field a spends in Ω, before exiting through the outflow boundary, is bounded by $\widehat{T} < +\infty$. As such particles follow pathlines of the flow, this is equivalent to assuming that each pathline is defined only for a finite time bounded by $\widehat{T}$ before exiting Ω. Therefore in (2.2) and (2.3) we have $|t - t_0| < \widehat{T}$, specifically

$$|u(S(t; x_0, t_0), t)| \le |u(x_0, t_0)| \exp\big(\widehat{T}\|c\|_{L^\infty(Q_T)}\big),$$

where $u(x_0, t_0)$ is either the initial condition ($x_0 \in \Omega, t_0 = 0$) or boundary condition ($x_0 \in \partial\Omega^-, t_0 \ge 0$). Therefore the exact solution u remains uniformly bounded for all t, although it may exponentially grow along each pathline ($c < 0$), which exist only for a bounded time $\widehat{T}$. We therefore have uniform boundedness of u even on a potentially infinite time interval $(0,T)$.

It is reasonable to assume that a 'good' numerical method will mimic the described behavior of the exact solution. Namely that whenever the exact solution remains uniformly bounded, so will the approximate solution and error of the method. As we will see, this is the case of the DG method.

3. Exponential scaling. The main tool in the analysis of [6] is a general form of the exponential scaling. The simple considerations of Section 1 required the assumption

$$c - \tfrac{1}{2}\mathrm{div}a \geq \gamma_0 > 0 \quad \text{on } Q_T \tag{3.1}$$

for some constant $\gamma_0 > 0$ in order to have uniform boundedness of the norm of the solution w.r.t. time. As we have seen in Section 2.1 this assumption is by no means necessary in the analysis of the continuous problem.

One possibility how to avoid assumption (3.1) in the analysis of (2.1) is the exponential scaling 'trick'. We write $u(x,t) = e^{\alpha t}w(x,t)$ for some $\alpha \in \mathbb{R}$. Substituting into (2.1) gives

$$e^{\alpha t}\frac{\partial w}{\partial t} + \alpha e^{\alpha t}w + e^{\alpha t}a\cdot\nabla w + e^{\alpha t}cw = 0.$$

Since $e^{\alpha t} > 0$ we can divide the equation by this common factor, obtaining the new problem for the unknown function w:

$$\frac{\partial w}{\partial t} + a\cdot\nabla w + (c+\alpha)w = 0. \tag{3.2}$$

In this equation we have the new reaction term $c + \alpha$. The new ellipticity condition for (3.2) now reads $c + \alpha - \frac{1}{2}\mathrm{div}a \geq \gamma_0 > 0$ and it can be satisfied by choosing α sufficiently large. One can then proceed to use the ellipticity to obtain estimates for w. The drawback of this approach is that in order to obtain estimates for u, one must multiply by the exponential factor $e^{\alpha t}$, the result being an estimate that depends exponentially on T.

For the stationary version of problem (2.1) the authors of [1] consider general exponential scaling with respect to space and set $u(x) = e^{\mu(x)}\tilde{u}(x)$. In our space-time setting this would correspond to taking

$$u(x,t) = e^{\mu(x)}\tilde{u}(x,t). \tag{3.3}$$

This is a generalized version of the simpler choice $\mu(x) = \mu_0\cdot x$ for some constant vector $\mu_0 \in \mathbb{R}^d$ considered e.g. in [4] and [7].

Substituting (3.3) into (2.1) and dividing by e^{μ} we get a new problem for $\tilde{u}$:

$$\frac{\partial\tilde{u}}{\partial t} + a\cdot\nabla\tilde{u} + (a\cdot\nabla\mu + c)\tilde{u} = 0.$$

The condition corresponding to (3.1) is now: There exists $\mu : \Omega \to \mathbb{R}$ such that

$$a\cdot\nabla\mu + c - \tfrac{1}{2}\mathrm{div}a \geq \gamma_0 > 0 \quad \text{on } Q_T. \tag{3.4}$$

The question then is when can such a function μ be found satisfying $a\cdot\nabla\mu \geq a_0 > 0$ so that this term can be used to dominate the other possibly negative terms of (3.4). The answer is in the following theorem, cf. [2] and [1].

THEOREM 3.1. *Let $a : \Omega \to \mathbb{R}^d$ be Lipschitz continuous. Then there exists a function $\mu \in W^{1,\infty}(\Omega)$ such that $a\cdot\nabla\mu \geq a_0 > 0$ if and only if the flow field a possesses neither closed curves nor stationary points.*

Proof. The proof itself is technical, here we only sketch the main arguments.

First let $a\cdot\nabla\mu \geq a_0 > 0$. Then clearly there cannot exist a point x where $a(x) = 0$. Assume that a possesses a closed curve $\beta : [0,1] \to \Omega$ parametrized by $s \in [0,1]$, i.e. $\beta(0) = \beta(1)$ and $\frac{\mathrm{d}}{\mathrm{d}s}\beta(s) = a(\beta(s))$ for all s. Then we have

$$0 = \mu(\beta(1)) - \mu(\beta(0)) = \int_0^1 \frac{\mathrm{d}\mu(\beta(s))}{\mathrm{d}s}\mathrm{d}s = \int_0^1 \nabla\mu(\beta(s))\cdot a(\beta(s))\,\mathrm{d}s \geq a_0 > 0,$$

which is a contradiction. Therefore a cannot contain closed curves.

The opposite implication is more technical, hence we only indicate the main ideas. A function μ_S is constructed in a neighborhood of each streamline S which satisfies $a\cdot\nabla\mu_S > 0$ on this neighborhood. This is done via the implicit function theorem on the neighborhood of each point of S separately and connecting the functions together on the neighborhood of the whole S. The set of all streamlines along with their considered neighborhoods form a covering from which a finite subcovering can be chosen and the local functions μ_S can then be 'glued' together using the related partition of unity. $\square$

3.1. General space-time exponential scaling. If we were to use the non-stationary version of the spatial exponential scaling (3.3), we would need to assume the statement of Theorem 3.1 holds for all t, i.e. that $a(\cdot,t)$ possesses neither closed curves nor stationary points for each $t \in (0,T)$. We view this as restrictive. Therefore we consider the general form

$$u(x,t) = e^{\mu(x,t)}\tilde{u}(x,t), \tag{3.5}$$

where $\mu : Q_T \to \mathbb{R}$ is an appropriate function. Substituting (3.5) into (2.1) gives

$$\frac{\partial\tilde{u}}{\partial t} + a\cdot\nabla\tilde{u} + \Big(\frac{\partial\mu}{\partial t} + a\cdot\nabla\mu + c\Big)\tilde{u} = 0 \tag{3.6}$$

after dividing by the common positive factor e^{μ}. The condition corresponding to (3.1) and (3.4) now reads: There exists $\mu : Q_T \to \mathbb{R}$ such that

$$\frac{\partial\mu}{\partial t} + a\cdot\nabla\mu + c - \tfrac{1}{2}\mathrm{div}a \geq \gamma_0 > 0 \quad \text{a.e. in } Q_T. \tag{3.7}$$

We will construct such a function μ in the following paragraph.

Up to now we worked only with the strong form of (2.1). The key step was to divide the whole equation by e^{μ}. The question is how to perform this operation in a weak formulation, where all terms are under integral signs. The solution is to take test functions of the form $v(x,t) = e^{-\mu(x,t)}\hat{v}(x,t)$ when using (3.5). Then the factors e^{μ} and $e^{-\mu}$ cancel each other and one thus obtains the weak form of (3.6). This is a key step in the analysis of [6].

3.2. Construction of the scaling function μ. If $c - \frac{1}{2}\mathrm{div}a$ is negative or changes sign frequently, we can use the expression $\mu_t + a\cdot\nabla\mu$ to dominate this term everywhere. If we choose μ_1 such that

$$\frac{\partial\mu_1}{\partial t} + a\cdot\nabla\mu_1 = 1 \quad \text{on } Q_T, \tag{3.8}$$

then by multiplying μ_1 by a sufficiently large constant, we can satisfy condition (3.7) for a chosen $\gamma_0 > 0$. Along pathlines equation (3.8) reads

$$\frac{\mathrm{d}\,\mu_1(S(t;x_0,t_0),t)}{\mathrm{d}t} = \Big(\frac{\partial\mu_1}{\partial t} + a\cdot\nabla\mu_1\Big)(S(t;x_0,t_0),t) = 1,$$

therefore

$$\mu_1(S(t;x_0,t_0),t) = t - t_0. \tag{3.9}$$

At the origin of the pathline, we have $\mu_1(S(t_0;x_0,t_0),t_0) = 0$ and the value of μ_1 along this pathline is simply the time elapsed since t_0. As we assumed in Section 2.1, the life-time of particles is bounded by $\widehat{T}$, hence $0 \le \mu_1(x,t) \le \widehat{T}$ for all $(x,t) \in Q_T$. In the analysis we need Lipschitz continuity of μ. This can be obtained under the assumption that there are no characteristic boundary points on the inlet boundary. We note that due to its Lipschitz continuity, μ_1 is differentiable with respect to x and t a.e. in Q_T, which justifies the previous considerations. The proof of the following theorem is rather technical, cf. [6]. Since μ_1 is defined very simply along pathlines, which are solutions of ordinary differential equations, the proof follows similar ideas as in the proof of dependence of a solution of an ODE on the initial condition.

THEOREM 3.2. *Let $a \in L^\infty(Q_T)$ be continuous with respect to time and Lipschitz continuous with respect to space. Let there exist a constant $a_{\min} > 0$ such that*

$$-a(x,t)\cdot \mathbf{n} \ge a_{\min}$$

for all $x \in \partial\Omega^-, t \in [0,T)$. Let the time any particle carried by the flow field $a(\cdot,\cdot)$ spends in Ω be uniformly bounded by $\widehat{T}$. Then μ_1 defined by (3.9) on $\overline{\Omega}\times[0,T)$ is uniformly Lipschitz continuous with respect to x and t and satisfies $0 \le \mu_1 \le \widehat{T}$.

4. Discontinuous Galerkin method. Now we introduce the DG discretization of (2.1). Let $\mathcal{T}_h$ be a triangulation of Ω, i.e. a partition of $\overline{\Omega}$ into closed simplices with mutually disjoint interiors with hanging nodes allowed. For $K \in \mathcal{T}_h$ let $h_K = \mathrm{diam}(K)$, $h = \max_{K\in\mathcal{T}_h} h_K$. For $K \in \mathcal{T}_h$ we define its inflow boundary by $\partial K^-(t) = \{x \in \partial K; a(x,t)\cdot \mathbf{n}(x) < 0\}$ where $\mathbf{n}(x)$ is the unit outer normal to ∂K. We will seek the discrete solution in the space $S_h = \{v_h; v_h|_K \in P^p(K), \forall K \in \mathcal{T}_h\}$, where $P^p(K)$ is the space of all polynomials on K of degree at most $p \in \mathbb{N}$. Given $K \in \mathcal{T}_h$ and $v_h \in S_h$ we define v_h^- as the trace of v_h on ∂K from the side of the element adjacent to K, or $v_h^- = 0$ if the face lies on $\partial\Omega$. Finally on ∂K we define the *jump* of v_h as $[v_h] = v_h - v_h^-$, where v_h is the trace from K.

The DG formulation of (2.1) then reads: We seek $u_h \in C^1([0,T);S_h)$ such that $u_h(0) = u_h^0 \approx u^0$ and

$$\Big(\frac{\partial u_h}{\partial t}, v_h\Big) + b_h(u_h,v_h) + c_h(u_h,v_h) = l_h(v_h), \quad \forall v_h \in S_h. \tag{4.1}$$

Here b_h, c_h and l_h are the *advection, reaction* and *right-hand side forms*, respectively, defined for u,v piecewise continuous on $\mathcal{T}_h$ as follows, [8], [3]:

$$\begin{aligned} b_h(u,v) &= \sum_{K\in\mathcal{T}_h}\int_K (a\cdot\nabla u)v\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_{\partial K^-}(a\cdot\mathbf{n})[u]v\,\mathrm{d}S,\\ c_h(u,v) &= \int_\Omega cuv\,\mathrm{d}x,\\ l_h(v) &= -\sum_{K\in\mathcal{T}_h}\int_{\partial K^-\cap\partial\Omega}(a\cdot\mathbf{n})u_D v\,\mathrm{d}x. \end{aligned}$$

4.1. Error estimates. We are interested in estimates of the DG error $e_h(t) := u(t)-u_h(t) = \eta(t_n)+\xi(t)$, where $\eta(t) = u(t)-\Pi_h u(t)$ and $\xi(t) = \Pi_h u(t) - u_h(t) \in S_h$. Here Π_h is the $L^2(\Omega)$–projection onto S_h.

As in Section 3.1, we wish to write $\xi = e^{\mu}\tilde{\xi}$ and test the error equation with $\phi = e^{-\mu}\tilde{\xi} = e^{-2\mu}\xi$ to obtain estimates for $\tilde{\xi}$. However, since $\phi(t) \notin S_h$ this is not possible. The solution is to test by $\Pi_h\phi(t) \in S_h$ and estimate the difference $\Pi_h\phi(t) - \phi(t)$, cf. [6]. A similar result is proved in the stationary case in [1] if $\mu \in W^{p+1,\infty}(\Omega)$. In [6] only the Lipschitz continuity of μ and standard approximation results are used in the proof of the following lemma.

LEMMA 4.1. *Let μ be globally bounded and Lipschitz continuous as in Theorem (3.2). Then there exists C independent of $h, t, \xi, \tilde{\xi}$ such that*

$$\|\Pi_h\phi(t) - \phi(t)\|_{L^2(K)} \le Ch_K \max_{x\in K} e^{-\mu(x,t)}\|\tilde{\xi}(t)\|_{L^2(K)},$$
$$\|\Pi_h\phi(t) - \phi(t)\|_{L^2(\partial K)} \le Ch_K^{1/2} \max_{x\in K} e^{-\mu(x,t)}\|\tilde{\xi}(t)\|_{L^2(K)}.$$

Now we come to the error analysis. Due to the consistency of the DG scheme, the exact solution u also satisfies (4.1). We subtract the equations for u and u_h, set $v_h = \Pi_h\phi(t)$ and rearrange the terms to get the error equation

$$\begin{aligned}\Big(\frac{\partial\xi}{\partial t}, \Pi_h\phi\Big) + b_h(\xi,\phi) + b_h(\xi, \Pi_h\phi - \phi) + b_h(\eta, \Pi_h\phi)\\ + c_h(\xi,\phi) + c_h(\xi, \Pi_h\phi - \phi) + c_h(\eta, \Pi_h\phi) + \Big(\frac{\partial\eta}{\partial t}, \Pi_h\phi\Big) = 0.\end{aligned} \tag{4.2}$$

The terms with ϕ are those where the factors e^{μ} and $e^{-\mu}$ cancel out as in Section 3.1 leading to the new reaction terms as in (3.6). Terms containing $\Pi_h\phi - \phi$ are estimated using Lemma 4.1 and η is estimated by standard approximation results. Altogether we have the following estimates, cf. [6].

LEMMA 4.2. *Let $\xi = e^{\mu}\tilde{\xi}, \phi = e^{-\mu}\tilde{\xi}$ and let μ be as in Theorem (3.2). Then*

$$\Big(\frac{\partial\xi}{\partial t}, \Pi_h\phi\Big) + b_h(\xi,\phi) + c_h(\xi,\phi) \ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\xi}\|^2 + \gamma_0\|\tilde{\xi}\|^2 + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\big\|[\tilde{\xi}]\big\|^2_{a,\partial K^-}$$

where $\|f\|_{a,\partial K^-} = \|\sqrt{|a\cdot\mathbf{n}|}f\|_{L^2(\partial K^-)}$.

LEMMA 4.3. *Let ξ, ϕ and μ be as above. Then*

$$\begin{aligned}\Big|b_h(\xi, \Pi_h\phi - \phi) + b_h(\eta, \Pi_h\phi) + c_h(\xi, \Pi_h\phi - \phi) + c_h(\eta, \Pi_h\phi) + \Big(\frac{\partial\eta}{\partial t}, \Pi_h\phi\Big)\Big|\\ \le Ch\|\tilde{\xi}\|^2 + Ch^{2p+1}\big(|u(t)|^2_{H^{p+1}} + |u_t(t)|^2_{H^{p+1}}\big) + \frac{1}{4}\sum_{K\in\mathcal{T}_h}\big\|[\tilde{\xi}]\big\|^2_{a,\partial K^-}.\end{aligned}$$

Now we come to the main theorem of [6] on the error of the DG scheme (4.1).

THEOREM 4.4. *Let the assumptions of Theorem 3.2 hold. Let the initial condition u_h^0 satisfy $\|u^0 - u_h^0\| \le Ch^{p+1/2}|u^0|_{H^{p+1}}$. Then there exists a constant C depending on $\widehat{T}$ but independent of h and T such that for h sufficiently small*

$$\begin{aligned}\max_{t\in[0,T]}\|e_h(t)\| + \sqrt{\gamma_0}\|e_h\|_{L^2(Q_T)} + \Big(\frac{1}{2}\int_0^T\sum_{K\in\mathcal{T}_h}\big\|[e_h(\vartheta)]\big\|^2_{a,\partial K^-}\,\mathrm{d}\vartheta\Big)^{1/2}\\ \le Ch^{p+1/2}\big(|u^0|_{H^{p+1}} + |u|_{L^2(H^{p+1})} + |u_t|_{L^2(H^{p+1})}\big).\end{aligned} \tag{4.3}$$

Proof. If we apply Lemmas 4.2 and 4.3 to (4.2), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\xi}(t)\|^2 + 2\gamma_0\|\tilde{\xi}(t)\|^2 + \frac{1}{2}\sum_{K\in\mathcal{T}_h}\big\|[\tilde{\xi}(t)]\big\|^2_{a,\partial K^-} \\ \le Ch\|\tilde{\xi}(t)\|^2 + Ch^{2p+1}\big(|u(t)|^2_{H^{p+1}} + |u_t(t)|^2_{H^{p+1}}\big).$$

Now we choose h small enough so that $Ch \le \gamma_0$ and the first right-hand side term can then be absorbed. We integrate over $(0,t)$, take the maximum over $t\in[0,T]$ and apply the estimate of the initial condition. Thus we get

$$\max_{t\in[0,T]}\|\tilde{\xi}(t)\|^2 + \gamma_0\int_0^T\|\tilde{\xi}(\vartheta)\|^2\,\mathrm{d}\vartheta + \frac{1}{2}\int_0^T\sum_{K\in\mathcal{T}_h}\big\|[\tilde{\xi}(\vartheta)]\big\|^2_{a,\partial K^-}\,\mathrm{d}\vartheta \\ \le Ch^{2p+1}\big(|u^0|^2_{H^{p+1}} + |u|^2_{L^2(H^{p+1})} + |u_t|^2_{L^2(H^{p+1})}\big). \tag{4.4}$$

Now we reformulate estimate (4.4) as an estimate of ξ instead of $\tilde{\xi}$. Because $\tilde{\xi} = e^{-\mu}\xi$, we can estimate for example

$$\|\tilde{\xi}(t)\|^2 \ge \min_{Q_T} e^{-2\mu(x,t)}\|\xi(t)\|^2 = e^{-2\max_{Q_T}\mu(x,t)}\|\xi(t)\|^2 \ge e^{-2\widehat{T}}\|\xi(t)\|^2$$

and similarly for the other norms in (4.4). If we multiply the resulting estimate by $e^{2\widehat{T}}$ and take the square root, we get inequality (4.3) for the discrete part $\tilde{\xi}$ of the error e_h. Finally, a similar estimate for η follows from standard approximation results which gives the estimate for $e_h = \xi + \eta$. □

The interpretation of Theorem 4.4 is the following. If one proceeds in a standard way, the need to use Gronwall's lemma arises. This leads to exponential growth in time. By using the exponential scaling argument we effectively apply Gronwall's lemma along pathlines, which exist only for a finite bounded time $\widehat{T}$, resulting in bounds uniform in T. This can be interpreted as application of Gronwall in the Lagrangian framework, not in the Eulerian. For future work, we plan to extend the analysis the nonlinear convective problems, by combining the presented ideas with the technique of [9] and [5].

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