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ON ROBUSTNESS OF FLUX RECONSTRUCTIONS - DISCONTINUOUS GALERKIN METHOD

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Abstract. We deal with the numerical solution of the Poisson equation. The equation is discretized with the aid of the incomplete interior penalty discontinuous Galerkin method. Guaranteed a posteriori upper bound based on the flux reconstruction can be derived. The main aim of this paper is to show that the robustness of a certain simple reconstruction depends at most on $p^{\frac{1}{2}}$ in one dimension, where p is the discretization polynomial degree. The theoretical results are verified by numerical experiments.

Key words. Flux reconstruction, polynomial robustness, mixed formulation, a posteriori error analysis, discontinuous Galerkin method

AMS subject classifications. 65N15, 65N30

1. Introduction. A posteriori error estimates are important and practical tools in numerical mathematics. They serve two main purposes in numerical discretization of PDEs: to provide the information about the discretization error for the current choice of discretization parameters and to provide the localization of the sources of high errors for upcoming possible adaptive procedures. For the survey of main a posteriori techniques for PDE discretizations see e.g. [2], [5], [9], [15], [17] and references cited therein.

Since higher order methods and *hp*-adaptive techniques start to be more and more popular, the question of robustness with respect to the discretization polynomial degree becomes very important. On the other hand and in contrast to the number of existing results devoted to the robustness with respect to the mesh-size, there are not many theoretical results devoted to the robustness with respect to the polynomial degree. A posteriori error techniques based on the local Neumann problem for *hp*-adaptive discretizations are discussed e.g. in [1] and [3]. For the analysis of the polynomial dependence of the technique based on the local residual estimators see e.g. [12]. It shall be pointed out that the efficiency of individual estimators proved in [12] behaves as p, where p is the underlying polynomial degree used in the finite element method (FEM) discretization.

Important class of approaches for deriving guaranteed a posteriori upper bounds is based on the hypercircle theorem, see [14]. The extension of these ideas to nonconforming discretizations can be found in e.g. [16]. The quality of the resulting error estimate depends heavily on the choice of the flux reconstruction. Among many approaches for flux reconstructions, the local mixed finite element technique is very popular, since it enables to reconstruct the fluxes based on local relatively cheap problems and since the resulting reconstruction is completely polynomially robust, i.e. the resulting estimators are efficient independently of the polynomial degree. The core of the proof of the polynomial robustness can be found in [7]. The extension of these ideas to wide class of discretization methods can be found in [11].

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The Poisson equation discretized by the incomplete interior penalty discontinuous Galerkin method (IIPG) is assumed in this paper. Moreover, even more simple and cheaper reconstruction following the ideas from [10] and [16] that can be easily evaluated directly, i.e. without the necessity to solve any local problems, is assumed in this paper. The main aim of this paper is to show its practical usefulness by presenting that the resulting local estimators efficiency depends on the polynomial degree as $p^{\frac{1}{2}}$ for one-dimensional problems.

2. Continuous problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain with a Lipschitz continuous boundary $\partial\Omega$. We use standard notation for Lebesque and Sobolev spaces. Moreover, we denote $H(\mathrm{div}, \Omega) = \{v \in L^2(\Omega)^d : \mathrm{div}\, v \in L^2(\Omega)\}$. Let us consider the following boundary value problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{2.1}$$

where $f \in L^2(\Omega)$.

Let us denote by $(.\,,.)$ and $\|.\|$ L^2-scalar product and norm, respectively.

Definition 2.1. *We say that the function $u \in H^1_0(\Omega)$ is the exact weak solution of problem (2.1), if*

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H^1_0(\Omega). \tag{2.2}$$

The existence and uniqueness of the weak solution follows from Lax-Milgram lemma. Since $f \in L^2(\Omega)$, it is possible to show that $\nabla u \in H(\mathrm{div}, \Omega)$.

3. Discrete problem.

3.1. Notation. We consider a space partition $\mathcal{T}_h$ consisting of a finite number of closed, d-dimensional simplices with mutually disjoint interiors and covering $\overline{\Omega}$, i.e., $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$. We assume conforming properties of the mesh, i.e., neighbouring elements share an entire edge or face. In the rest of the paper we speak only about edges, but we mean edges or faces depending on the dimension d. We denote the vertices of the mesh by a and edges by e and we denote the set of edges as $\mathcal{F}_h$. We set $h_K = \mathrm{diam}(K)$ and $h = \max_K h_K$. We assume shape regularity of elements, i.e., $h_K/\rho_K \le C_S$ for all $K \in \mathcal{T}_h$, where ρ_K is the radius of the largest d-dimensional ball inscribed into K and constant C_S does not depend on $\mathcal{T}_h$ for $h \in (0, h_0)$. Moreover, we assume the local quasi-uniformity of the mesh, i.e., $h_K \le C_G h_{K'}$ for neigbouring elements K and K', where constant C_G does not depend on $\mathcal{T}_h$ for $h \in (0, h_0)$ again. For each edge e, let $n = n_e$ denote a unit normal vector to e with arbitrary but fixed direction for the inner edges and with outer direction on $\partial\Omega$. Moreover, for each $K \in \mathcal{T}_h$, n_K is the unit outer normal vector to K. In order to simplify the notation, we set $(.\,,.)_M$ and $\|.\|_M$ the local $L^2(M)$-scalar products and norms, respectively, where $M \subset \overline{\Omega}$ is either some union of elements $K \in \mathcal{T}_h$ or edges $e \in \mathcal{F}_h$, e.g. $M = K$ or $M = \partial K$.

We define broken Sobolev space

$$H^s(\Omega, \mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^s(K), K \in \mathcal{T}_h\}. \tag{3.1}$$

For $v \in H^1(\Omega, \mathcal{T}_h)$ we need to define one-sided values, jumps and mean values on the inner edge $e \in \mathcal{F}_h$, $e \not\subset \partial\Omega$

$$\begin{aligned} v_L(x) &= \lim_{\epsilon \to 0+} v(x - \epsilon n), \quad x \in e, \\ v_R(x) &= \lim_{\epsilon \to 0+} v(x + \epsilon n), \quad x \in e, \\ \langle v \rangle(x) &= \frac{v_L + v_R}{2}, \quad x \in e, \\ [v]\,(x) &= v_L - v_R, \quad x \in e. \end{aligned} \tag{3.2}$$

On boundary edge $e \in \mathcal{F}_h$, $e \subset \partial\Omega$

$$v_L(x) = \langle v \rangle(x) = [v](x) = \lim_{\epsilon \to 0+} v(x - \epsilon n), \quad x \in e. \tag{3.3}$$

3.2. IIPG discretization. For $u \in H^2(\Omega, \mathcal{T}_h)$, $v \in H^1(\Omega, \mathcal{T}_h)$ we set the discrete form corresponding to IIPG formulation

$$A_h(u, v) = \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K - \sum_{e \in \mathcal{F}_h} (\langle \nabla u \rangle \cdot n, [v])_e + \sum_{e \in \mathcal{F}_h} (\alpha [u], [v])_e, \tag{3.4}$$

where the penalty parameter α is defined by

$$\alpha = \frac{C_W}{\bar{h}_e}, \tag{3.5}$$

where $\bar{h}_e$ is some intermediate value between h_K and $h_{K'}$ for neighbouring elements $K, K' \in \mathcal{T}_h$ sharing edge $e \in \mathcal{F}_h$. The detailed derivation of the form (3.4) can be found in [8].

For $v \in H^1(\Omega, \mathcal{T}_h)$, let us define mesh-dependent norm

$$|\!|\!| v |\!|\!|^2 = \sum_{K \in \mathcal{T}_h} \|\nabla v\|_K^2 + \sum_{e \in \mathcal{F}_h} \|\alpha^{1/2} v\|_e^2. \tag{3.6}$$

Now, we define the space of discontinuous piecewise polynomial functions

$$V_h = \{v \in L^2(\Omega) : v|_K \in P_p(K), K \in \mathcal{T}_h\}, \tag{3.7}$$

where the space $P_p(K)$ denotes the space of polynomials up to the degree $p \geq 1$. Moreover, let us define space

$$V = H_0^1(\Omega) + V_h = \{v + v_h : v \in H_0^1(\Omega), v_h \in V_h\}. \tag{3.8}$$

The space V provides the minimal extension of the space $H_0^1(\Omega)$ and V_h. Since the exact solution u of problem (2.2) satisfies $u \in H_0^1(\Omega)$ and $\nabla u \in H(\mathrm{div}, \Omega)$, the form $A_h(.,.)$ is consistent in u, i.e.

$$A_h(u, v) = (f, v) \quad \forall v \in V. \tag{3.9}$$

Lemma 3.1. *Let the shape-regularity and the local quasi-uniformity of the mesh be satisfied. Let*

$$C_W \geq C_G C_M (1 + C_I), \tag{3.10}$$

where C_G is the constant from local quasi-uniformity, C_M is the constant from trace inequality, see e.g. [8, Lemma 2.19], and C_I is the constant from inverse inequality. Then A_h is positive definite on V_h, i.e.

$$A_h(v_h, v_h) \geq \frac{1}{2} \|\!|v_h|\!\|^2, \quad v_h \in V_h. \tag{3.11}$$

Proof. The proof can be found in [8]. □

We assume in the following text that the constant C_W is chosen large enough to satisfy (3.10).

Now we are able to define IIPG solution of problem (2.2).

DEFINITION 3.2. *We say that the function $u_h \in V_h$ is the approximate solution of (2.2), if*

$$A_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{3.12}$$

Assuming C_W satisfies (3.10), we can apply Lemma 3.1 and then the existence and uniqueness of the approximate solution follows again from the Lax-Milgram lemma.

3.3. Mixed formulation and numerical fluxes. Since DG formulations are more natural for first order problems, it is suitable to reformulate the original problem (2.1) into the first order system

$$\begin{aligned} \nabla u - \sigma &= 0 \\ -\operatorname{div} \sigma &= f. \end{aligned} \tag{3.13}$$

Following the idea from [4] we can integrate (3.13) over each individual element $K \in \mathcal{T}_h$, apply Green's theorem and replace values of u and σ by the so-called numerical fluxes $\hat{u}$ and $\hat{\sigma}$

$$\begin{aligned} (\nabla u, w)_K + (\hat{u} - u, n_K \cdot w)_{\partial K} - (\sigma, w)_K = 0, \quad w \in L^2(K)^d \\ (\sigma, \nabla v)_K - (\hat{\sigma}, v)_{\partial K} = (f, v)_K, \quad v \in H^1(K). \end{aligned} \tag{3.14}$$

When we eliminate the term $(\sigma, \nabla v)_K$ from both equations and sum the equation over all elements $K \in \mathcal{T}_h$ we arive to

$$\sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K - (\hat{\sigma}, v)_{\partial K} + (\hat{u} - u, n_K \cdot \nabla v)_{\partial K} = (f, v) \quad v \in H^1(\Omega, \mathcal{T}_h) \tag{3.15}$$

The numerical fluxes corresponding to the IIPG discretization are

$$\begin{aligned} \hat{u}|_e &= \langle u \rangle + \frac{1}{2}[u] n \cdot n_K, \quad e \in \mathcal{F}_h, e \not\subset \partial\Omega \\ \hat{u}|_e &= [u] n \cdot n_K, \quad e \in \mathcal{F}_h, e \subset \partial\Omega \\ \hat{\sigma}|_e &= \langle \nabla u \rangle - \alpha [u] n, \quad e \in \mathcal{F}_h. \end{aligned} \tag{3.16}$$

Then (3.15) is equivalent to consistency equation (3.9).

4. A posteriori error estimate.

4.1. Error measure. The natural choice for the error measure for conforming methods is the dual norm of the residual. The definition of a similar suitable error measure is more delicate task for non-conforming methods, since the spaces, norms and formulations may differ for the original continuous problem and for the discrete problem.

Here, we employ the consistency of the form $A_h(.,.)$ that provides common formulation for both problems and define the dual norm of residual with respect to this (extended) formulation, cf. (3.9).

The next difficulty arises from the choice of the dual space and the choice of the corresponding norm. The most natural way how to interpret the residual containing $A_h(.,.)$ is to consider the residual as an element of dual space V^* to V, since the space V serves as the test function space in (3.9) as well as in (3.12). But this choice could not be used directly with the norm $\|\nabla .\|$ on the space V, since V contains V_h and V_h contains non-zero piece-wise constant functions that annihilate on $\|\nabla .\|$, i.e. $\|\nabla .\|$ is just semi-norm on V. But the consistency (3.9) and discrete formulation (3.12) imply the Galerkin orthogonality

$$A_h(u - u_h, v_h) = 0, \quad \forall v_h \in V_h. \tag{4.1}$$

This leads to the idea to employ space V/V_h, i.e. the space V factorized by V_h, instead of V. It is possible to define $\|\nabla .\|$ as a norm on V/V_h.

Now, we can define the error measure

$$\mathrm{Err}(u_h) = \|\mathrm{Res}(u_h)\|_{(V/V_h)^*}, \tag{4.2}$$

where

$$\|\mathrm{Res}(u_h)\|_{(V/V_h)^*} = \sup_{v \in V/V_h} \frac{A_h(u - u_h, v)}{\|\nabla v\|} = \sup_{v \in V/V_h} \frac{(f, v) - A_h(u_h, v)}{\|\nabla v\|}.$$

Let us define $s \in H_0^1(\Omega)$ as $H_0^1(\Omega)$-orthogonal projection of the approximate solution u_h, i.e.

$$(\nabla s, \nabla v) = \sum_{K \in \mathcal{T}_h} (\nabla u_h, \nabla v)_K, \quad \forall v \in H_0^1(\Omega). \tag{4.3}$$

Then it is possible to show that

$$\mathrm{Err}(u_h) = \|\nabla u - \nabla s\|. \tag{4.4}$$

4.2. Flux reconstruction. We define the reconstruction σ_h element-wise similarly as in [10]. We seek $\sigma_h|_K \in RTN_p(K)$, where $RTN_p(K) = xP_p(K) + P_p(K)^d$ is Raviart-Thomas-Nedelec space, see e.g. [6], such that

$$\begin{aligned} \sigma_h|_e &= \langle \nabla u \rangle - \alpha [u] n \quad \forall e \in \mathcal{F}_h, \\ (\sigma_h, w_h)_K &= (\nabla u_h, w_h)_K \quad \forall w_h \in P_{p-1}(K)^d. \end{aligned} \tag{4.5}$$

These conditions represent natural degrees of freedom for $RTN_p(K)$ space, see [6, Proposition 2.3.4].

The reconstruction σ_h defined by (4.5) satisfies following important property

$$(f + \operatorname{div} \sigma_h, v_h) = 0 \quad \forall v_h \in V_h.$$

The proof can be done in the same way as in [10].

4.3. Upper bound. Let us denote partial estimators

$$\begin{aligned}\eta_{R,K} &= C_P h_K \|f + \operatorname{div} \sigma_h\|_K, \\ \eta_{F,K} &= \|\sigma_h - \nabla u_h\|_K,\end{aligned} \tag{4.6}$$

where C_P is the constant from Poincare inequality

$$\|v - (v,1)_K\|_K \leq C_P h_K \|\nabla v\|_K, \quad \forall v \in H^1(K), \tag{4.7}$$

see [13].

Now, we are able to present the upper a posteriori error estimate.

THEOREM 4.1. *Let $u_h \in V_h$ be the approximate solution obtained by (3.12) and σ_h be the reconstruction obtained from u_h by (4.5). Then*

$$Err(u_h)^2 \leq \eta^2 = \sum_{K \in \mathcal{T}_h} (\eta_{R,K} + \eta_{F,K})^2. \tag{4.8}$$

The Proof of Theorem 4.1 follows the same idea as the proof in [10], where the main idea follows the hyper-circle method. For the overview of a posteriori error estimates based on the hyper-circle method see e.g. [17]. The complete proof is rather long and technical. Therefore the proof is skipped here.

4.4. Lower bound. We limit ourselves to $d = 1$ in this section. It is possible to see that the error estimator $\eta_{R,K}$ converges one degree faster than $\eta_{F,K}$ for piecewise smooth f and therefore this term is much smaller in many practical situations and a posteriori error estimate (4.8) is usually dominated by $\eta_{F,K}$. The aim of this section is to present that the local individual estimator $\eta_{F,K}$ obtained with the aid of reconstruction (4.5) is locally efficient and how this efficiency depends on the polynomial degree p. For the purpose of the efficiency analysis we suppose a traditional assumption that $f \in V_h$. Otherwise, classical oscillation term appears additionally in the efficiency results.

We will use following notation in this section. We will denote by $\lesssim$ the inequality up to some generic constant that is independent of solutions u and u_h, local mesh sizes h_K and polynomial degree p. We will denote by ω_K a patch consisting of elements sharing at least a vertex with element K. Moreover, we define local version of the error measure $\mathrm{Err}(u_h)$

$$\mathrm{Err}_{\omega_K}(u_h) = \sup_{\substack{v \in V/V_h \\ \mathrm{supp}(v) \subset \omega_K}} \frac{(f,v) - A_h(u_h, v)}{\|v'\|},$$

where v' denotes the derivative of v.

Now we are ready to present the local efficiency result.

THEOREM 4.2. *Let $f \in V_h$. Let $u_h \in V_h$ be the approximate solution obtained by (3.12) and let σ_h be the reconstruction obtained from u_h by (4.5). Then*

$$\eta_{F,K} \lesssim p^{\frac{1}{2}} \, Err_{\omega_K}(u_h). \tag{4.9}$$

The proof of Theorem 4.2 is very long and very technical. Therefore the proof is skipped here and will be published in forthcoming papers.

Global efficiency estimate is a direct consequence of Theorem 4.2.

THEOREM 4.3. *Let $f \in V_h$. Let $u_h \in V_h$ be the approximate solution obtained by (3.12) and let σ_h be the reconstruction obtained from u_h by (4.5). Then*

$$\eta^2 = \sum_{K \in \mathcal{T}_h} (\eta_{R,K} + \eta_{F,K})^2 \lesssim p \, Err(u_h)^2. \tag{4.10}$$

5. Numerical experiments. Let us show that the estimate from Theorem 4.1 is reliable and efficient for $d = 1$.

5.1. Problem setting and error measure. The computation of the individual a posteriori error estimators can be made directly according to (4.6). On the other hand, the computation of the error measures $\mathrm{Err}(u_h)$ or $\mathrm{Err}_{\omega_K}(u_h)$ is difficult even if the exact solution is known, since these error measures are defined as suprema over infinite dimensional spaces. Following (4.4) we approximate these error measures by computing these suprema over finite-dimensional FEM space $V_h^+ \subset H_0^1(\Omega)$ with four times denser mesh than the mesh for V_h and polynomial degree $p+2$ instead of p. We construct spaces $V_{h,M}^+$ as subspaces of V_h^+ containing functions with supports restricted to $M \subset \overline{\Omega}$. We compute the approximation of the Riesz representative of residual $z \in V_h^+$ satisfying

$$(\nabla z, \nabla v_h) = \langle \mathrm{Res}(u_h), v_h \rangle = (f, v_h) - A_h(u_h, v_h) \quad \forall v_h \in V_h^+. \tag{5.1}$$

Then $\mathrm{Err}(u_h) \approx \mathrm{Err}_h^+(u_h) = \|\nabla z\|$. The localized versions $\mathrm{Err}_M(u_h)$ are approximated analogically with the aid of $V_{h,M}^+$ instead of V_h^+.

Let us denote global estimators

$$\eta_R^2 = \sum_{K \in \mathcal{T}_h} \eta_{R,K}^2, \qquad \eta_F^2 = \sum_{K \in \mathcal{T}_h} \eta_{F,K}^2, \tag{5.2}$$

approximate effectivity indices

$$\mathrm{Eff} = \frac{\eta}{\mathrm{Err}_h^+(u_h)}, \qquad \mathrm{Eff}_R = \frac{\eta_R}{\mathrm{Err}_h^+(u_h)}, \qquad \mathrm{Eff}_F = \frac{\eta_F}{\mathrm{Err}_h^+(u_h)} \tag{5.3}$$

and its local counterparts for element $K \in \mathcal{T}_h$

$$\mathrm{Eff}_{R,K} = \frac{\eta_{R,K}}{\mathrm{Err}_{h,\omega_K}^+(u_h)}, \qquad \mathrm{Eff}_{F,K} = \frac{\eta_{F,K}}{\mathrm{Err}_{h,\omega_K}^+(u_h)}. \tag{5.4}$$

We restrict ourselves to $d = 1$, $\Omega = (0, 1)$ and $f = e^x$.

5.2. Global efficiency. We test the error estimate (4.8) with respect to the mesh refinement. The polynomial degree is set as $p = 3$. We assume a sequence of successively refined equidistant meshes started with $h = 1/10$ and halved in each step.

$1/h$	$\mathrm{Err}_h^+(u_h)$	η	Eff	η_R	Eff_R	η_F	Eff_F
10	5.5844 − 6	5.6271 − 6	1.01	1.1269 − 9	0.00	5.6260 − 6	1.01
20	7.0084 − 7	7.0362 − 7	1.00	3.5235 − 11	0.00	7.0358 − 7	1.00
40	8.7782 − 8	8.7959 − 7	1.00	1.1020 − 12	0.00	8.7958 − 8	1.00
80	1.0984 − 8	1.0995 − 8	1.00	9.4526 − 14	0.00	1.0995 − 8	1.00
160	1.3737 − 9	1.3745 − 9	1.00	1.7090 − 13	0.00	1.3744 − 9	1.00
320	1.7206 − 10	1.7206 − 10	1.00	3.2426 − 13	0.00	1.7180 − 10	1.00

TABLE 5.1
Global h-performance, $p = 3$

We also test the error estimate (4.8) with respect to the changing polynomial degree p. We assume equidistant mesh with $h = 1/10$ and $p = 1, \dots, 7$.

p	$\mathrm{Err}_h^+(u_h)$	η	Eff	η_R	Eff_R	η_F	Eff_F
1	5.0865 − 2	5.1600 − 2	1.01	2.1169 − 5	0.00	5.1578 − 2	1.01
2	6.1459 − 4	6.6591 − 4	1.08	1.7890 − 7	0.00	6.6573 − 4	1.08
3	5.5844 − 6	5.6271 − 6	1.01	1.1269 − 9	0.00	5.6260 − 6	1.01
4	3.2762 − 8	3.5444 − 8	1.08	5.6630 − 12	0.00	3.5439 − 8	1.08
5	1.7734 − 10	1.7813 − 10	1.00	6.4088 − 14	0.00	1.7808 − 10	1.00
6	8.0068 − 13	8.7070 − 13	1.09	1.4471 − 13	0.18	7.4584 − 13	0.93

TABLE 5.2
Global p-performance, $h = 1/10$

We can see from Table 5.1 that the effectivity indices are tending to one for decreasing h. We observe from Table 5.2 that two regimes for odd and even polynomial degrees appear. For both regimes the effectivity indices stagnate with increasing p. Moreover, we can see that η_R converges faster to zero than other terms as expected, since this term is equivalent to $L^2(\Omega)$ orthogonal projection error of function f, cf. (4.6).

5.3. Local efficiency. We test the robustness of efficiency estimates (4.9) with respect to decreasing h. The polynomial degree is set as $p = 3$. We assume a sequence of successively refined equidistant meshes started with $h = 1/10$ and halved in each step. For each mesh we take element $K = [0.4, 0.4 + h]$ and we investigate local efficiency on this element.

$1/h$	$\mathrm{Err}_{h,\omega_K}^+(u_h)$	$\eta_{R,K}$	$\mathrm{Eff}_{R,K}$	$\eta_{F,K}$	$\mathrm{Eff}_{F,K}$
10	2.7482 − 6	3.1326 − 10	0.00	1.5624 − 6	0.57
20	2.3299 − 7	6.7506 − 12	0.00	1.3467 − 7	0.58
40	2.0327 − 8	1.4741 − 13	0.00	1.1755 − 8	0.58
80	1.7886 − 9	1.8961 − 14	0.00	1.0325 − 9	0.58
160	1.5761 − 10	2.5994 − 14	0.00	9.0973 − 11	0.58
320	1.3898 − 11	1.1566 − 14	0.00	8.0288 − 12	0.58

TABLE 5.3
Local h-performance, $p = 3$, $K = [0.4, 0.4 + h]$

We also test the robustness of efficiency estimates (4.9) with respect to the changing polynomial degree p. We assume equidistant mesh with $h = 1/10$ and $p = 1, \dots, 7$. Similarly as in the previous tests, we take $K = [0.4, 0.5]$ and we investigate local efficiency on this element.

We can see that the effectivity indices in Table 5.3 are uniformly bounded for decreasing h. We can observe again in Table 5.4 two regimes for odd and even polynomial degrees, but the effectivity indices stagnate with increasing p. Again, the term $\eta_{R,K}$ converges faster to zero than other terms as expected.

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p	$\mathrm{Err}^{+}_{h,\omega_K}(u_h)$	$\eta_{R,K}$	$\mathrm{Eff}_{R,K}$	$\eta_{F,K}$	$\mathrm{Eff}_{F,K}$
1	$2.5401-2$	$5.8845-6$	0.00	$1.4324-2$	0.57
2	$2.9950-4$	$4.9730-8$	0.00	$1.8488-4$	0.62
3	$2.7482-6$	$3.1326-10$	0.00	$1.5624-6$	0.57
4	$1.5751-8$	$1.5743-12$	0.00	$9.8416-9$	0.62
5	$8.6655-11$	$3.1663-14$	0.00	$4.9450-11$	0.57
6	$3.5099-13$	$6.4333-14$	0.18	$2.0438-13$	0.58

TABLE 5.4
Local p-performance, $h=1/10$, $K=[0.4,0.5]$

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