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ON A POSTERIORI ERROR ESTIMATES FOR SPACE–TIME DISCONTINUOUS GALERKIN METHOD

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Abstract. We deal with nonlinear nonstationary convection–diffusion problem. We discretize this problem by discontinuous Galerkin method in space and in time and, assuming the error is measured as a mesh dependent dual norm of residual, we present a posteriori estimate to this error measure. This a posteriori error estimate is cheap, robust with respect to degeneration to hyperbolic problem and fully computable. Moreover, we present a local asymptotic efficiency of this estimate.

Key words. nonlinear convection–diffusion problem, a posteriori error analysis, discontinuous Galerkin method

AMS subject classifications. 65M60, 65M70

1. Introduction. We consider nonstationary nonlinear convection–diffusion equation, which represents a model problem for the system of the compressible Navier-Stokes equations. This problem we discretize by discontinuous Galerkin finite element method in space as well as in time and derive a posteriori error analysis.

The class of discontinuous Galerkin methods seems to be one of the most promising candidates to construct high order accurate schemes for solving convection-diffusion problems. For a survey about DG methods, see [2], [3] or [14]. A priori analysis of discontinuous Galerkin methods was presented in many papers, see, e.g. [5], [11] and [15].

In this paper we shall focus on a posteriori numerical analysis of the proposed problem. Our aim is to derive a guaranteed, cheap and fully computable upper bound to chosen error measure that provides local efficiency at least asymptotically. To achieve these properties we use the technique of so-called equilibrated flux reconstruction. For the description of basic idea in the context of elliptic problems see e.g. [6] or more recent paper [8], where robustness with respect to the polynomial degree is shown. This technique, which is usually used for stationary problems only, we apply to nonstationary problem. Robust a posteriori estimates for linear unsteady advection–diffusion problems can be found in [16], see also references therein. We have been largely influenced by [4] and [7], where only lower order time discretizations are considered, and by [1], where discontinuous Galekin time discretization is analyzed and nodal superconvergence is derived via a posteriori error estimates.

2. Continuous problem. Let $\Omega \subset R^d$ $(d = 1, 2, 3)$ be a bounded polyhedral domain with Lipschitz continuous boundary $\partial\Omega$ and $T > 0$. Let us consider the

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following initial–boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla \cdot \sigma(u, \nabla u) &= f \quad \text{in } \Omega \times (0,T), \\ u &= 0 \quad \text{in } \partial\Omega \times (0,T), \\ u &= 0 \quad \text{in } \Omega \times \{0\}. \end{aligned} \tag{2.1}$$

We assume that the right–hand side satisfies $f \in C(0,T,L^2(\Omega)$. Moreover, we assume that $\sigma(v, \nabla v) = K(v)\nabla v - F(v)$, where K is a bounded matrix–valued function and F is a once continuously differentiable vector–valued function. Such a problem can be viewed as a model problem for compressible Navier–Stokes equations.

Let us denote by $(.,.)$ and $\|.\|$ the $L^2(\Omega)$ scalar product and norm, respectively. Let us denote the time derivative $u' = \frac{\partial u}{\partial t}$. Let us define spaces $X = L^2(0,T,H_0^1(\Omega))$,

$$Y = \{v \in X : v' \in L^2(0,T,L^2(\Omega)), v(0) = 0\}. \tag{2.2}$$

DEFINITION 2.1. *We call $u \in Y$ the weak solution of problem (2.1), if*

$$\int_0^T (f,v) - (u',v) - (\sigma(u,\nabla u), \nabla v) dt = 0, \quad \forall v \in X. \tag{2.3}$$

We assume that problem (2.3) has a unique weak solution.

3. Discretization. We consider a space partition $\mathcal{T}_h$ consisting of a finite number of closed, d-dimensional simplices K with mutually disjoint interiors and covering $\bar{\Omega}$, i.e.

$$\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K. \tag{3.1}$$

We assume conforming properties, i.e. neighbouring elements share an entire edge or face. We set $h_K = \mathrm{diam}(K)$ and $h = \max_K h_K$. By ρ_K we denote the radius of the largest d-dimensional ball inscribed into K. We assume shape regularity of elements, i.e.

$$h_K/\rho_K \le C, \quad \forall K \in \mathcal{T}_h, \tag{3.2}$$

where the constant does not depend on $\mathcal{T}_h$ for $h \in (0, h_0)$. We define unit normal vector n to arbitrary edge (or face for $d = 3$) of $\mathcal{T}_h$. For inner edges the direction is arbitrary, for outer edges we assume n be unit outer normal vector. By n_K we denote unit outer normal to element $K \in \mathcal{T}_h$.

We set the space for the semi–discrete solution

$$X_h = \{v \in L^2(\Omega) : v|_K \in P^p(K)\}, \tag{3.3}$$

where the space $P^p(K)$ denotes the space of polynomials up to the degree $p \ge 1$ on K. For a function $v \in X_h$ we define on inner edges the one–sided limits

$$v_L(x) = \lim_{t \to 0+} v(x - nt), \quad v_R(x) = \lim_{t \to 0+} v(x + nt), \tag{3.4}$$

jump and mean value

$$[v] = v_L - v_R, \quad \langle v \rangle = \frac{1}{2}(v_L + v_R). \tag{3.5}$$

On outer edges we define

$$[v] = \langle v \rangle = v_L = \lim_{t \to 0+} v(x - nt). \tag{3.6}$$

In order to discretize problem (2.3) in time, we consider a time partition $0 = t_0 < t_1 < \ldots < t_r = T$ with time intervals $I_m = (t_{m-1}, t_m)$, time steps $\tau_m = |I_m| = t_m - t_{m-1}$ and $\tau = \max_{m=1,\ldots,r} \tau_m$. The approximate solution will be sought in the space of piecewise polynomial functions

$$X_h^\tau = \{v \in L^2(0, T, X_h) : v|_{I_m} = \sum_{j=0}^{q} v_{j,m} t^j,\ v_{j,m} \in X_h\}. \tag{3.7}$$

For a function $v \in X_h^\tau$ we define the one–sided limits

$$v_\pm^m = v(t_m \pm) = \lim_{t \to t_m \pm} v(t) \tag{3.8}$$

and the jumps

$$\{v\}_m = v_+^m - v_-^m. \tag{3.9}$$

We omit the subscript $\pm$ for continuous functions, since $v(t_m \pm) = v(t_m)$.

To simplify further notation we define local L^2 scalar products and L^2 norms. Let $M \subset \overline{\Omega}$, e.g. $M = K$ or $M = \partial K$, and $1 \le m \le r$. Then we define

$$(u, v)_M = \int_M uv dx, \quad (u, v)_{M,m} = \int_{I_m} \int_M uv dx dt, \tag{3.10}$$

with corresponding norms $\|.\|_M$, $\|.\|_{M,m}$.

3.1. Derivation of the space discretization. Let us consider auxiliary stationary problem

$$\begin{aligned} -\nabla \cdot \sigma(u, \nabla u) &= \tilde{f} \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \partial\Omega. \end{aligned} \tag{3.11}$$

Considering the space discretization of (3.11) our aim is to find a suitable formulation describing $A_h(u_h, v_h) \approx -(\nabla \cdot \sigma(u_h, \nabla u_h), v_h)$ for $u_h, v_h \in X_h$. To do so, we follow classical approach described in [2], where the second order differential problem is decomposed into the system of first order and then each of the resulting relations is discretized by usual approach with numerical fluxes well known from the finite volume method.

We denote $w = \nabla u$ and from problem (3.11) we obtain first order system

$$\begin{aligned} -\nabla \cdot \sigma(u, w) &= \tilde{f}, \\ \nabla u &= w. \end{aligned} \tag{3.12}$$

To discretize this problem we assume $K \in \mathcal{T}_h$, $v_h \in X_h$ and we get for the exact solution u and w from the first equation of (3.12)

$$(\tilde{f}, v_h)_K = (-\nabla \cdot \sigma(u, w), v_h)_K \approx (\sigma(u, w), \nabla v_h)_K - (\hat{\sigma} \cdot n_K, v_h)_{\partial K}, \tag{3.13}$$

where $\hat{\sigma} = \hat{\sigma}(u,w)$ is a numerical flux approximating $\sigma(u,w)$ on ∂K. Similarly we gain from the second equation of (3.12)

$$
\begin{aligned}
(K(u)w, \nabla v_h)_K &= (K(u)\nabla u, \nabla v_h)_K = (\nabla u, K(u)^T \nabla v_h)_K \\
&= -(u, \nabla \cdot (K(u)^T \nabla v_h))_K + (uK(u)n_K, \nabla v_h)_{\partial K} \\
&\approx -(u, \nabla \cdot (K(u)^T \nabla v_h))_K + (\hat{u}K(u)n_K, \nabla v_h)_{\partial K} \\
&= (K(u)\nabla u, \nabla v_h)_K + ((\hat{u}K(u) - uK(u))n_K, \nabla v_h)_{\partial K},
\end{aligned} \tag{3.14}
$$

where $\widehat{u} = \hat{u}(u)$ is again a numerical flux approximating u on ∂K. Now, since $\sigma(u,w) = K(u)w - F(u)$ it is possible to eliminate w and we get primal formulation

$$
\begin{aligned}
(\tilde{f}, v)_K &\approx (\sigma(u,w), \nabla v_h)_K - (\hat{\sigma} \cdot n_K, v_h)_{\partial K} \\
&\approx (K(u)\nabla u - F(u), \nabla v_h)_K \\
&\quad + ((\hat{u} - u)n_K, K(u)^T \nabla v_h)_{\partial K} - (\hat{\sigma} \cdot n_K, v_h)_{\partial K}.
\end{aligned} \tag{3.15}
$$

With different choices of the numerical fluxes $\hat{u}$ and $\hat{\sigma}$ we obtain different variants of discontinuous Galerkin discretization of auxiliary problem (3.11). The choice of the numerical flux $\hat{\sigma}$ on the edge e

$$
\begin{aligned}
\hat{\sigma} &= \langle K(u)\nabla u\rangle + \alpha h_e^{-1}[u]n + F(u_L), \quad \text{if } \langle F'(u)\cdot n\rangle > 0, \\
\hat{\sigma} &= \langle K(u)\nabla u\rangle + \alpha h_e^{-1}[u]n + F(u_R), \quad \text{if } \langle F'(u)\cdot n\rangle \le 0,
\end{aligned} \tag{3.16}
$$

where $F(u_R)$ on $\partial\Omega$ is set to $F(0)$ and the choice of numerical flux $\hat{u}$

$$
\begin{aligned}
\hat{u} &= \langle u\rangle + \theta[u]n \cdot n_K, \quad e \notin \partial\Omega, \\
\hat{u} &= 2\theta[u]n \cdot n_K, \quad e \in \partial\Omega.
\end{aligned} \tag{3.17}
$$

results in classical interior penalty Galerkin discretization of diffusion term $-\nabla \cdot (K(u)\nabla u)$ and upwind discretization of convective term $\nabla \cdot F(u)$. Moreover, the choice of the parameter θ leads to SIPG ($\theta = 0$), IIPG ($\theta = 1/2$) or NIPG ($\theta = 1$) variant. For other variants of discontinuous Galerkin method and their numerical fluxes see, e.g. [2]. For the purpose of a posteriori numerical analysis we assume for the rest of this paper that the numerical fluxes $\hat{\sigma}$ and $\hat{u}$ are consistent and that the numerical flux $\hat{\sigma}$ is conservative.

3.2. Derivation of the time discretization and fully discrete problem. Considering the time discretization we can proceed in a very similar way as in space discretization. We assume auxiliary problem

$$
\begin{aligned}
u' &= \tilde{f}, \quad \text{in } (0,T), \\
u(0) &= 0.
\end{aligned} \tag{3.18}
$$

To discretize this problem we assume $v \in X_h^\tau$ and we get

$$
\begin{aligned}
(\tilde{f}, v)_{K,m} &= (u', v)_{K,m} = -(u, v')_{K,m} + (u_-^m, v_-^m)_K - (u_+^{m-1}, v_+^{m-1})_K \\
&\approx -(u, v')_{K,m} + (u_-^m, v_-^m)_K - (u_-^{m-1}, v_+^{m-1})_K \\
&= (u', v)_{K,m} + (\{u\}_{m-1}, v_+^{m-1})_K,
\end{aligned} \tag{3.19}
$$

where again we can consider u_-^m as an approximation of $u(t_m)$ by numerical flux. Such a numerical flux is consistent and conservative.

Combining the ideas of space and time discretization we gain following scheme:

Definition 3.1. *We say that the function $u_h^\tau \in X_h^\tau$ is the discrete solution of (2.3) obtained by space–time discontinuous Galerkin finite element method, if the following conditions are satisfied*

$$\begin{aligned}(3.20)\quad &\int_{I_m} ((u_h^\tau)', v) + A_h(u_h^\tau, v)dt + (\{u_h^\tau\}_{m-1}, v_+^{m-1}) = \int_{I_m} (f, v)dt \\ &\qquad\qquad\qquad\qquad \forall m = 1, \dots, r,\ \forall v \in X_h^\tau, \\ &(u_h^\tau)_-^0 = 0,\end{aligned}$$

where

$$\begin{aligned}(3.21)\qquad A_h(u, v) = &\sum_K (K(u)\nabla u - F(u), \nabla v)_K \\ &-(\hat{\sigma} \cdot n_K, v)_{\partial K} + ((\hat{u} - u)n_K, K(u)^T \nabla v)_{\partial K}\end{aligned}$$

with suitably chosen numerical fluxes $\hat{\sigma}$ and $\hat{u}$.

Since the numerical fluxes are assumed consistent, it is possible to see that the sufficiently smooth exact solution u, i.e. $u \in Y \cap L^2(0, T, H^2(\Omega))$, satisfies (3.20) too.

4. A posteriori analysis. In this section we shall propose suitable error measure and we shall derive a posteriori error estimate of this measure.

4.1. Error measure. Let $d_{K,m} > 0$ be a user dependent parameter associated to space-time element $K \times I_m$, e.g. $d_{K,m}^2 = h_K^2 + \tau_m^2$ or $d_{K,m} = 1$ or $d_{K,m} = h_K$ or $d_{K,m}^2 = (h_K^{-2} + \tau_m^{-2})^{-1}$. As we will see later, the complete analysis will be independent of the choice of $d_{K,m}$, since it will be included into the error measure as well as into the resulting a posteriori error estimate. It is possible to expect that this parameter could emphasize certain aspects of the error measure, but this question is still open. Let us define the space

$$(4.1)\qquad Y^\tau = \{v \in X : v'|_{I_m} \in L^2(I_m, L^2(\Omega))\}$$

of piecewise continuous functions with respect to time. We define the norm

$$\begin{aligned}(4.2)\qquad \|v\|_{Z,K,m}^2 &= \frac{h_K^2 \|\nabla v\|_{K,m}^2 + \tau_m^2 \|v'\|_{K,m}^2}{d_{K,m}^2}, \\ \|v\|_Z^2 &= \sum_{K,m} \|v\|_{Z,K,m}^2.\end{aligned}$$

Since $Y^\tau \subset X$, we gain from (2.3) that the exact solution $u \in Y$ satisfies

$$\begin{aligned}(4.3)\qquad &\int_{I_m} (f, v) - (u', v) - (\sigma(u, \nabla u), \nabla v)dt - (\{u\}_{m-1}, v_+^{m-1}) = 0 \\ &\qquad\qquad\qquad\qquad \forall m = 1, \dots, r,\ \forall v \in Y^\tau.\end{aligned}$$

The existence of the solution u of problem (4.3) comes clearly from the existence of the solution of problem (2.3). Assuming σ monotone, i.e.

$$(4.4)\qquad (\sigma(u, \nabla u) - \sigma(v, \nabla v), \nabla u - \nabla v) \geq 0,$$

it is possible to show that the solution of problem (4.3) is unique.

It is natural to define error measure as residual of (4.3)

$$
\begin{aligned}
\text{(4.5)} \qquad \operatorname{Res}(w) = \sup_{0\neq v\in Y^\tau} \frac{1}{\|v\|_Z} \Bigg(\sum_{K,m} (f,v)_{K,m} - (w',v)_{K,m} \\
-(\sigma(w,\nabla w),\nabla v)_{K,m} - (\{w\}_{m-1}, v_+^{m-1})_K \Bigg)
\end{aligned}
$$

for $w \in X_h^\tau$, $w_-^0 = 0$.

Unfortunately, such an error measure is suitable for measuring error for $u_h^\tau \in Y^\tau$, but the discrete solution $u_h^\tau \in X_h^\tau \not\subset Y^\tau$. To overcome this difficulty, we measure in addition the distance of u_h^τ from Y^τ. To do so, we use

$$
\text{(4.6)} \qquad J_{K,m}(v) = \frac{d_{K,m}^2}{h_K^3} \|[v]\|_{\partial K,m}^2, \quad J(v) = \sum_{K,m} J_{K,m}(v).
$$

$J(v)$ is in fact a weighted element–wise version of the classical penalization term well known from discontinuous Galerkin method. It is evident that for arbitrary $v \in X_h^\tau$ holds: $J(v) = 0$, iff $v \in Y^\tau$. Using the partial error measures Res and J we can construct the complete error measure

$$
\text{(4.7)} \qquad \operatorname{EST}(u_h^\tau) = \sqrt{J(u_h^\tau)} + \operatorname{Res}(u_h^\tau).
$$

It is possible to show that $\operatorname{EST}(u_h^\tau) = 0$, iff u_h^τ is equal to the exact solution u.

4.2. Reconstruction of the solution with respect to time. To be able to produce a posteriori error estimates we need to reconstruct the solution u_h^τ in such a way that the reconstruction R_h^τ is conforming with respect to time, i.e. $R_h^\tau \in C(0,T,L^2(\Omega))$ with $R_h^\tau(0) = 0$, and that $u_h^\tau \approx R_h^\tau$.

Let $r_m \in P^{q+1}(I_m)$ be the right Radau polynomial on I_m, i.e. $r_m(t_{m-1}) = 1$, $r_m(t_m) = 0$ and r_m is orthogonal on P^{q-1}. Then there exists polynomial reconstruction $R_h^\tau = R_h^\tau(u_h^\tau)$ such that on I_m

$$
\text{(4.8)} \qquad R_h^\tau(t) = u_h^\tau(t) - \{u_h^\tau\}_{m-1} r_m(t), \quad \forall t \in I_m.
$$

Then the resulting function R_h^τ is continuous with respect to time and satisfies the initial condition, i.e. $R_h^\tau(0) = 0$. Moreover,

$$
\begin{aligned}
\int_{I_m} ((R_h^\tau)',v)dt &= \int_{I_m} ((u_h^\tau)',v) - r_m'(\{u_h^\tau\}_{m-1},v)dt \\
\text{(4.9)} \qquad &= \int_{I_m} ((u_h^\tau)',v)dt + \int_{I_m} r_m(\{u_h^\tau\}_{m-1},v')dt \\
&\quad -r_m(t_m)(\{u_h^\tau\}_{m-1},v_-^m) + r_m(t_{m-1})(\{u_h^\tau\}_{m-1},v_+^{m-1}) \\
&= \int_{I_m} ((u_h^\tau)',v)dt + (\{u_h^\tau\}_{m-1},v_+^{m-1}), \quad \forall v \in P^q(I_m,L^2(\Omega)).
\end{aligned}
$$

Such a reconstruction is used to show equivalence among Radau IIA Runge–Kutta method, Radau collocation method and discontinuous Galerkin method. For the details see, e.g. [9] and [10]. Such a reconstruction is also used for proving a posteriori nodal superconvergence in [1].

4.3. Reconstruction of the solution with respect to space. For similar reasons we reconstruct also the spatial fluxes of the solution in such a way that $\sigma_h^\tau \in L^2(0,T,H(\mathrm{div}))$ and $\sigma(u_h^\tau, \nabla u_h^\tau) \approx \sigma_h^\tau$. Let $RTN_p(K)$ be the Raviar-Thomas-Nedelec space of order p, i.e. $RTN_p(K) = P_p(K)^d + xP_p(K)$. Then we seek $\sigma_h^\tau|_{K\times I_m} \in P^q(I_m, RTN_p(K))$ such that

$$
\begin{aligned}
(4.10)\quad &(\sigma_h^\tau \cdot n, v)_{e,m} = (\hat{\sigma} \cdot n, v)_{e,m}, \quad \forall v \in P^q(I_m, P^p(e)),\ \forall e \subset \partial K,\\
&(\sigma_h^\tau, \nabla v)_{K,m} = (\sigma(u_h^\tau, \nabla u_h^\tau), \nabla v)_{K,m} + ((\hat{u} - u_h^\tau)n_k, K(u_h^\tau)^T \nabla v)_{\partial K,m},\\
&\qquad\qquad \forall v \in P^q(I_m, P^p(K)).
\end{aligned}
$$

The reconstruction (4.10) is a space–time version of the reconstruction described in [6]. From this we can see that such a reconstruction σ_h^τ exists and using (2.3), (4.8) and (4.10) we find that σ_h^τ satisfies approximation property $\sigma(u_h^\tau, \nabla u_h^\tau) \approx \sigma_h^\tau$ in the following sense:

$$
\begin{aligned}
(4.11)\qquad &(f - (R_h^\tau)' + \nabla \cdot \sigma_h^\tau, v)_{K,m}\\
&= (f - (u_h^\tau)' + \nabla \cdot \sigma_h^\tau, v)_{K,m} - (\{u_h^\tau\}_{m-1}, v_+^{m-1})_K\\
&= (f - (u_h^\tau)', v)_{K,m} - (\{u_h^\tau\}_{m-1}, v_+^{m-1})_K\\
&\quad -(\sigma_h^\tau, \nabla v)_{K,m} + (\sigma_h^\tau \cdot n_K, v)_{\partial K,m}\\
&= (f - (u_h^\tau)', v)_{K,m} - (\{u_h^\tau\}_{m-1}, v_+^{m-1})_K\\
&\quad -(\sigma(u_h^\tau, \nabla u_h^\tau), \nabla v)_{K,m} - ((\hat{u} - u_h^\tau)n_k, K(u_h^\tau)^T \nabla v)_{\partial K,m}\\
&\quad +(\hat{\sigma} \cdot n_K, v)_{\partial K,m} = 0, \quad \forall v \in P^q(I_m, P^p(K)).
\end{aligned}
$$

4.4. Upper bound. Now, we are ready to derive upper bound to $\mathrm{EST}(u_h^\tau)$, for $u_h^\tau \in X_h^\tau$. Since EST consists of J and Res, we can provide the upper bound individually. Moreover, $J(u_h^\tau)$ can be considered as upper bound to itself, since it depends only on known function $u_h^\tau \in X_h^\tau$. It remains to provide a suitable upper bound to $\mathrm{Res}(u_h^\tau)$, $u_h^\tau \in X_h^\tau$, only.

LEMMA 4.1. *Let $u \in Y$ be the solution of (2.3) and assume $u_h^\tau \in X_h^\tau$ to be arbitrary. Let R_h^τ be the reconstruction obtained from u_h^τ by (4.8) and σ_h^τ be the reconstruction obtained from u_h^τ by (4.10). Then*

$$
\begin{aligned}
(4.12)\ Res(u_h^\tau) = &\sup_{0\neq v\in Y^\tau} \frac{1}{\|v\|_Z} \Bigg(\sum_{K,m} (f,v)_{K,m} - ((u_h^\tau)', v)_{K,m}\\
&-(\sigma(u_h^\tau, \nabla u_h^\tau), \nabla v)_{K,m} - (\{u_h^\tau\}_{m-1}, v_+^{m-1})_K \Bigg)\\
\leq &\Bigg(\sum_{K,m} \Bigg(C_P d_{K,m} \|f - (R_h^\tau)' + \nabla \cdot \sigma_h^\tau\|_{K,m} +\\
&\frac{d_{K,m}}{h_K} \|\sigma_h^\tau - \sigma(u_h^\tau, \nabla u_h^\tau)\|_{K,m} + \frac{d_{K,m}}{\tau_m} \|(R_h^\tau - u_h^\tau)'\|_{K,m} \Bigg)^2 \Bigg)^{1/2},
\end{aligned}
$$

where the constant C_P is the constant from Poincare inequality.

Using the definition of Res (4.5) and in particular the definition of the norm $\|.\|_Z$ defined by (4.2), properties of the reconstructions R_h^τ and σ_h^τ expressed by (4.9) and

(4.11) it is quite simple to prove Lemma 4.1, but still the proof is rather long. For these reasons we skip the proof in this paper.

Now, we are ready to present fully computable guaranteed upper bound for $\mathrm{EST}(u_h^\tau)$.

THEOREM 4.2 (Upper bound). *Let $u \in Y$ be the solution of (2.3) and assume $u_h^\tau \in X_h^\tau$ to be arbitrary. Let R_h^τ be the reconstruction obtained from u_h^τ by (4.8) and σ_h^τ be the reconstruction obtained from u_h^τ by (4.10). Then*

$$EST(u_h^\tau) \leq J(u_h^\tau)^{1/2} + \left(\sum_{K,m} \left(C_P d_{K,m} \|f - (R_h^\tau)' + \nabla \cdot \sigma_h^\tau\|_{K,m} + \right.\right.$$

$$\left.\left. \frac{d_{K,m}}{h_K} \|\sigma_h^\tau - \sigma(u_h^\tau, \nabla u_h^\tau)\|_{K,m} + \frac{d_{K,m}}{\tau_m} \|(R_h^\tau - u_h^\tau)'\|_{K,m} \right)^2 \right)^{1/2} \tag{4.13}$$

In our case, where we assume the spatial mesh consists from simplices, we can bound $C_P \leq 1/\pi$, see e.g. [13].

It should be mentioned that the estimate (4.13) is independent of the choice of $u_h^\tau \in X_h^\tau$. In other words, (4.13) holds for arbitrary function $w \in X_h^\tau$ instead of u_h^τ and not only for the discrete solution of problem (3.20). This property can be exploited in further a posteriori considerations, where other aspects of the computation are taken into account, e.g. numerical solution of the linear and nonlinear problems coming from the discretization (3.20), see, e.g. [12].

4.5. Asymptotic local lower bound. The goal of this section is to show that local individual terms from a posteriori estimate (4.13) are locally effective, i.e. provide local lower bound, at least in asymptotic sense, i.e. the bound holds up to some generic constant C that might depend on the constants coming from the problem, e.g. on $|\Omega|$, on shape regularity of the mesh $\mathcal{T}_h$ and on polynomial degree of functions involved, e.g. on p and q, but is independent of the exact solution u, the discrete solution u_h^τ and space–time mesh sizes h_K and τ_m. Most importantly this constant is independent of Peclet number, i.e. a posteriori error estimate is robust for singularly perturbed problems.

To be able to express locality of the result we will need following notation. Let $\mathcal{T}_K$ be a patch consisting of elements surrounding K and K itself. Let $M \subset \overline{\Omega}$, e.g. $M = K$ or $M = \mathcal{T}_K$. We define local version of space Y^τ

$$Y_{M,m}^\tau = \{v \in Y^\tau : \mathrm{supp}(v) \subset \overline{M \times I_m}\}, \tag{4.14}$$

local version of $\mathrm{Res}(w)$

$$\mathrm{Res}_{M,m}(w) = \sup_{0 \neq v \in Y_{M,m}^\tau} \frac{1}{\|v\|_Z} \left(\sum_{K,m} (f, v)_{K,m} - (w', v)_{K,m} \right.$$

$$\left. -(\sigma(w, \nabla w), \nabla v)_{K,m} - (\{w\}_{m-1}, v_+^{m-1})_K \right), \tag{4.15}$$

local version of $J(w)$

$$J_{M,m}(w) = \sum_{K \subset M,m} J_{K,m}(w) \tag{4.16}$$

and local version of $\mathrm{EST}(w)$

$$\mathrm{EST}_{M,m}(w) = J_{M,m}(w)^{1/2} + \mathrm{Res}_{M,m}(w). \tag{4.17}$$

For the purpose of the effectivity analysis let us assume f to be a space–time polynomial. Otherwise, it is necessary to deal with the classical oscillation term

$$\sup_{0\neq v\in Y^\tau} \frac{1}{\|v\|_Z} \int_0^T (f - f_h^\tau, v)dt, \tag{4.18}$$

where f_h^τ is some polynomial approximation of f, not necessarily of degree p in space and q in time. Moreover, $\sigma(u_h^\tau, \nabla u_h^\tau)$ is not polynomial in general even if u_h^τ is. We define $\bar\sigma \in P^q(I_m, RTN_p(K))$ on $K \times I_m$

$$\begin{aligned} (\bar\sigma(u_h^\tau, \nabla u_h^\tau) \cdot n, v)_{e,m} &= (\sigma(u_h^\tau, \nabla u_h^\tau) \cdot n, v)_{e,m}, \\ &\qquad \forall v \in P^q(I_m, P^p(e)),\ \forall e \subset \partial K, \\ (\bar\sigma(u_h^\tau, \nabla u_h^\tau), \nabla v)_{K,m} &= (\sigma(u_h^\tau, \nabla u_h^\tau), \nabla v)_{K,m}, \\ &\qquad \forall v \in P^q(I_m, P^p(K)), \end{aligned} \tag{4.19}$$

and for the purpose of the effectivity analysis we assume that

$$\frac{d_{K,m}}{h_K} \|\bar\sigma(u_h^\tau, \nabla u_h^\tau) - \sigma(u_h^\tau, \nabla u_h^\tau)\|_{K,m} \le C\, \mathrm{EST}_{\mathcal{T}_K,m}(u_h^\tau). \tag{4.20}$$

THEOREM 4.3 (Local effectivity estimate). *Let $u \in Y$ be the solution of (2.3) and assume $u_h^\tau \in X_h^\tau$ to be arbitrary. Let R_h^τ be the reconstruction obtained from u_h^τ by (4.8) and σ_h^τ be the reconstruction obtained from u_h^τ by (4.10). Let f be a space–time polynomial and let (4.20) holds. Then*

$$\begin{aligned} &d_{K,m}^2 \|f - (R_h^\tau)' - \nabla\cdot\sigma_h^\tau\|_{K,m}^2 + \frac{d_{K,m}^2}{\tau_m^2}\|R_h^\tau - u_h^\tau\|_{K,m}^2 \\ &\quad + \frac{d_{K,m}^2}{h_K^2}\|\sigma_h^\tau - \sigma(u_h^\tau, \nabla u_h^\tau)\|_{K,m}^2 + J_{K,m}(u_h^\tau) \le C\, EST_{\mathcal{T}_K,m}(u_h^\tau)^2. \end{aligned} \tag{4.21}$$

The proof of Theorem 4.3 is very technical and quite long. For these reasons we skip it in this paper.

Using shape regularity of the mesh it is possible to show that

$$\begin{aligned} \sum_{K,m} \mathrm{Res}_{K,m}(u_h^\tau) &\le \sum_{K,m} \mathrm{Res}_{\mathcal{T}_K,m}(u_h^\tau) \le C\,\mathrm{Res}(u_h^\tau), \\ \sum_{K,m} J_{K,m}(u_h^\tau) &\le \sum_{K,m} J_{\mathcal{T}_K,m}(u_h^\tau) \le C\, J(u_h^\tau). \end{aligned} \tag{4.22}$$

Combining these facts with the local effectivity result (4.21) we gain following theorem

THEOREM 4.4 (Global effectivity estimate). *Let $u \in Y$ be the solution of (2.3) and assume $u_h^\tau \in X_h^\tau$ to be arbitrary. Let R_h^τ be the reconstruction obtained from u_h^τ by (4.8) and σ_h^τ be the reconstruction obtained from u_h^τ by (4.10). Let f be a space–time polynomial and let (4.20) holds. Then*

$$\begin{aligned} &\sum_{K,m}\Big(d_{K,m}\|f - (R_h^\tau)' - \nabla\cdot\sigma_h^\tau\|_{K,m} + \frac{d_{K,m}}{\tau_m}\|R_h^\tau - u_h^\tau\|_{K,m} \\ &\quad + \frac{d_{K,m}}{h_K}\|\sigma_h^\tau - \sigma(u_h^\tau, \nabla u_h^\tau)\|_{K,m}\Big)^2 + J(u_h^\tau) \le C\, EST(u_h^\tau)^2. \end{aligned} \tag{4.23}$$

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