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SEMI-IMPLICIT METHODS BASED ON INFLOW IMPLICIT AND OUTFLOW EXPLICIT TIME DISCRETIZATION OF ADVECTION

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Abstract. We introduce several numerical methods for the solution of advection equation using semi-implicit time discretization in which the inflow fluxes are discretized implicitly and the outflow fluxes explicitly. We derive the so called κ-scheme and show it is 2^{nd} order accurate and unconditionally stable in 1D and 2D case for tensor grids with a special choice of κ giving 3^{rd} order accurate scheme for constant speed in 1D. Moreover, we present a 2^{nd} order accurate and unconditionally stable Corner Transport scheme in 2D case for tensor grids that is 3^{rd} order accurate for constant velocity. We discuss several improved properties of these schemes when compared to analogous fully explicit and fully implicit schemes.

Key words. advection equation, numerical solution, semi-implicit time discretization

AMS subject classifications. 65M06, 65M08, 65M12

1. Introduction. Linear advection equation is an important part of many mathematical models with interesting applications, therefore many textbooks are dealing with its numerical solution in details [15, 8, 12]. One of the most used class of numerical methods are so called κ-schemes based on fully explicit time discretization. An attractive variant of κ-scheme is the QUICKEST scheme that is formally third order accurate for 1D advection with constant speed [6].

In many situations the fully explicit schemes are considered inappropriate due to their restrictive stability condition on the choice of time discretization step. The first contribution of this paper is an introduction of fully implicit κ-schemes in 1D with a derivation of their accuracy and less restrictive stability condition. However, the main motivation here is, firstly, to construct semi-implicit κ-schemes that are based on so called IIOE (Inflow Implicit / Outflow Explicit) time discretization [9, 10] and, secondly, to compare them with fully explicit and fully implicit schemes.

The main results of this paper are conclusions about several improved properties of semi-implicit IIOE schemes. In 1D case they are unconditionally stable for arbitrary choice of parameter κ including one particular choice which gives the 3^{rd} order accurate method for constant speed. Furthermore, for 2D case with tensor grids, the 2^{nd} order accuracy and unconditional stability is preserved also for the simplest dimension by dimension application of IIOE κ-schemes that is not the case for fully explicit and fully implicit schemes. Finally, we present in 2D a 2^{nd} order accurate and unconditionally stable semi-implicit scheme that is 3^{rd} order accurate for constant velocity by using an approach of so called Corner Transport scheme [7, 1, 8]. The derived schemes confirm improved properties also for chosen numerical experiments.

The topic is divided into four sections. The Section 2 describes the fully explicit, fully implicit and semi-implicit IIOE κ-schemes for 1D case together with their properties. The Section 3 derives IIOE schemes for 2D case on tensor grids. Finally, the Section 4 presents numerical experiments that illustrate the properties of IIOE schemes for several 2D examples.

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2. One dimensional case. The one dimensional form of advection equation can be written as

$$\partial_t u(x,t) + V(x)\partial_x u(x,t) = 0\,, \quad u(x,0) = u^0(x)\,, \quad x \in \Omega. \tag{2.1}$$

We consider $\Omega = (-1,1)$, $h = 2/M$ with M given, $x_i = -1+ih$ and $x_{i+1/2} = x_i + h/2$, $i = -1, 0, \ldots, M+1$. Furthermore let $\tau > 0$ be a given time step and $t^n = n\tau$, $t^{n+1/2} = t^n + 0.5\tau$, $n = 0, 1, \ldots, N$ with N chosen. We use standard indexing for discrete values like $V_i = V(x_i)$, $V_{i+1/2} = V(x_{i+1/2})$ and so on.

Our aim is to find approximate values $u_i^n \approx u(x_i, t^n)$. The values u_i^0 are determined from initial function $u^0(x)$. The values u_0^n and/or u_M^n shall be determined from Dirichlet boundary conditions if $V_0 \geq 0$ and/or if $V_M \leq 0$. For the case of outflow boundaries, the auxiliary values (or variables) $u_{-1}^n = 2u_0^n - u_1^n$ and/or $u_{M+1}^n = 2u_M^n - u_{M-1}^n$ are introduced to be used later in numerical schemes to determine u_0^n and/or u_M^n.

Our general numerical scheme to solve (2.1) is given by

$$u_i^{n+1} + \frac{\tau V_{i+1/2}}{h}\left(u_{i+1/2}^{n+1/2} - u_i^{n+1/2}\right) - \frac{\tau V_{i-1/2}}{h}\left(u_{i-1/2}^{n+1/2} - u_i^{n+1/2}\right) = u_i^n\,. \tag{2.2}$$

The scheme (2.2) can be formally viewed as a finite volume discretization of (2.1) using $V\partial_x u = \partial_x(Vu) - u\partial_x V$, see e.g. [3, 9, 10] for more details.

To obtain any particular scheme of the form (2.2) we will specify an approximation of the differences $(u_{i\pm1/2}^{n+1/2} - u_i^{n+1/2})$ in (2.2). We consider here only approximations that depend at most on three consecutive values from $(u_{i-2}^*, u_{i-1}^*, u_i^*, u_{i+1}^*, u_{i+2}^*)$ where $* = n$ or $* = n+1$. If such approximations are defined using strictly the known values for $* = n$, the scheme is called fully explicit. If the approximations are based purely on unknown values for $* = n+1$, we speak about fully implicit schemes. Our aim is to derive a semi-implicit scheme when in general the differences in (2.2) are approximated either explicitly or implicitly depending on the sign of $V_{i\pm1/2}$.

In all schemes discussed in this section we use the following approximation of gradients $\partial_x u_i^* \approx \partial_x u(x_i, t^*)$

$$h\,\partial_x u_i^* = 0.5(1-\kappa_i)(u_i^* - u_{i-1}^*) + 0.5(1+\kappa_i)(u_{i+1}^* - u_i^*)\,, \tag{2.3}$$

where the parameter κ_i is free to choose with a natural choice being $\kappa_i \in [-1,1]$. Such scheme is called "κ-scheme" in [14, 15] for fully explicit case. In what follows we introduce fully explicit, fully implicit and semi-implicit variants of κ-schemes.

To derive a fully explicit variant of (2.2) one can use finite Taylor series approximations as in [8, 12, 5]. Particularly, expressing $u_i^{n+1/2} = u_i^n + 0.5\tau\partial_t u_i^n$ and using $\partial_t u = -V\partial_x u$ one gets

$$u_i^{n+1/2} = u_i^n - 0.5\tau V_i \partial_x u_i^n. \tag{2.4}$$

Applying similar treatment for $u_{i+1/2}^{n+1/2}$ one arrives to the approximations

$$u_{i+1/2}^{n+1/2} = \begin{cases} u_i^n + 0.5(h - \tau V_i)\partial_x u_i^n & V_{i+1/2} > 0 \\ u_{i+1}^n - 0.5(h + \tau V_{i+1})\partial_x u_{i+1}^n & V_{i+1/2} < 0 \end{cases}, \tag{2.5}$$

where also the upwind principle has been used.

The scheme (2.2) together with (2.3) - (2.5) will be called the fully explicit κ-scheme. This scheme gives in the case of constant velocity V the well-known particular variants [15, 8], namely Lax-Wendroff for $\kappa_i \equiv \kappa = \text{sign}(V)$, Beam-Warming for $\kappa = -\text{sign}(V)$, and Fromm scheme for $\kappa = 0$.

Next we comment on the accuracy and stability conditions of fully explicit κ-scheme.

One can show that the scheme (2.2) - (2.5) is formally second order accurate for arbitrary value of κ_i in (2.3) and for variable smooth velocity V if $V_{i-1/2}V_{i+1/2} \geq 0$. We do not comment here the cases $V_{i-1/2}V_{i+1/2} < 0$ that we propose to treat by a first order accurate discretization as in e.g. [3].

The accuracy of numerical scheme can be proved by putting the values $u(x_{i\pm k}, t^*)$, $k = -2, -1 \ldots, 2$ of the exact solution of (2.1) into the numerical scheme, expressing these values by Taylor series with respect to $u(x_i, t^n)$, and replacing $\partial_t u(x_i, t^n)$ and $\partial_{tt} u(x_i, t^n)$ by space derivatives using the relations (obtained from $\partial_t u = -V \partial_x u$)

$$\partial_{tt} u = -V \partial_{tx} u \,, \quad \partial_{tx} u = -V \partial_{xx} u - V' \partial_x u \,. \tag{2.6}$$

One can show that the nonzero terms in the Taylor series start with $h^p t^q$ where $p+q \geq 2$, so the method is second order accurate. Note that also the values $V(x_{i\pm1/2})$ and $V(x_{i\pm1})$ shall be replaced by Taylor series with respect to $V(x_i)$. This Taylor series analysis was done by the author using software package Mathematica [16], the code is available by request. Note that analogous analysis is realized for all schemes in this paper.

It is interesting to note that the result on accuracy order is valid even without upwind principle, i.e. a replacement of (2.5) by a definition based on downwind principle would give the same accuracy results. As we discuss later, the correct upwind choice is required due to stability condition.

Concerning some practical suggestions which value of κ_i to choose in (2.3), the constant value 0 is often preferred which gives for several examples (but not for all) the smallest so called "phase error" [15, 8], see also some analogous numerical experiments later. Note that a variable choice of κ_i with respect to i is used e.g. as a part of so called limiter procedure [8] to reduce non-physical oscillations in numerical solutions.

An interesting option is the so called QUICKEST scheme [6] with a variable choice

$$\kappa_i = \text{sign}(V_i) \, \frac{h - 2\tau |V_i|}{3h} \,, \tag{2.7}$$

that gives the 3^{rd} order accurate scheme in the case of constant velocity. We search for analogous suggestions later in the case of fully implicit and semi-implicit schemes.

Additionally to the accuracy also a stability of numerical schemes must be studied. We investigate for the case of constant velocity $V_i \equiv V$ the so called von Neumann stability analysis, see e.g. [13, 15, 8]. Although a stability condition can be derived for some schemes using analytical methods [13, 15], we present here briefly an approach proposed and used in [1] where such condition is derived numerically. The advantage of such approach is that it can be applied to any numerical scheme studied in this paper. As we use it in the form published elsewhere, we reduce our description to a minimum and refer to literature for more practical details [1, 11].

To derive a stability condition of numerical scheme, one introduces a grid function $\epsilon_i^n = \epsilon(x_i, t^n)$ defined by

$$\epsilon(x, t) = \exp(-\lambda t) \exp(\imath x) \,, \quad x \in R \,, \; t \geq 0 \,, \tag{2.8}$$

where $\imath$ is the imaginary number, and the parameter λ shall be found. The values ϵ_i^n are supposed to fulfill the numerical scheme. Using trivial relations

$$\epsilon_{i\pm j}^n = \exp(\pm \imath j \Delta x) \epsilon_i^n \,, \quad \epsilon_i^{n+1} = S \epsilon_i^n \,, \; S := \exp(-\lambda \tau) \,, \tag{2.9}$$

where S denotes the so called amplification factor, the *stability condition of numerical scheme* is derived by requiring that $|S| \le 1$ [13, 1, 15, 8, 11].

We derive now such condition in the case of constant and positive velocity V (the case $V < 0$ can be studied analogously) for the scheme (2.2) -(2.5) that takes the form

$$\begin{aligned} u_i^{n+1} = u_i^n + \frac{C}{4} \big(&(-1+\kappa+C(1-\kappa))u_{i-2}^n + (5-3\kappa+C(-1+3\kappa))u_{i-1}^n + \\ &(-3+3\kappa-C(1+3\kappa))u_i^n + (-1-\kappa+C(1+\kappa))u_{i+1}^n \big) . \end{aligned} \tag{2.10}$$

where the nondimensional parameter $C := \tau|V|/h$ is the so called Courant number.

We replace in (2.10) u_i^n with ϵ_i^n etc., divide (2.10) by $\epsilon_i^n \neq 0$, and use (2.9) to obtain

$$\begin{aligned} S = 1 + \frac{C}{4} \big(&(-1+\kappa+C(1-\kappa)) \exp(-2\imath\Delta x) + (5-3\kappa+C(-1+3\kappa)) \exp(-\imath\Delta x) + \\ &(-3+3\kappa-C(1+3\kappa)) + (-1-\kappa+C(1+\kappa)) \exp(\imath\Delta x) \big) . \end{aligned} \tag{2.11}$$

Investigating the value $|S|$ in (2.11) numerically for a representative set of values C, κ, and $\Delta x \in (-\pi, \pi)$, the stability condition

$$\kappa \in [-1, 1] \quad \text{and} \quad C \in [0, 1] \tag{2.12}$$

is obtained. Note that it is in the agreement with available analogous theoretical results for fully explicit κ-scheme in e.g. [15].

We are now interested if a second order accurate fully implicit variant of (2.2) analogous to (2.4) - (2.5) can be derived with less restrictive stability condition. It is now rather straightforward to derive such fully implicit scheme, one has to use

$$u_i^{n+1/2} = u_i^{n+1} + 0.5\tau V_i \partial_x u_i^{n+1} \tag{2.13}$$

$$u_{i+1/2}^{n+1/2} = \begin{cases} u_i^{n+1} + 0.5\,(h + \tau V_i)\,\partial_x u_i^{n+1} & V_{i+1/2} > 0 \\ u_{i+1}^{n+1} - 0.5\,(h - \tau V_{i+1})\,\partial_x u_{i+1}^{n+1} & V_{i+1/2} < 0 \end{cases} . \tag{2.14}$$

Note that (2.13) - (2.14) differ not only by the time index $n+1$ to related formulas (2.4) - (2.5) of fully explicit κ-scheme. Using Taylor series analysis one can show that (2.2) with (2.3) and (2.13) - (2.14) is second order accurate for arbitrary κ_i in the case of variable velocity if $V_{i-1/2}V_{i+1/2} \ge 0$. The choice

$$\kappa_i = \operatorname{sign}(V_i) \frac{h + 2\tau|V_i|}{3h} \tag{2.15}$$

gives formally the third order accurate scheme for a constant velocity case.

Concerning the stability condition for the case of constant velocity $V_i \equiv V > 0$, one can show that for $\kappa_i \equiv \kappa \le 0$ the fully implicit κ-scheme is unconditionally stable for arbitrary positive Courant number C . This can be seen as an advantage when compared to fully explicit κ-schemes. The price to pay is that a system of linear algebraic equations has to be solved in each time step. In the case of time independent velocity as in (2.1) the matrix of such system does not change in time and has in general a 4-diagonal form if $V(x)$ does not change its sign for $x \in [-1, 1]$.

Furthermore, the choices $\kappa > 0$ give more restrictive stability condition. For instance the value $\kappa = 1/3$ gives a stable scheme for $C \in [0, 2]$, the third order accurate scheme (2.15) is stable only for $C \in [0, 0.5]$. The choice $\kappa = 1$ gives unstable numerical scheme for $C \in [0, 1]$.

Finally, we present the semi-implicit scheme based on IIOE time discretization (Inflow Implicit and Outflow Explicit). Following [9, 10, 4] we use in (2.2) with (2.3)

$$u^{n+1/2}_{i-1/2} - u^{n+1/2}_{i} = \begin{cases} u^{n+1}_{i-1} - u^{n+1}_{i} + 0.5\, h\partial_x u^{n+1}_{i-1} & V_{i-1/2} > 0 \\ -0.5\, h\partial_x u^{n}_{i} & V_{i-1/2} < 0 \end{cases}, \tag{2.16}$$

$$u^{n+1/2}_{i+1/2} - u^{n+1/2}_{i} = \begin{cases} 0.5\, h\partial_x u^{n}_{i} & V_{i+1/2} > 0 \\ u^{n+1}_{i+1} - u^{n+1}_{i} - 0.5\, h\partial_x u^{n+1}_{i+1} & V_{i+1/2} < 0 \end{cases}. \tag{2.17}$$

Note that (2.16) - (2.17) is derived by different and simpler approach than (2.4) - (2.5) or (2.13) - (2.14), e.g. it does not contain the parameters τ and $V_{i\pm1}$. Concerning the accuracy one can show that the scheme is second order accurate for arbitrary κ_i and for variable velocity if $V_{i-1/2}V_{i+1/2} \geq 0$. It is important to note (see also a related discussion in 2D case later) that such analysis exploits only the first relation in (2.6), and not the second one. This is possible due to the fact that the IIOE scheme contains "mixed" discrete values like u^{n+1}_{i-1} that helps to cancel the mixed derivative $\partial_{tx}u^n_i$ in Taylor series analysis that is not possible for e.g. fully explicit scheme. An implementation of this analysis in Mathematica is available by request. Finally the choice

$$\kappa_i = \mathrm{sign}(V_i)\frac{h - \tau|V_i|}{3h} \tag{2.18}$$

gives in the case of constant velocity the third order accurate scheme. Note that to prove it, the both relations in (2.6) are exploited that will be important in 2D case.

The most important results is concerning the stability condition of semi-implicit scheme. It can be shown that the κ-scheme of IIOE form for constant velocity is unconditionally stable for arbitrary Courant number and arbitrary κ_i. We see this as an advantage with respect to fully implicit κ-scheme. Moreover, the matrix of resulting system of linear algebraic equations has a more convenient 3-diagonal form with off-diagonals strictly either below or up to the main diagonal (if $V(x)$ does not change its sign), so the system can be solved in one step using a forward or backward substitution.

We note that the accuracy and stability conditions remain valid for a simpler finite difference form of all presented schemes when the values $V_{i-1/2}$ and $V_{i+1/2}$ in (2.2), (2.5), (2.14), and (2.16) - (2.17) are replaced by V_i.

3. Two-dimensional case. The representative advection equation takes form

$$\partial_t u(x,y,t) + \vec{V}(x,y)\cdot\nabla u(x,y,t) = 0\,, \quad u(x,y,0) = u^0(x,y) \tag{3.1}$$

that shall be accompanied by appropriate boundary conditions.

We consider here numerical methods only for a structured tensor grid having a uniform space discretization step h and a uniform time discretization step τ. The computational domain is a square $\Omega = (-1,1)^2$ and the notations from one-dimensional case are extended as usual.

Firstly, the following approximation of gradients $(\partial_x u^*_{ij}, \partial_y u^*_{ij}) \approx \nabla u(x_i, y_j, t^*)$ is used,

$$\begin{aligned} h\,\partial_x u^*_{ij} &= 0.5(1-\kappa^x_{ij})(u^*_{ij} - u^*_{i-1\,j}) + 0.5(1+\kappa^x_{ij})(u^*_{i+1\,j} - u^*_{ij})\,, \\ h\,\partial_y u^*_{ij} &= 0.5(1-\kappa^y_{ij})(u^*_{ij} - u^*_{i\,j-1}) + 0.5(1+\kappa^y_{ij})(u^*_{i\,j+1} - u^*_{ij})\,, \end{aligned} \tag{3.2}$$

where the parameters κ^x_{ij} and κ^y_{ij} are free to choose, e.g. $\kappa^x_{ij} \in [-1,1]$ and $\kappa^y_{ij} \in [-1,1]$.

Let $\vec{V} = (V, W)$. We present the numerical scheme obtained by the dimension by dimension extension of 1D case that takes the form

$$
\begin{aligned}
(3.3) \quad u_{ij}^{n+1} &+ \frac{\tau V_{i+1/2\,j}}{h}\left(u_{i+1/2\,j}^{n+1/2} - u_{ij}^{n+1/2}\right) - \frac{\tau V_{i-1/2\,j}}{h}\left(u_{i-1/2\,j}^{n+1/2} - u_{ij}^{n+1/2}\right) + \\
&+ \frac{\tau W_{i\,j+1/2}}{h}\left(u_{i\,j+1/2}^{n+1/2} - u_{ij}^{n+1/2}\right) - \frac{\tau W_{i\,j-1/2}}{h}\left(u_{i\,j-1/2}^{n+1/2} - u_{ij}^{n+1/2}\right) = u_{ij}^{n}\,,
\end{aligned}
$$

where in the 1^{st} line of (3.3) the index j is fixed and the one-dimensional definitions of previous Section with respect to i shall be used, and analogously for the 2^{nd} line.

We discuss now particular schemes of the form (3.3) and their order of accuracy. From [1, 8] one can easily show that a fully explicit variant of (3.3) can not be in general second order accurate even for constant velocity (V, W). The reason is a missing possibility to cancel the mixed derivative $\partial_{xy} u(x_i, t^n)$ in Taylor series analysis that occur due to (analogously to 1D case given by (2.6))

$$
(3.4) \qquad \partial_{tt} u = -V \partial_{tx} u - W \partial_{ty} u\,,
$$

$$
(3.5) \qquad \partial_{tx} u = -V \partial_{xx} u - W \partial_{xy} u\,,\; \partial_{ty} u = -V \partial_{xy} u - W \partial_{yy} u\,.
$$

To obtain a second order accurate fully explicit scheme, the so called Corner Transport scheme is proposed in [7, 8] that exploits at least one diagonal value of numerical solution, e.g. the value $u_{i-1\,j-1}^{n}$ is used if a constant velocity $\vec{V}$ is such that $V > 0$ and $W > 0$. We will use this idea to obtain a third order accurate Corner Transport IIOE scheme later.

Next we present for clarity the dimension by dimension extension of the IIOE κ-scheme of the form (3.2) - (3.3) that is given by

$$
(3.6) \qquad u_{i-1/2\,j}^{n+1/2} - u_{ij}^{n+1/2} = \begin{cases} u_{i-1\,j}^{n+1} - u_{ij}^{n+1} + 0.5\, h \partial_x u_{i-1\,j}^{n+1} & V_{i-1/2\,j} > 0 \\ -0.5\, h \partial_x u_{ij}^{n} & V_{i-1/2\,j} < 0 \end{cases},
$$

$$
(3.7) \qquad u_{i+1/2\,j}^{n+1/2} - u_{ij}^{n+1/2} = \begin{cases} 0.5\, h \partial_x u_{ij}^{n} & V_{i+1/2\,j} > 0 \\ u_{i+1\,j}^{n+1} - u_{ij}^{n+1} - 0.5\, h \partial_x u_{i+1\,j}^{n+1} & V_{i+1/2\,j} < 0 \end{cases},
$$

$$
(3.8) \qquad u_{i\,j-1/2}^{n+1/2} - u_{ij}^{n+1/2} = \begin{cases} u_{i\,j-1}^{n+1} - u_{ij}^{n+1} + 0.5\, h \partial_y u_{i\,j-1}^{n+1} & W_{i\,j-1/2} > 0 \\ -0.5\, h \partial_y u_{ij}^{n} & W_{i\,j-1/2} < 0 \end{cases},
$$

$$
(3.9) \qquad u_{i\,j+1/2}^{n+1/2} - u_{ij}^{n+1/2} = \begin{cases} 0.5\, h \partial_y u_{ij}^{n} & W_{i\,j+1/2} > 0 \\ u_{i\,j+1}^{n+1} - u_{ij}^{n+1} - 0.5\, h \partial_y u_{i\,j+1}^{n+1} & W_{i\,j+1/2} < 0 \end{cases}.
$$

One can show that the scheme (3.3) with (3.6) - (3.9) is second order accurate for variable velocity and for arbitrary κ_{ij}^{x} and κ_{ij}^{y} in (3.2). Such result can be obtained due to the fact that, opposite to fully explicit scheme, the relation (3.5) is not used in Taylor series analysis. This property has been proved using the software Mathematica and confirmed also in numerical experiments as given in the next Section. More detailed publication of these results is in a preparation.

Moreover, the scheme (3.3) - (3.9) is unconditionally stable for constant velocity vector and for arbitrary values of κ_{ij}^{x} and κ_{ij}^{y} that was proved also using Mathematica.

The scheme (3.3) - (3.9) can be further simplified without loosing all listed properties by replacing all values $V_{i\pm1/2\,j}$ with V_{ij} and the values $W_{i\,j\pm1/2}$ with W_{ij} in

(3.3) and (3.6) - (3.9). Such simplification can be seen as a finite difference form of IIOE κ-scheme.

Unfortunately, there exists no special choice of κ parameters to obtain the third order accuracy for the case of constant velocity vector. The reason is, analogously to 1D case, that the relation (3.5) must be used for such accuracy results, and the mixed derivative $\partial_{xy}u$ can not be canceled in the Taylor series analysis. It is an interesting question if the approach of Corner Transport scheme can be used with IIOE time discretization to obtain a numerical scheme with the desired property in 2D case.

The answer is positive and in what follows we present such scheme in the finite difference form that was derived using software Mathematica. It is derived directly from Taylor series analysis and it does not take the form (3.3). We denote

$$C = \frac{\tau V_{ij}}{h}\,,\; D = \frac{\tau W_{ij}}{h}\,,\; C^+ = \max\{C,0\}\,,\; C^- = \min\{C,0\}\,.$$

The Corner Transport IIOE scheme can be written as follows,

$$\begin{aligned}
(3.10)\qquad & (12+8|C|+C^2+8|D|+D^2+2|CD|)\,u_{ij}^{n+1} + \\
&+ C^+(2u_{i-2\,j}^{n+1} - 10u_{i-1\,j}^{n+1} + C^+(u_{i-2\,j}^{n+1} - 2u_{i-1\,j}^{n+1})) + \\
&+ D^+(2u_{i\,j-2}^{n+1} - 10u_{i\,j-1}^{n+1} + D^+(u_{i\,j-2}^{n+1} - 2u_{i\,j-1}^{n+1})) + \\
&- C^-(2u_{i+2\,j}^{n+1} - 10u_{i+1\,j}^{n+1} - C^-(u_{i+2\,j}^{n+1} - 2u_{i+1\,j}^{n+1})) + \\
&- D^-(2u_{i\,j+2}^{n+1} - 10u_{i\,j+1}^{n+1} - D^-(u_{i\,j+2}^{n+1} - 2u_{i\,j+1}^{n+1})) + \\
&+ 2D^+(C^+(u_{i-1\,j-1}^{n+1} - u_{i-1\,j}^{n+1} - u_{i\,j-1}^{n+1}) - C^-(u_{i+1\,j-1}^{n+1} - u_{i+1\,j}^{n+1} - u_{i\,j-1}^{n+1})) + \\
&- 2D^-(C^+(u_{i-1\,j+1}^{n+1} - u_{i-1\,j}^{n+1} - u_{i\,j+1}^{n+1}) - C^-(u_{i+1\,j+1}^{n+1} - u_{i+1\,j}^{n+1} - u_{i\,j+1}^{n+1})) = \\
& 12u_{ij}^n + C^+(2u_{ij}^n + 2u_{i-1\,j}^n - 4u_{i+1\,j}^n) - C^-(2u_{ij}^n + 2u_{i+1\,j}^n - 4u_{i-1\,j}^n) + \\
&+ D^+(2u_{ij}^n + 2u_{i\,j-1}^n - 4u_{i\,j+1}^n) - D^-(2u_{ij}^n + 2u_{i\,j+1}^n - 4u_{i\,j-1}^n) + \\
&+ C^2(u_{i+1\,j}^n - 2u_{ij}^n + u_{i-1\,j}^n) + D^2(u_{i\,j+1}^n - 2u_{ij}^n + u_{i\,j-1}^n) + \\
&+ |CD|(u_{i-1\,j}^n + u_{i+1\,j}^n + u_{i\,j-1}^n + u_{i\,j+1}^n - 2u_{ij}^n) + \\
&- (C^-D^- + C^+D^+)(u_{i-1\,j+1}^n + u_{i+1\,j-1}^n) + (C^-D^+ + C^+D^-)(u_{i-1\,j-1}^n + u_{i+1\,j+1}^n)\,.
\end{aligned}$$

Note that due the fact that either $C^+ = 0$ or $C^- = 0$, and similarly for D, only some terms in above are nonzero.

The Taylor series analysis of (3.10) (from which the scheme (3.8) was derived) proves that this scheme is 2^{nd} order accurate for variable velocity case and the 3^{rd} order accurate for constant velocity vector $\vec{V}$ in (3.1). Moreover, it is unconditionally stable for arbitrary velocity and arbitrary positive time step.

4. Numerical experiments. In what follows we illustrate the properties of semi-implicit IIOE methods for some benchmark examples. Note that linear algebraic systems in all examples are solved by Gauss-Seidel iterations using so called fast sweeping method [17]) where at most 2 sweeps (i.e. 8 Gauss-Seidel iterations) used.

4.1. Translation by constant velocity. To illustrate the formal order of accuracy of all methods we start with numerical examples of advection equation with constant velocity $\vec{V} = (0.8, 0.9)$. The exact solution is given simply by

$$u(x, y, t) = u^0(x - 0.8t, y - 0.9t), \tag{4.1}$$

where $u^0 = u^0(x, y)$ is a given initial function. To check the implementation of methods we begin with the choice $u^0(x, y)$ being randomly chosen quadratic function. Using the exact solution (4.1) to define Dirichlet boundary conditions, we obtain with (3.3) and (3.6) - (3.9) for all interesting choices of κ^*_{ij} in (3.2) the exact solution up to a machine accuracy for any chosen T, N, and M. Choosing as the initial function some cubic polynomial, only the Corner Transport IIOE scheme (3.10) gives numerical solutions differing from the exact solution purely by rounding errors.

Next we test analogous example where we choose

$$u^0(x, y) = \exp(-((x - x_0)^2 + (y - y_0)^2)/0.04) \tag{4.2}$$

with $(x_0, y_0) = (-0.5, -0.4)$ and the zero Dirichlet boundary conditions that differ negligibly from the exact boundary conditions given by (4.1). Analogous initial and boundary conditions are typical for examples of e.g. contaminant transport [2].

In Figure 4.1 we illustrate a typical behavior of the IIOE κ-scheme for three choices of parameters κ^*_{ij}. We can confirm for this example the analogous behavior of phase error as reported e.g. in [15, 8] for fully explicit κ-scheme. Such numerical error is visible for coarse grids e.g. from a wrong speed of the maximum of numerical solution, see Figure 4.1. Particularly for the choice $\kappa = 1$ this speed is underestimated and for $\kappa = -1$ it is overestimated, the choice $\kappa = 0$ seems to give a good compromise.

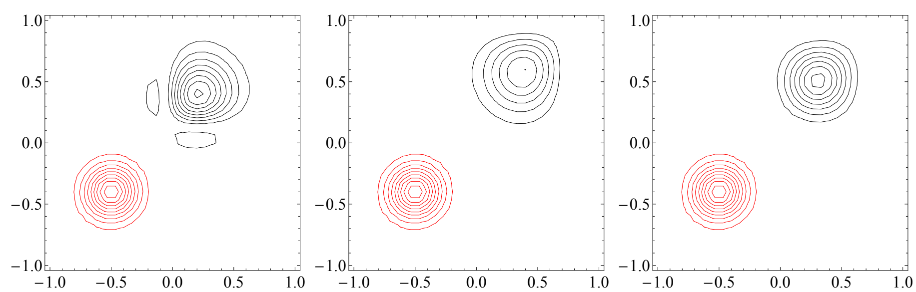

FIG. 4.1. *Numerical solutions for translation of Gaussian with constant velocity for the coarse grid ($M = 30$, $N = 50$). The red contour lines represent the initial function for values* 0.1, 0.2 *up to* 0.9, *the black contour lines represent the numerical solutions for values* 0.1 *up to* 0.8, *with the addition of value* −0.1 *in the left picture, consult the Table 4.1. The left picture represent the numerical solution at $T = 1$ for $\kappa^*_{ij} \equiv 1$, the middle one for $\kappa^*_{ij} \equiv -1$ and the right one for $\kappa^*_{ij} \equiv 0$.*

The Table 4.1 illustrates three types of errors for the previous numerical experiments - the values of extrema, where the exact extrema are 0 and 1, and the discrete error e given by

$$e = \tau\, h \sum_{n=1}^{N} \sum_{i,j=1}^{M} \left| u^n_{ij} - u(x_i, y_j, t^n) \right| . \tag{4.3}$$

Analogous example is tested with the Corner Transport IIOE scheme, see Table 4.2 for results and Figure 4.2 for pictures. Note that the Experimental Order of Convergence (EOC) for this example is larger than 2. To check also the stability condition, we compute this example with five times larger time step τ. No instabilities are observed and the EOC is analogous.

M	min	max	$e10^{-2}$	min	max	$e10^{-2}$	min	max	$e10^{-2}$
30	-1.6E-1	.83	4.40	-8.7E-2	.60	5.22	-2.6E-2	.74	1.76
60	-1.8E-2	.97	1.09	-4.9E-2	.87	1.72	-4.7E-3	.94	0.40

TABLE 4.1

The minimum, maximum and the error (4.3) for the numerical solutions of translation of Gaussian for $\kappa^*_{ij} \equiv 1$ (the 2^{nd} - 4^{th} columns), $\kappa^*_{ij} \equiv -1$ (the 5^{th} - 7^{th} ones), and $\kappa^*_{ij} \equiv 0$. Note that $N = 5/3M$.

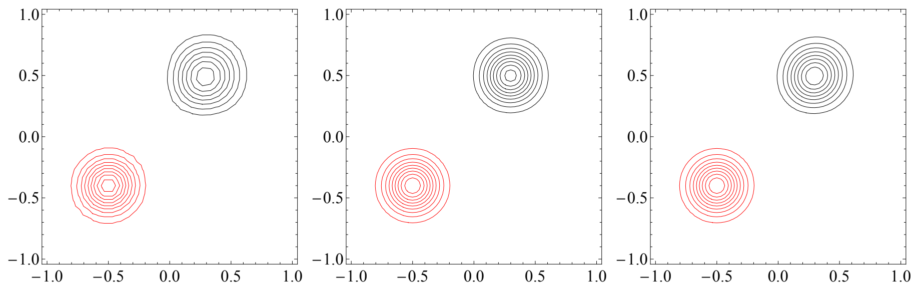

FIG. 4.2. *Numerical solutions analogous to Figure 4.1 using Corner Transport IIOE scheme. The left picture represent the numerical solution at $T = 1$ for $M = 30$ and $N = 50$, the middle one for $M = 60$ and $N = 100$ and the right one for $M = 60$ and $N = 20$.*

M	N	min	max	$e.10^{-2}$	N	min	max	$e.10^{-2}$
30	50	-1.5E-2	.77	1.43	10	-2.6E-2	.65	3.0
60	100	-4.2E-4	.95	.23	20	-3.5E-3	.89	.60
120	200	-2.2E-4	.99	.032	40	-2.2E-4	.98	.087

TABLE 4.2

Analogous results to Table 4.1 for the Corner Transport IIOE scheme for two different Courant numbers, consult the Table 4.1 for notations.

The next example takes the same initial function in (4.2) for $(x_0, y_0) = (-0.5, 0.)$ with a variable velocity field defined by

$$\vec{V}(x, y) = (-2\pi y, 2\pi x)\,. \tag{4.4}$$

The stop time is $T = 1.5$, so the initial profile will rotate one and half time, namely

$$u(x, y, t) = e^{-((x\cos(2\pi t)+y\sin(2\pi t)-x_0)^2+(x\sin(2\pi t)+y\cos(2\pi t)-x_0)^2)/0.04}.$$

The results are summarized in Table 4.3 and in Figure 4.3 where we compare the κ-scheme for $\kappa^*_{ij} \equiv 0$ and the Corner Transport scheme for same settings. For the latter case we present also the results with three times larger time step.

We note that for chosen examples the choice $\kappa_i \equiv 0$ gives better results than $\kappa_i = \text{sign}(V_i)$ or $\kappa_i = -\text{sign}(V_i)$ in (3.6) - (3.9), but our experiences (to be published elsewhere) is that e.g. the single vortex benchmark example [3] gives the smallest error at $t = T$ for the choice $\kappa_i = \text{sign}(V_i)$.

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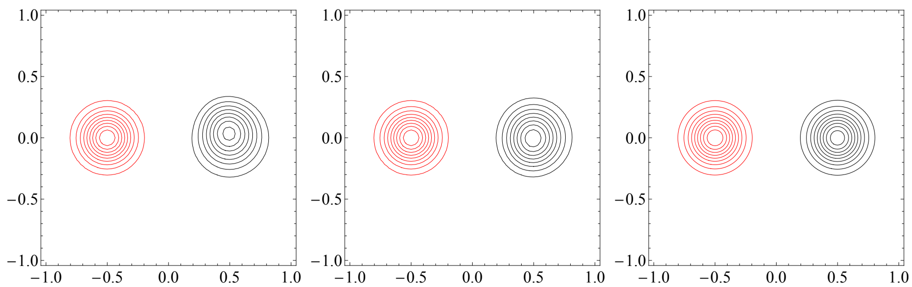

FIG. 4.3. *Numerical solutions for one and half rotation of Gaussian with analogous explanation as for Figure 4.1. The left picture represents the numerical solution at $T = 1.5$ for the κ-scheme with $\kappa^*_{ij} \equiv 0$ using $M = 60$ and $N = 600$, the middle one for the Corner Transport scheme with identical M and N, and the right one for the Corner Transport scheme with $M = 120$ and $N = 1200$.*

M	min	max	$e10^{-2}$	min	max	$e10^{-2}$	min	max	$e10^{-2}$
60	-2.3E-2	.85	1.9	-4.5E-3	.88	1.00	-1.0E-2	.84	1.5
120	-9.1E-4	.97	0.48	-5.3E-5	.98	.15	-1.4E-4	.97	.24

TABLE 4.3

The minimum, maximum and the error (4.3) for the numerical solutions of rotation of Gaussian for $\kappa^*_{ij} \equiv 0$ and $N = 10M$ (the 2^{nd} - 4^{th} columns), the Corner Transport scheme for $N = 10M$ (the 5^{th} - 7^{th} columns), and for $N = 10/3M$.

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