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TWO-GRID ALGORITHMS FOR PRICING AMERICAN OPTIONS BY A PENALTY METHOD

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Abstract. In this manuscript we present two-grid algorithms for the American option pricing problem with a smooth penalty method where the variational inequality, associated with the optimal stopping time problem, is approximated with a nonlinear Black-Scholes equation. In order to compute the numerical solution of the latter unconstrained problem we must solve a system of nonlinear algebraic equations resulting from the discretization by e.g. the finite difference or the finite element method. We propose two-grid algorithms as we first solve the nonlinear system on a coarse grid with mesh size h^c and further a linearized system on a fine grid with mesh size h^f, satisfying $h^f = \mathcal{O}((h^c)^{2^k})$, $k = 1, 2, \ldots$, where k is the number of Newton iterations. Numerical experiments illustrate the computational efficiency of the algorithms.

1. Introduction. Pricing American-type of options is an issue of serious importance in computational finance as these are the common type of financial instruments on the derivative market. The holder of an American call (put) option has the right to buy (sell) an asset at the prescribed strike price *before or on* a given expiry date. The *early exercise* feature of the option is the distinguishing characteristic of this option contract which postulates the optimal stopping time problem further reformulated as a variational inequality. Analytical solutions of the American pricing problem are seldom available even for very simplistic cases and there is a strong demand for computationally efficient numerical methods, see e.g. Ševčovič et al. [10].

Let S stand for the underlying asset price process, following a standard geometric Brownian motion with volatility σ and drift equal to the interest rate r while t denotes the time variable. The American option value with payoff $V^*(S)$ and maturity T satisfies the backward parabolic linear complementarity problem (LCP) [2, 6, 10]

$$\begin{cases} LV(S,t)\cdot(V(S,t)-V^*(S)) = 0, \\ LV(S,t) := V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rV_S - rV \geq 0, \\ V(S,t) - V^*(S) \geq 0, \end{cases}$$

a.e. in $(0,\infty)\times[0,T)$. Further, we consider the penalized problem which approximates the LCP for some sufficiently small positive parameter ϵ

$$V_t^\epsilon + \frac{1}{2}\sigma^2 S^2 V_{SS}^\epsilon + rSV_S^\epsilon - rV + g(S,V^\epsilon) = 0,\ (S,t)\in(0,S_{\max})\times[0,T) \tag{1.1}$$

with far field boundary location $S_{\max}$. We consider pricing an American put option with strike price K and the following conditions on the parabolic boundary:

$$V^\epsilon(S,T) = V^*(S) := \max(K-S,0),\quad V^\epsilon(0,t) = K,\quad V^\epsilon(S_{\max},t) = 0. \tag{1.2}$$

The penalty method is a widely-used technique in constrained nonlinear programming which guarantees in an asymptotic sense the fulfilment of constraints by including in the objective function an additional penalty term, see Grossmann et al.

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[3]. Thus the constrained programming problem is embedded in a family of unconstrained variational problems, depending on some parameter present in the penalty term acting which acts against the optimization goal if constraints are violated.

The exterior penalty method with nonsmooth penalty term

$$g(S, V^{\epsilon}) = \epsilon^{-1} \max(V^* - V, 0)$$

is thoroughly investigated in numerical analysis and well-known in computational finance, cf. e.g. [2, 4]. It is certainly attractive for its exponential convergence rate in the penalty parameter. However, this approach does *not guarantee that the early exercise constraint is strictly satisfied* by the solution.

Nielsen et al. [11] propose an interior penalty method for pricing American put option pricing where the penalty term takes the form

$$g(S, V_{\epsilon}) = \frac{\epsilon \mathrm{C}}{V_{\epsilon} + \epsilon - q(S)}, \quad q(S) = K - S. \tag{1.3}$$

The early exercise constraint $V^{\epsilon} - V^* \geq 0$ is not violated and moreover the penalty function $g(S, V_{\epsilon})$ is smooth. The interior penalty approach (1.3) is considered in a large number of articles, showing satisfactory numerical results, cf. [7, 11, 12], and convergence theory has been recently established, see Zhang and Wang [16].

In this paper we propose two-grid algorithms for implicit difference schemes for the initial-boundary value problem (IBVP) (1.1),(1.2) with interior penalization (1.3). After the seminal works of O. Axelsson [1] and J. Xu [15] two-grid finite element methods were further developed in many papers, cf. e.g. [8, 13], and these showed remarkable computational efficiency. Here, because of the nature of the problem, our error estimates are measured in maximum norm, similarly as for differential equations with boundary layers. Section 2 briefly presents the two-grid method and, on this base, in Section 3 we formulate two algorithms. Finally, various numerical experiments are presented in the last section.

Notations. Further on, by $\mathcal{C}$ we denote a generic positive constant, independent of mesh size, and by $\|\cdot\|, \|\cdot\|_h$ we denote the continuous and discrete maximum norms.

2. The two-grid method. In this section for clarity of the presentation we consider the forward price $u(x,t)$ by applying the log transformation $S = Ke^x$ and abusing the time notation where $t = T - \tau$ now stands for time-to-maturity. The IBVP (1.1),(1.2) now reads as follows:

$$u_t - \frac{1}{2}\sigma^2 u_{xx} + \left(\frac{1}{2}\sigma^2 - r\right) u_x + ru - g(Ke^x, u) = 0, \ (x,t) \in \mathrm{I} \times (0, T]$$
$$u(-L, t) = K, \ u(L, t) = 0, \ u(x, 0) = V^*(Ke^x)$$

for $\mathrm{I} := (-L, L), \ L = \ln(S_{\max}/K)$.

Further, for a given integer N we define $\triangle t = T/N, \ t^n = n\triangle t$. Following backward Euler time discretization we consider the following ODE problem:

Find $p^{n+1}(x), \ n = 0, \dots, N-1$ *such that*

$$\begin{aligned} -\frac{1}{2}\sigma^2 p^{n+1}_{xx} + \left(\frac{1}{2}\sigma^2 - r\right) p^{n+1}_x + \left(r + \frac{1}{\triangle t}\right) p^{n+1} - g(Ke^x, p^{n+1}) = \frac{1}{\triangle t} p^n \\ p^{n+1}(-L) = K, \ \ p^{n+1}(L) = 0, \ \ p^0(x) = V^*(Ke^x). \end{aligned} \tag{2.1}$$

From (1.3) we have

$$g'_{V_\epsilon}(S, V_\epsilon) = \frac{-\epsilon \mathrm{C}}{(V_\epsilon + \epsilon - q(S))^2} < 0$$

and from the maximum principle for the boundary value problem (BVP) (2.1)

$$\|p^{n+1}(x)\| \le K + Q(\|g(Ke^x, 0)\| + \frac{1}{\triangle t}\|p^n(x)\|) := \zeta,$$

where the constant Q depends on σ and r while ζ - on $\triangle t$ and the known $p^n(x)$.

Let us now consider Newton's linearization for the BVP (2.1), $p^{(k)}(x) \approx p^{n+1}(x)$:

$$\mathcal{L}p^{(k+1)} := -\frac{1}{2}\sigma^2 p^{(k+1)}_{xx} + \left(\frac{1}{2}\sigma^2 - r\right) p^{(k+1)}_x + \left(r + \frac{1}{\triangle t}\right) p^{(k+1)} - g'_p(Ke^x, p^{(k+1)})$$

$$= \left(r + \frac{1}{\triangle t}\right) p^{(k)} - g(Ke^x, p^{(k)}) + \frac{1}{\triangle t} p^n(x) - g'_p(Ke^x, p^{(k)}) p^{(k)} =: f(p^{(k)}) \tag{2.2}$$

$$p^{(k+1)}(-L) = K, \; p^{(k+1)}(L) = 0.$$

The following lemma asserts the convergence of the Newton's iteration (2.2) with initial guess $p^{(0)}$ s.t. $\|p^{(0)} - p^{n+1}\| < \rho$.

Lemma 2.1. *Let $Q^{-1}\theta\rho < 1$, where*

$$\theta = \max_{x \in \mathrm{I}, |\psi| \le \zeta + 2\rho} \left| \frac{\partial^2 g(Ke^x, \psi)}{\partial \psi^2} \right|.$$

Then the linearization process (2.2) is convergent and the following estimate holds

$$\|p^{(k+1)} - p^{n+1}\| \le Q\theta^{-1}(Q^{-1}\theta\rho)^{2^k}, \quad k = 0, 1, 2, \ldots$$

For a given integer m, we define $h = 2L/m$, $x_i = -L + ih$, $i = 0, 1, \ldots, m$ and further consider the finite difference analogue of (2.2), $P_i^{(k)} \approx p^{(k)}(x_i)$

$$\begin{aligned} \mathcal{L}_h P_i^{(k+1)} &:= -\frac{1}{2}\sigma^2 P^{(k+1)}_{\overline{x}x_i} + \left(\frac{1}{2}\sigma^2 - r\right) P^{(k+1)}_{\mathring{x}_i} + \left(r + \frac{1}{\triangle t}\right) P_i^{(k+1)} \\ &- g'_{P_i}(Ke^{x_i}, P_i^{(k)}) P_i^{(k+1)} = f(P_i^{(k)}), \quad P_0^{(k+1)} = K, \quad P_m^{(k+1)} = 0, \end{aligned} \tag{2.3}$$

where $P_{\mathring{x}_i}$ and $P_{\overline{x}x_i}$ are the second-order compact finite difference approximations of p_x and p_{xx} for $i = 2, \ldots, k$ respectively.

Lemma 2.2. *Let $p^{(k+1)}$ be the solution of problem (2.2) and $P^{(k+1)}$ be the solution of (2.3). Then the following error estimate holds*

$$\|P^{(k+1)} - p^{(k+1)}\| \le \mathcal{C}h^2.$$

For the convergence of the scheme (2.3) we have the following theorem.

Theorem 1. *There exist constants ρ_0 and h_0 s.t. if $h \le h_0$ and $\|p^{(0)} - p^{n+1}\| \le \rho \le \rho_0$ then the following estimate holds*

$$\|P^{(k+1)} - p^{n+1}\|_h \le \mathcal{C}h^2 + Q\theta^{-1}(Q^{-1}\theta\rho)^{2^k}, \quad k = 0, 1, 2 \ldots \tag{2.4}$$

Next, we consider the mesh function $P_i^{n+1} \approx p^{n+1}(x_i)$ and approximate the problem (2.1) in the following standard way for $i = 2, \ldots, m-1$:

$$-\frac{1}{2}\sigma^2 P^{n+1}_{\overline{x}x_i} + \left(\frac{1}{2}\sigma^2 - r\right) P^{n+1}_{\mathring{x}_i} + \left(r + \frac{1}{\triangle t} P_i^{n+1}\right) - g(e^{x_i}, P_i^{n+1}) = \frac{1}{\triangle t} P_i^n. \tag{2.5}$$

Following the estimate (2.4) we consider two-grid algorithms for the pricing of the American options. Let us define two spatial grids: a coarse grid with step size h^c and a fine grid with step size h^f where $h^c \gg h^f$ holds. The two-grid method is initiated with solving the discrete problem (2.5) on the coarse grid and then interpolate this coarse-grid solution P^{n+1} for which we have the estimate

$$\|I(P^{n+1}) - p^{n+1}\|_{h^c} \leq \|I(P^{n+1}) - P^{n+1}\|_{h^c} + \|P^{n+1} - p^{n+1}\|_{h^c} \leq \mathcal{C}(h^c)^2,$$

where $I(P^{n+1})$ is the interpolant of the discrete solution P^{n+1} of problem (2.5). If in the iterative process (2.2) we consider one Newton iteration $k = 1$ and the initial guess $p^{(0)} := I(P^{n+1})$ then in (2.4) we have $(Q^{-1}\theta\rho)^2 = \mathcal{C}h^4$. Therefore, if one solves the linearized problem (2.2) on a fine mesh with step $h^f = (h^c)^2$ then the right-hand side of (2.4) takes the form $\mathcal{C}(h^c)^4$.

THEOREM 2. *Let the assumptions of Theorem 1 hold. Then for the error of the two-grid method with $h^f = (h^c)^2$ we have*

$$\|P^{n+1} - u\|_{h^c} \leq \mathcal{C}(\triangle t + (h^c)^4).$$

3. Two-grid algorithms for American option pricing. The outlined theoretical considerations, which refer to the log-price of the underlying for clarity, prompt the application of the two-grid method for the American option pricing problem.

In this section we present our two-grid algorithms for the IBVP (1.1),(1.2) after time reversal with interior penalty term (1.3). Let us define two non-uniform spatial grids - a coarse mesh $\overline{\omega}_c$ and a fine grid $\overline{\omega}_f$

$$\overline{\omega}_c = \{S_1 = 0,\ S_{i+1} = S_i + h_i^c,\ i = 1, \dots, m_c - 1,\ S_{m_c} = S_{\max}\},$$
$$\overline{\omega}_f = \{S_1 = 0,\ S_{i+1} = S_i + h_i^f,\ i = 1, \dots, m_f - 1,\ S_{m_f} = S_{\max}\},$$

where $m_f \gg m_c$ and the discrete solution, computed on the mesh $\overline{\omega}_*$ is denoted by $P_{i,*}^n = V(S_i, t^n)$. Let us now write down the considered finite difference approximations of the first derivative for $h_i = S_{i+1} - S_i,\ \hbar_i = 0.5(h_i + h_{i-1})$

$$(P_{\check{S}})_i^n = \frac{P_{i+1}^n - P_i^n}{h_i}, \quad (P_{\hat{S}})_i^n = \frac{P_i^n - P_{i-1}^n}{h_{i-1}}, \quad (P_{\mathring{S}})_i^n = \frac{h_{i-1}P_{S}{}_i^n + h_i P_{\overline{S}}{}_i^n}{2\hbar_i}$$

where $(P_{\check{S}})_i^n, (P_{\hat{S}})_i^n$ are of first order and $(P_{\mathring{S}})_i^n$ of second on a smooth grid. The second derivative is further approximated as

$$(P_{SS})_i^n = \left((P_{\check{S}})_i^n - (P_{\hat{S}})_i^n\right)/\hbar_i.$$

After backward Euler time discretization of (1.1) and application of the maximal use of central differencing with flag $\chi := \mathrm{H}(\sigma^2 S_i - r h_i)$ (H stands for the Heaviside function), see Wang and Forsyth [14], we get the following system of nonlinear equations for $n = 0, \dots, N-1$ and $i = 1, \dots, m-1$:

$$\begin{aligned} &\frac{P_i^{n+1} - P_i^n}{\triangle t} - \frac{\sigma^2 S_i^2}{2}(P_{SS})_i^{n+1} \\ &\quad -rS_i\left(\chi(P_{\mathring{S}})_i^{n+1} + (1-\chi)(P_{\check{S}})_i^{n+1}\right) + rP_i^{n+1} - g(S_i, P_i^{n+1}) = 0, \\ &\quad P(0, t^{n+1}) = K,\ P(S_m, t^{n+1}) = 0,\ P(S_i, 0) = V^*(S_i). \end{aligned} \tag{3.1}$$

We find P^{n+1} by initiating a Newton's iteration process with initial guess $P^{(0)} = P^n$, where the Newton increment on the $(k+1)$-th step $\triangle^{(k+1)} = P^{(k+1)} - P^{(k)}$ is the solution of the following tridiagonal system of linear equations

$$-A_i\triangle_{i-1}^{(k+1)} + C_i^{(k)}\triangle_i^{(k+1)} - B_i\triangle_{i+1}^{(k+1)} = \frac{P_i^n}{\triangle t} + A_iP_{i-1}^{(k)} - \widetilde{C}_iP_i^{(k)} + B_iP_{i+1}^{(k)} + F_i^{(k)}, \tag{3.2}$$

where $A_1 = A_N = B_1 = B_N = 0$, $C_1^{(k)} = C_N^{(k)} = 1$, $F_N^{(k)} = K$, $F_N^{(k)} = 0$ and

$$A_i = \frac{S_i}{2\hbar_i h_{i-1}}\left(\sigma^2 S_i - \chi r h_i\right), \quad B_i = \frac{S_i}{2\hbar_i h_i}\left(\sigma^2 S_i + \chi r h_{i-1}\right) + (1-\chi)\frac{rS_i}{h_i},$$
$$C_i^{(k)} = \widetilde{C}_i + \frac{\epsilon \mathrm{C}}{(P_i^{(k)} + \epsilon - q_i)^2}, \quad \widetilde{C}_i = \frac{1}{\triangle t} + A_i + B_i + r, \quad F_i^{(k)} = \frac{\epsilon \mathrm{C}}{P_i^{(k)} + \epsilon - q_i},$$

The iteration process is terminated when reaching the desired tolerance i.e. we set $P^{n+1} := P^{(k+1)}$ when $\max_i\{|\triangle_i^{(k+1)}|/(\max\{1, P_i^{(k+1)}\})\} < \mathsf{tol}$.

At each iteration k, in view of the definition of χ, $M := tridiag[-A_i, C_i^{(k)}, -B_i]$ is strictly diagonally dominant and $A_i, C_i^{(k)}, B_i > 0$, i.e. it is an M-matrix (inverse monotone). The fully implicit upwind (first order) scheme is unconditionally monotone under the mild restriction $\mathrm{C} \geq rK$, see Nielsen et al. [11]. Analogously, the same result holds true for the fully implicit discretization (3.1).

Following Section 2 we propose the following space two-grid algorithm.

Algorithm 1 (**A1**) At each time level $n = 0, 1, \dots$ we perform the two steps:

step 1. Set $P_c^{(0)} := P_c^n$ and compute P_c^{n+1} by (3.1) through Newton's iterations (3.2) on the coarse mesh $\overline{\omega}_c$.

step 2. Set $P_f^{(0)} := I(P_c^{n+1})$, where $I(P_c)$ is the interpolant of P_c on the fine grid, perform *only one* Newton's iteration (3.2) on the fine mesh $\overline{\omega}_f$ and get P_f^{n+1}.

Further, in order to accelerate the computational process we propose the following space-time two-grid algorithm. Let us define two time steps - a coarse $\triangle t^c$ ($n\triangle t^c = T$) step and a fine $\triangle t^f$ step such that $\triangle t^c \gg \triangle t^f$ and $\triangle t^c = j\triangle t^f$ (or $nj\triangle t^f = T$), where j is a positive integer.

Algorithm 2 (**A2**) At each time level $n = 0, 1, \dots$ we perform the two steps:
step 1. The same as step 1 in Algorithm 1, $\triangle t := \triangle t^c$.

step 2. For $l = 0, \dots, j-1$ on the fine space mesh $\overline{\omega}_f$ with fine time step $\triangle t^f$, compute P_f^{n+1}, solving the difference scheme:

$$\begin{aligned} &\frac{P_{f,i}^{l+1} - P_{f,i}^l}{\triangle t^f} - \frac{\sigma^2 S_i^2}{2}(P_{f_{SS}})_i^{l+1} \\ &-rS_i[\chi(P_{f_{\hat{S}}})_i^{l+1} + (1-\chi)(P_{f_S})_i^{l+1}] + rP_{f,i}^{l+1} = g(S_i, P^*), \quad i = 2, \dots, m_f - 1, \end{aligned} \tag{3.3}$$
$$P(0, t_l^f) = K, \quad P(S_N, t_l^f) = 0, \quad P(S_i, 0) = V^*(S_i), \; i = 1, \dots, m_f,$$
$$t_l^f = n\triangle t^c + (l+1)\triangle t^f, \qquad P_f^0 := I(P_c^n), \quad P^* = \begin{cases} I(P_c^{l+1}), & l = j-1, \\ P_{f,i}^l, & l < j-1. \end{cases}$$

Here, at step 2, we advance in time (between two coarse levels) by a fine step $\triangle t^f$, solving the semi-implicit scheme (3.3) which is monotone under an additional step-size condition $\triangle t^f \leq \varepsilon/(rK)$, cf. also Nielsen et al. [11]. This scheme is, however, very cheap in terms of computational resources because of the applied LU decomposition of the system matrix prior to the time loop. In contrast to **A1**, *using* ***A2*** *we perform Newton iterations only on the coarse time levels.*

4. Numerical experiments. In this section we verify the experimental convergence rate and computational efficiency of the presented space and space-time two-grid methods, **A1** and **A2**, respectively.

4.1. The space-time grid partition. The numerical experiments are first computed on a uniform space grid as a benchmark test. Further, we consider a smooth nonuniform grid, cf. in 't Hout et al. [5] - uniform inside $[S_r, S_l] = [K/2, 3K/2]$, which is the region of interest for practitioners where *the at-the-money, at-the-strike, option value and the free boundary are located*, and nonuniform outside with stretching parameter $c = K/10$:

$$S_i := \phi(\xi_i) = \begin{cases} S_l + c\sinh(\xi_i), & \xi_{\min} \leq \xi_i < 0, \\ S_l + c\xi_i, & 0 \leq \xi_i \leq \xi_{\text{int}}, \\ S_r + c\sinh(\xi_i - \xi_{\text{int}}), & \xi_{\text{int}} \leq \xi_i < \xi_{\max}. \end{cases} \tag{4.1}$$

The uniform partition of $[\xi_{\min}, \xi_{\max}]$ is defined trough $\xi_{\min} = \xi_0 < \ldots < \xi_N = \xi_{\max}$:

$$\xi_{\min} = \sinh^{-1}\left(\frac{-S_l}{c}\right), \quad \xi_{\text{int}} = \frac{S_r - S_l}{c}, \quad \xi_{\max} = \xi_{\text{int}} + \sinh^{-1}\left(\frac{S_{\max} - S_r}{c}\right).$$

In view of Theorems 1 and 2 the likely local consistency is $O(\triangle t + |h^f|^2 + |h^c|^4)$, $|h| = \max_i h_i$ at the grid nodes in the region of interest, where the central stencil is used for first derivative approximation. Let us note that the first order upwinding is applied only for very small values of S where the problem is convection-dominated. Thus, in order to verify this assertion and to obtain the optimal accuracy, it is naturally to choose $\triangle t = \min_i (h_i^f)^2$ and $m_f = (m_c - 1)^2/S_{\max} + 1$ (i.e. $h^f = (h^c)^2$ in the case of uniform mesh) when computing with **A1**. Further, we set $\triangle t = T/\lceil T/(\min_i h_i^f)^2\rceil$, where $\lceil u\rceil$ is the smallest integer greater than or equal to u in order to get exactly the desired final time T and avoiding time interpolation.

As for **A2**, again referring to Theorems 1 and 2, we further expect second order consistency in time with respect to the coarse time grid i.e. $O((\triangle t^c)^2 + \triangle t^f + |h^f|^2 + |h^c|^4)$. Thus, the optimal accuracy will be obtained if we set $\triangle t^f = (\triangle t^c)^2$, $\triangle t^c = \min_i (h^c)^2$ and $m_f = (m_c - 1)^2/S_{\max} + 1$. Since we have to get embedded coarse-fine meshes in time and also the final time T we select $\triangle t^c = T/\lceil T/(\min_i h_i^c)^2\rceil$, $\triangle t^f = \triangle t^c/j$, where $j = \max\{\lceil \triangle t^c/(\min_i h_i^f)^2\rceil, \lceil rK\triangle t^c/\epsilon\rceil\}$, respecting the stability condition for the semi-implicit scheme.

4.2. Computational results for (1.1)-(1.3). We consider pricing an American put option with strike $K = 50$, maturity $T = 1$ and interest rate $r = 0.1$ where the constant in the interior penalty term is selected as $\mathrm{C} = rK$. The tolerance of the Newton's iterative scheme is $\mathsf{tol} = 1.e - 8$ and for the coarse-fine grid transition we use the shape-preserving piecewise cubic interpolation of the solution and in the opposite direction - from fine to coarse - we use linear interpolation.

We list the option values at-the-money i.e. at $S = K$ and time-to-maturity T (corresponding to $t = 0$ in the backward problem (1.1)-(1.3)). Here diff stands for the difference in the value $P(K, T)$ from the previous grid refinement level and therefore for the numerical convergence rate (CR) we use the double-mesh principle. These are computed when solving numerically (1.1),(1.3) for fixed ϵ which is an approximation of the LCP and thus penalization error is present.

Example 1. (*moderate volatility*) Let $\sigma = 0.2$, $\epsilon = 0.001$ and the far field boundary location $S_{\max} = 100$. First, we compute the numerical solution of (1.1)-(1.3) with the one-grid Newton process (step 1 of **A1**, **A2**) on the uniform and the smooth non-uniform spatial grid (4.1). The results are provided in Table 4.1.

Table 4.1
One-grid computations, Example 1

	Uniform mesh, $\triangle t \sim h^2$				Non-uniform mesh, $\triangle t \sim \min_i \{h_i^2\}$			
m_c	$P_c(K,1)$	diff	CR	CPU	$P_c(K,1)$	diff	CR	CPU
161	2.26865			0.02	2.32293			0.02
321	2.36986	-1.01e-1		0.03	2.38820	-6.53e-2		0.05
641	2.40003	-3.02e-2	1.75	0.11	2.40545	-1.72e-2	1.92	0.13
1281	2.40882	-8.79e-3	1.78	0.48	2.41024	-4.79e-3	1.85	0.61
2561	2.41113	-2.32e-3	1.92	3.02	2.41151	-1.26e-3	1.92	3.35
5121	2.41174	-6.05e-4	1.94	16.99	2.41183	-3.29e-4	1.94	21.17
10241	2.41189	-1.57e-4	1.95	135.41	2.41192	-8.48e-5	1.96	181.01
20481	2.41193	-4.01e-5	1.97	1150.01	2.41194	-2.16e-5	1.97	1462.04

We observe that the rate of convergence at $S = K$ is about 2. For the same number of spatial grid nodes the accuracy of the scheme on the nonuniform grid (4.1) is better than the one on the uniform one. On the other side the CPU time is more than the corresponding uniform one since, because of the choice of the time step (dependent on $\min_i h_i$), we solve on much more fine grid in time. Nevertheless, we observe the better computational performance with the nonuniform grid (4.1) - we get better accuracy for much smaller CPU time in comparison with the uniform grid, compare the option values on the uniform grid with nodes $m_c = 20481$ in Table 4.1 and for $m_c = 10241$ on the nonuniform grid (4.1).

In Table 4.2 we present the computational results with our algorithms **A1** and **A2** on the considered uniform and nonuniform grid. We observe that the space-time two-grid algorithm **A2** is more efficient than the space two-grid approach **A1** if $\min_i(h_i^f)^2 < \epsilon/(rK)$, i.e. on fine grids. The computational efficiency of the two-grid technique is noticeable as we compare the last line of Table 4.1 with last two lines in Table 4.2 since we get the same precision as one-grid computations but spending much less computational resources.

Table 4.2
Two-grid computations, Example 1

	Uniform mesh				Non-uniform mesh			
m_c	$P_f(K,1)$	diff	CR	CPU	$P_f(K,1)$	diff	CR	CPU
161(**A1**)	2.34701			0.05	2.37606			0.18
321(**A1**)	2.40710	-6.01e-2		0.32	2.40933	-3.33e-2		0.73
641(**A1**)	2.41162	-4.52e-3	3.73	10.15	2.41177	-2.45e-3	3.77	16.30
641(**A2**)					2.41185			24.49
1281(**A1**)	2.41193	-3.07e-4	3.88	493.02	2.41194	-1.65e-4	3.89	678.81
1281(**A2**)	2.41193			45.88	2.41194			85.59

In order to check the rate of convergence of **A2** on the non-uniform grid (4.1), we set $\triangle t^c = \min_i(h_i^c)^2$, $m_f = (m_c - 1)^2/S_{\max} + 1$ and fix $\triangle t^f \sim 5.e - 6$ for all computations. The results are listed in Table 4.3.

TABLE 4.3
Two-grid algorithm ***A2*** *on the grid (4.1), Example 1*

m_c	$P_f(K,1)$	diff	CR	CPU
161	2.4116149			9.90
321	2.4119249	-3.09994e-4		20.17
641	2.4119445	-1.95796e-5	3.9855	79.48
1281	2.4119457	-1.21910e-6	4.0048	333.59

The rate of convergence of the two-grid algorithms is fourth order on the coarse space at the strike $S = K$. We infer that these are much more efficient than the one-grid procedure and we observe best performance of **A2** on fine grids.

On Figures 4.1, 4.2 we visualize, respectively, the numerical option price and the hedge factor Delta $= P_{\mathring{S}}(S,t)$, computed by **A1** with $m_c = 320$, $m_f = 1025$ and $\triangle t \sim \min_i(h_i^f)^2$ at (time-to-maturity) $t = 0$ (at expiry i.e. the payoff) and at $t = T$.

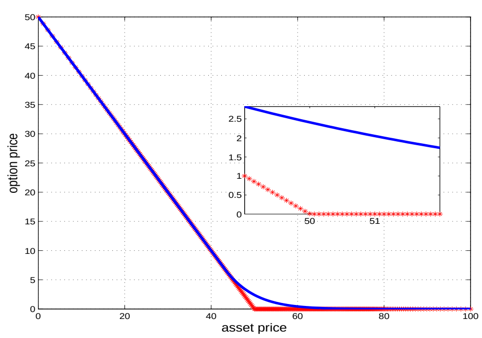

FIG. 4.1. *Option price for* $\sigma = 0.2$

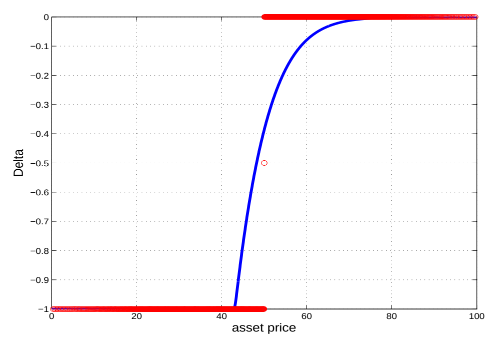

FIG. 4.2. *Delta of the option for* $\sigma = 0.2$

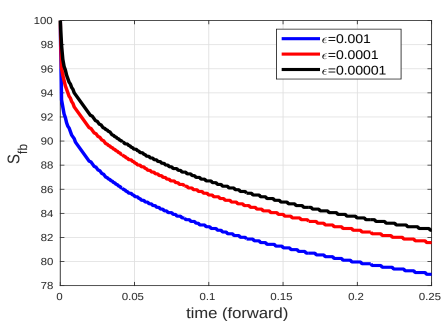

FIG. 4.3. *Free boundary for* $\sigma = 0.3$

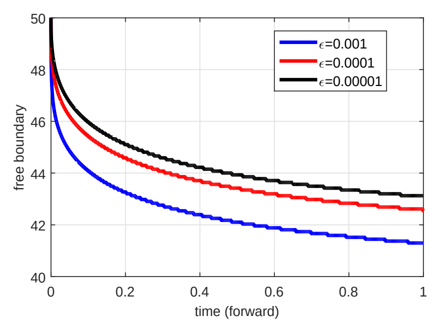

FIG. 4.4. *Free boundary for* $\sigma = 0.2$

The time evolution of the free boundary S_{fb}, separating the continuation and stopping regions, is shown on Fig. 4.3 for various ϵ. For $\epsilon = 0.00001$ we observe a plot, analogous to the one computed with the PSOR method, see Ševčovič et al. [9], and convergent behaviour for $\epsilon \to 0$ since the penalized equation (1.1),(1.3) approximates the LCP in the penalty parameter. Fig. 4.4 visualizes the free boundary for $\sigma = 0.2$.

Example 2. (*small volatility*) Here we set $\sigma = 0.05$ and $S_{\max} = 100$ in order to compute the rate of convergence of the two-grid approach. In Table 4.4 we present the computational results with **A2** for $\triangle t^c = \min_i (h_i^c)^2$, $m_f = (m_c - 1)^2/S_{\max} + 1$ and $\triangle t^f = 5.e - 6$ on the non-uniform grid (4.1) for different values of ϵ.

TABLE 4.4
Two-grid computations with ***A2*** *for various* ϵ*, Example 2*

	$\epsilon = 0.01$			$\epsilon = 0.001$			$\epsilon = 0.0005$		
m_c	$P_c(K,1)$	diff	CR	$P_c(K,1)$	diff	CR	$P_c(K,1)$	diff	CR
161	0.25949			0.22710			0.22508		
321	0.26438	-4.89e-3		0.23167	-4.51e-3		0.22941	-4.34e-3	
641	0.26468	-2.99e-4	4.02	0.23202	-3.52e-4	3.70	0.22975	-3.38e-4	3.68
1281	0.26470	-1.87e-5	4.00	0.23204	-2.08e-5	4.08	0.22977	-2.10e-5	4.01

On Figures 4.5, 4.6 we visualize the option value at $T = 1$, the payoff and the corresponding Delta. These are computed with the two-grid space-time algorithm **A2** for $m^c = 320$ and $\epsilon = 0.001$.

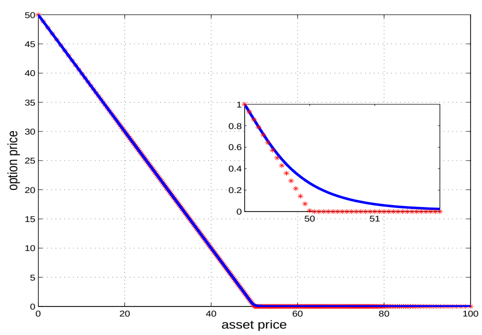

FIG. 4.5. *Option price for* $\sigma = 0.05$

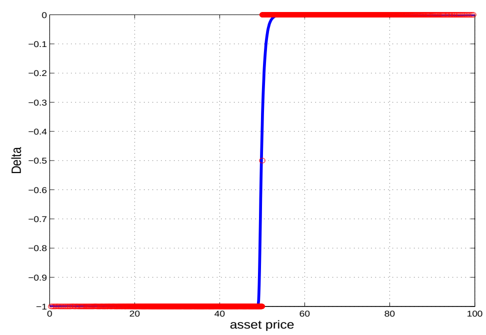

FIG. 4.6. *Delta of the option for* $\sigma = 0.05$

Example 3. (*substantial volatility*) Let us now consider the model problem (1.1)-(1.3) for $\sigma = 0.8$ with far field boundary location $S_{\max} = 400$. We compare the efficiency of the presented two-grid algorithms and simultaneously verify the rate of convergence. To this aim the relation between grid parameters for **A2** are the same as in Example 2. The results for $\epsilon = 0.001$ on the non-uniform grid (4.1) are given in Table 4.5 as we observe the superior performance of **A2** in comparison with **A1**.

TABLE 4.5
Two-grid computations, Example 3

	A1				A2			
m_c	$P_c(K,1)$	diff	CR	CPU	$P_c(K,1)$	diff	CR	CPU
400	13.02472			0.39	*13.14786*			*12.27*
800	13.13939	-1.15e-1		1.68	13.14807	-2.12e-4		28.29
1600	*13.14753*	-8.14e-3	3.82	*41.57*	13.148088	-1.33e-5	4.00	116.54
3200	13.14807	-5.38e-4	3.92	1961.88	13.148089	-8.88e-7	3.90	534.47

Finally, let us note that all experiments were done with Matlab 2013a. We focus on the computational efficiency of the presented two-grid algorithms for the American option pricing problem and we do not discuss the interior penalty method in detail.

Conclusion. In this paper we realize two-grid Newton algorithms for pricing American options with the interior penalty method. We infer the superior computational efficiency of the considered methods over the standard one-grid computations. Because of the suggested practical importance of this approach we consider future improvements and extensions as well as deriving rigorous theoretical estimates.

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