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DISCONTINUOUS GALERKIN METHOD FOR THE FLUID-STRUCTURE INTERACTION*

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Abstract. The subject of the paper is the brief description of the stability analysis of the space-time discontinuous Galerkin method for the numerical solution of a model nonlinear parabolic problem in a time dependent domain. The analyzed method is used for the solution of compressible flow in a time-dependent domain and interacted with an elastic body, applied to the simulation of vocal fold flow-induced vibrations.

Key words. time-dependent domain, ALE formulation, ALE space-time discontinuous Galerkin method, stability, interaction of compressible flow with elastic structure polynomial

AMS subject classifications. 65M60, 76M10

1. Introduction. Most of the results on the solvability and numerical analysis of nonstationary partial differential equations (PDEs) are obtained under the assumption that a space domain is independent of time. However, in practice it is necessary to work out an accurate, efficient and robust, theoretically based numerical method for the solution of nonlinear initial boundary value problems in time dependent domains. A typical example is the solution of the interaction between compressible flow described by the Navier-Stokes equations and an elastic body.

The subject of this paper is the description of the analysis of a simplified model of compressible flow represented by a nonlinear parabolic problem in a time dependent domain solved by the space-time discontinuous Galerkin method (STDGM) and then applied to the interaction of compressible flow and dynamic elasticity.

2. Formulation of a model problem. Let $d = 2$ or 3. We consider the following scalar initial boundary value problem in a time-dependent bounded domain $\Omega_t \subset \mathbb{R}^d$, where $t \in [0,T]$ and $T > 0$: Find a function $u = u(x,t)$ with $x \in \Omega_t$, $t \in (0,T)$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \mathrm{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\, t \in (0,T), \tag{2.1}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\, t \in (0,T), \tag{2.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{2.3}$$

We assume that $f_s \in C^1(\mathbb{R}),\ f_s(0) = 0$,

$$|f_s'| \le L_f, \quad s = 1,\dots,d, \tag{2.4}$$

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$$\beta : I\!R \to [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty, \tag{2.5}$$

$$|\beta(u_1) - \beta(u_2)| \le L_\beta |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}. \tag{2.6}$$

Condition (2.6) completes the problem, but it is not necessary for the stability analysis of the numerical method. The convection terms $f_s(u)$ correspond to the Euler inviscid fluxes $\boldsymbol{f}_s$ in (5.3) in the Navier-Stokes system and the diffusion expression $\beta(u)\nabla u$ represents a simplification of the viscous terms $\boldsymbol{R}_s$ in (5.4).

3. ALE-space time DG discretization. We consider a partition $0 = t_0 < t_1 < \cdots < t_M = T$ and set $\tau_m = t_m - t_{m-1}$, $I_m = (t_{m-1}, t_m)$, $\overline{I}_m = [t_{m-1}, t_m]$ for $m = 1, \ldots, M$, $\tau = \max_{m=1,\ldots,M} \tau_m$.

The space-time discontinuous Galerkin method has an advantage that on every time interval $\overline{I}_m$ it is possible to consider a different space partition, i. e. triangulation with standard properties. It allows us to consider an ALE mapping

$$\mathcal{A}_{h,t}^{m-1} : \overline{\Omega}_{t_{m-1}} \xrightarrow{\text{onto}} \overline{\Omega}_t \text{ for } t \in [t_{m-1}, t_m),\ h \in (0, \overline{h}). \tag{3.1}$$

separately on each time interval $[t_{m-1}, t_m)$ for $m = 1, \ldots, M$ and the resulting ALE mapping in $[0, T]$ may be discontinuous at time instants t_m, $m = 1, \ldots, M-1$. We assume that $\mathcal{A}_{h,t}^{m-1}$ is in space a piecewise affine mapping on a triangulation $\hat{\mathcal{T}}_{h,t_{m-1}}$, continuous in space variable $X \in \Omega_{t_{m-1}}$ and continuously differentiable in time $t \in [t_{m-1}, t_m)$ and $\mathcal{A}_{h,t_{m-1}}^{m-1} = \mathrm{Id}$ (identical mapping). Hence, we assume that all domains Ω_t are polygonal (polyhedral). We also use the simple notation $\mathcal{A}_t(X)$. For every $t \in [t_{m-1}, t_m)$ we define the conforming triangulation

$$\mathcal{T}_{h,t} = \left\{ K = \mathcal{A}_{h,t}^{m-1}(\hat{K});\ \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \right\} \text{ in } \Omega_t. \tag{3.2}$$

We define the domain velocity $\tilde{\boldsymbol{z}}(\boldsymbol{X}, t) = \frac{\partial}{\partial t}\mathcal{A}_t(\boldsymbol{X})$, $\quad t \in [0, T]$, $\boldsymbol{X} \in \Omega_0$, $\boldsymbol{z}(\boldsymbol{x}, t) = \tilde{\boldsymbol{z}}(\mathcal{A}_t^{-1}(\boldsymbol{x}), t)$, $t \in [0, T]$, $\boldsymbol{x} \in \Omega_t$.

By $\mathcal{F}_{h,t}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{h,t}$. It consists of the set of all inner faces $\mathcal{F}_{h,t}^I$ and the set of all boundary faces $\mathcal{F}_{h,t}^B$: $\mathcal{F}_{h,t} = \mathcal{F}_{h,t}^I \cup \mathcal{F}_{h,t}^B$. Each $\Gamma \in \mathcal{F}_{h,t}$ will be associated with a unit normal vector $\boldsymbol{n}_\Gamma$. By $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)} \in \mathcal{T}_{h,t}$ we denote the elements adjacent to the face $\Gamma \in \mathcal{F}_{h,t}^I$. Moreover, for $\Gamma \in \mathcal{F}_{h,t}^B$ the element adjacent to this face will be denoted by $K_\Gamma^{(L)}$. We shall use the convention, that $\boldsymbol{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$.

If $v \in H^1(\Omega_t, \mathcal{T}_{h,t}) := \{v;\ v|_K \in H^1(K)\ \forall K \in \mathcal{T}_{h,t}\}$ and $\Gamma \in \mathcal{F}_{h,t}$, then $v_\Gamma^{(L)}$ and $v_\Gamma^{(R)}$ will denote the traces of v on Γ from the side of elements $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$, respectively. We set $h_K = \operatorname{diam} K$ for $K \in \mathcal{T}_{h,t}$, $h(\Gamma) = \operatorname{diam} \Gamma$ for $\Gamma \in \mathcal{F}_{h,t}$ and $\langle v \rangle_\Gamma = \frac{1}{2}\left(v_\Gamma^{(L)} + v_\Gamma^{(R)}\right)$, $[v]_\Gamma = v_\Gamma^{(L)} - v_\Gamma^{(R)}$, for $\Gamma \in \mathcal{F}_{h,t}^I$. Moreover, by ρ_K we denote the diameter of the largest ball inscribed into $K \in \mathcal{T}_{h,t}$. For a set $\omega \in I\!R$ and functions $a, b : \omega \to I\!R$ we put $(a, b)_\omega = \int_\omega ab \,\mathrm{d}x$.

Further, let p, $q \ge 1$ be integers. We introduce discrete function spaces. For every $m = 1, \ldots, M$ we consider the space

$$S_h^{p,m-1} = \left\{ \varphi \in L^2(\Omega_{t_{m-1}});\ \varphi|_{\hat{K}} \in P^p(\hat{K})\ \forall \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \right\}, \tag{3.3}$$

where $p \ge 1$ is an integer and $P^p(\hat{K})$ is the space of all polynomials on $\hat{K}$ of degree $\le p$.

We set

$$S_{h,\tau}^{p,q} = \Big\{\varphi;\ \varphi\left(\mathcal{A}_{h,t}^{m-1}(X), t\right) = \sum_{i=0}^{q} \vartheta_i(X)\, t^i, \quad \vartheta_i \in S_h^{p,m-1},$$
$$X \in \Omega_{t_{m-1}},\ t \in \overline{I}_m,\ m = 1, \dots, M\Big\}.$$

Now we introduce the following discrete forms:

Diffusion form

$$a_h(u,\varphi,t) := \sum_{K\in\mathcal{T}_{h,t}} \int_K \beta(u)\nabla u \cdot \nabla\varphi \,\mathrm{d}x \tag{3.4}$$
$$- \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \left(\langle\beta(u)\nabla u\rangle \cdot \boldsymbol{n}_\Gamma\,[\varphi] + \theta\,\langle\beta(u)\nabla\varphi\rangle \cdot \boldsymbol{n}_\Gamma\,[u]\right) \mathrm{d}S$$
$$- \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma \left(\beta(u)\nabla u \cdot \boldsymbol{n}_\Gamma\,\varphi + \theta\beta(u)\nabla\varphi \cdot \boldsymbol{n}_\Gamma\,u - \theta\beta(u)\nabla\varphi \cdot \boldsymbol{n}_\Gamma\,u_D\right) \mathrm{d}S,$$
$$\theta = -1,\ 0,\ 1$$

Interior and boundary penalty

$$J_h(u,\varphi,t) := c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^I} h(\Gamma)^{-1} \int_\Gamma [u]\,[\varphi]\,\mathrm{d}S + c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u\,\varphi\,\mathrm{d}S, \tag{3.5}$$

$$A_h(u,\varphi,t) = a_h(u,\varphi,t) + \beta_0\, J_h(u,\varphi,t), \tag{3.6}$$

Convection forms

$$b_h(u,\varphi,t) := -\sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} f_s(u) \frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \tag{3.7}$$
$$+ \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)\,[\varphi]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \boldsymbol{n}_\Gamma)\,\varphi\,\mathrm{d}S,$$

$$d_h(u,\varphi,t) := -\sum_{K\in\mathcal{T}_{h,t}} \int_K (\boldsymbol{z}\cdot\nabla u)\varphi\,\mathrm{d}x, \tag{3.8}$$

Right-hand side form

$$\ell_h(\varphi,t) := \sum_{K\in\mathcal{T}_{h,t}} \int_K g\varphi\,\mathrm{d}x + \beta_0\, c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma u_D\,\varphi\,\mathrm{d}S. \tag{3.9}$$

A suitable choice of the constant c_W plays a role in the proof of (4.6). The ALE time derivative $D_t U$ of a function $U = U(x,t)$ for $x \in \Omega_t$ and $t \in [0,T]$ is defined by the relations $D_t U(x,t) = \frac{\partial\tilde{U}}{\partial t}(X,t)$, $\quad \tilde{U}(X,t) = U(\mathcal{A}_t(X),t)$, $X \in \Omega_{t_{m-1}}$, and $x = \mathcal{A}_t(X) \in \Omega_t$.

In (3.7), H is a numerical flux with the following properties:

(H1) $H(u,v,\boldsymbol{n})$ is defined in $I\!R^d \times B_1$, where $B_1 = \{\boldsymbol{n} \in \mathbb{R}^d;\, |\boldsymbol{n}| = 1\}$, and is Lipschitz-continuous with respect to u, v.

(H2) H is consistent: $H(u,u,\boldsymbol{n}) = \sum_{s=1}^{d} f_s(u) n_s, \quad u \in \mathbb{R},\ \boldsymbol{n} \in B_1$,

(H3) H is conservative: $H(u,v,\boldsymbol{n}) = -H(v,u,-\boldsymbol{n}), \quad u,v \in \mathbb{R},\ \boldsymbol{n} \in B_1$.

In what follows, in the stability analysis the property **(H2)** is used. (Assumptions **(H1)** and **(H3)** are important for error estimation, but here it is not necessary.)

For a function φ defined in $\bigcup_{m=1}^{M} I_m$ we denote

$$\varphi_m^{\pm} = \varphi(t_m\pm) = \lim_{t\to t_m\pm} \varphi(t), \quad \{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-), \tag{3.10}$$

if the one-sided limits $\varphi_m^{\pm}$ exist.

Now we define the ALE-STDG approximate solution. A function U is an approximate solution of problem (2.1)–(2.3), if $U \in S_{h,\tau}^{p,q}$ and

$$\int_{I_m} \big((D_t U, \varphi)_{\Omega_t} + A_h(U,\varphi,t) + b_h(U,\varphi,t) + d_h(U,\varphi,t)\big)\,\mathrm{d}t \tag{3.11}$$

$$+(\{U\}_{m-1}, \varphi_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} \ell_h(\varphi,t)\,\mathrm{d}t \quad \forall \varphi \in S_{h,\tau}^{p,q},$$

$$m = 1, \dots, M,$$

$$U_0^- \in S_h^{p,0}, \quad (U_0^- - u^0, v_h) = 0 \quad \forall v_h \in S_h^{p,0}. \tag{3.12}$$

We can mention that the expression $(\{U\}_{m-1}, \varphi_{m-1}^{+})_{\Omega_{t_{m-1}}}$ sticks the scheme on intervals I_{m-1} and I_m. It is important that identity (3.11) is satisfied by the exact solution u.

Now we shall briefly describe the stability analysis of the ALE-STDG scheme.

4. Analysis of the stability. The stability of the ALE-STDGM was analyzed in [1] and [2] in the simplified situation, when $q = 1$. Here we are concerned with a general case when $q > 0$ is arbitrary. We assume the boundedness of the Jacobian matrices and Jacobians of the mappings $\mathcal{A}_{h,t}^{m-1}$ and $(\mathcal{A}_{h,t}^{m-1})^{-1}(x)$. Moreover, we assume that there exists a constant $c_z > 0$ such that $|\boldsymbol{z}(x,t)|,\ |\mathrm{div}\boldsymbol{z}(x,t)| \le c_z$ for $x \in \Omega_t,\ t \in (0,T)$.

Important relations are multiplicative trace inequality and the inverse inequality hold: There exist constants $c_M, c_I > 0$ independent of v, h, t and K such that

$$\|v\|_{L^2(\partial K)}^2 \le c_M \left(\|v\|_{L^2(K)}\, |v|_{H^1(K)} + h_K^{-1} \|v\|_{L^2(K)}^2\right), \tag{4.1}$$

$$v \in H^1(K),\ K \in \mathcal{T}_{h,t},\ h \in (0,\overline{h}),\ t \in [0,T],$$

$$|v|_{H^1(K)} \le c_I\, h_K^{-1} \|v\|_{L^2(K)}, \quad v \in P^p(K),\ K \in \mathcal{T}_{h,t},\ h \in (0,\overline{h}),\ t \in [0,T]. \tag{4.2}$$

We introduce the notation of norms: $\|\cdot\|_{\Omega_t}$ - $L^2(\Omega_t)$ norm,

$$\|\varphi\|_{DG,t} = \Big(\sum_{K\in\mathcal{T}_{h,t}} |\varphi|_{H^1(K)}^2 + J_h(\varphi,\varphi,t)\Big)^{1/2}. \tag{4.3}$$

$$\|u_D\|_{DGB,t} = \Big(c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma |u_D|^2\,\mathrm{d}S\Big)^{1/2} \tag{4.4}$$

If we use $\varphi := U$ as a test function in (3.11), we get the basic identity

$$\int_{I_m} \big((D_t U, U)_{\Omega_t} + A_h(U,U,t) + b_h(U,U,t) + d_h(U,U,t)\big)\,\mathrm{d}t \tag{4.5}$$

$$+(\{U\}_{m-1}, U_{m-1}^{+})_{\Omega_{t_{m-1}}} = \int_{I_m} \ell_h(U,t)\,\mathrm{d}t.$$

4.1. Important estimates. The analysis is based on the following inequalities valid for suitable values of c_W for numbers $\theta = -1,\ 0,\ 1$:

$$\int_{I_m} (a_h(U,U,t) + \beta_0\, J_h(U,U,t))\, \mathrm{d}t \qquad (4.6)$$
$$\geq \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,t}\, \mathrm{d}t - \frac{\beta_0}{2} \int_{I_m} \|u_D\|^2_{DGB,t}\, \mathrm{d}t.$$

For each $k_1, k_2, k_3 > 0$ there exists a constant $c_b, c_d > 0$ such that we have

$$\int_{I_m} |b_h(U,U,t)| \mathrm{d}t \leq \frac{\beta_0}{2k_1} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t + c_b \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t, \qquad (4.7)$$

$$\int_{I_m} |d_h(U,U,t)|\, \mathrm{d}t \leq \frac{\beta_0}{2k_2} \int_{I_m} \|U\|^2_{DG,t}\, \mathrm{d}t + \frac{c_d}{2\beta_0} \int_{I_m} \|U\|^2_{\Omega_t}\, \mathrm{d}t, \qquad (4.8)$$

$$\int_{I_m} |\ell_h(U,t)|\, \mathrm{d}t \leq \frac{1}{2} \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}\right)\, \mathrm{d}t \qquad (4.9)$$
$$+ \frac{\beta_0 k_3}{2} \int_{I_m} \|u_D\|^2_{DGB,t}\, \mathrm{d}t + \frac{\beta_0}{2k_3} \int_{I_m} \|U\|^2_{DG,t}\, \mathrm{d}t,$$

$$\int_{I_m} (D_t U, U)_{\Omega_t}\, \mathrm{d}t + \left(\{U\}_{m-1}, U^+_{m-1}\right)_{\Omega_{t_{m-1}}} \qquad (4.10)$$
$$\geq \frac{1}{2}\|U^-_m\|^2_{\Omega_{t_m}} + \frac{1}{2}\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t - \left(U^-_{m-1}, U^+_{m-1}\right)_{\Omega_{t_{m-1}}}.$$

There exists a constant $C_{T2} > 0$ such that

$$\|U^-_m\|^2_{\Omega_{t_m}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} \qquad (4.11)$$
$$+ \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{DG,t} \mathrm{d}t$$
$$\leq C_{T2} \left(\int_{I_m} \|g\|^2_{\Omega_t} \mathrm{d}t + \int_{I_m} \|u_D\|^2_{DGB,t} \mathrm{d}t + \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \right).$$

Now we have to estimate the following problematic term $\int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t$. To this end we introduce the discrete characteristic function. For $m = 1, \ldots, M$ we use the following notation: $U = U(x,t)$, $x \in \Omega_t$, $t \in I_m$, $\tilde{U} = \tilde{U}(X,t) = U(\mathcal{A}_t(X), t)$, $X \in \Omega_{t_{m-1}}$ $t \in I_m$ - the approximate solution transformed to the reference domain $\Omega_{t_{m-1}}$. For $s \in I_m$ we denote $\tilde{\mathcal{U}}_s = \tilde{\mathcal{U}}_s(X,t)$, $X \in \Omega_{t_{m-1}}$, $t \in I_m$ - the discrete characteristic function to $\tilde{U}$ at a point $s \in I_m$. It is defined as $\tilde{\mathcal{U}}_s \in P^q(I_m; S_h^{p,m-1})$ such that

$$\int_{I_m} (\tilde{\mathcal{U}}_s, \varphi)_{\Omega_{t_{m-1}}}\, \mathrm{d}t = \int_{t_{m-1}}^{s} (\tilde{U}, \varphi)_{\Omega_{t_{m-1}}}\, \mathrm{d}t \qquad (4.12)$$
$$\forall \varphi \in P^{q-1}(I_m; S_h^{p,m-1}),$$
$$\tilde{\mathcal{U}}_s(X, t^+_{m-1}) = \tilde{U}(X, t^+_{m-1}),\ X \in \Omega_{t_{m-1}}. \qquad (4.13)$$

The existence and uniqueness of the discrete characteristic function $\tilde{U}(X,t)$ is proved in the monograph [5].

On the basis of complicated estimates, the following estimate is proved: There exists a constant $C_{T4} > 0$ independent of h and τ such that

$$\int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \leq C_{T4} \tau_m \left(\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \int_{I_m} (\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}) \mathrm{d}t \right). \qquad (4.14)$$

Now, if (4.14) is substituted into (4.11) and the discrete Gronwall inequality is applied, we obtain the unconditional stability of the ALE-STDGM:

There exists a constant $C_1^* > 0$ independent of h, τ, m such that

$$\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^m \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|U\|^2_{DG,t}\,\mathrm{d}t \tag{4.15}$$
$$\leq C_1^* \left(\|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_{I_j} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\mathrm{d}t \right),$$
$$m = 1, \dots, M,\ h \in (0, \overline{h}).$$

In this estimate the left-hand side depending on norms of the approximate solution is bounded by expressions dependent on data. This means that the ALE-STDGM is stable.

5. Application of the STDGM to flow-induced vocal folds vibration. The ALE-STDGM is applied to the numerical study of vocal folds vibrations excited by airflow in a vocal tract model, see Figures 6.1 and 6.2.

In the vocal tract Ω_t the air flow is considered. It is describe by the ALE form of the compressible Navier-Stokes system:

$$\frac{D^A \boldsymbol{w}}{Dt} + \sum_{s=1}^2 \frac{\partial \boldsymbol{g}_s(\boldsymbol{w})}{\partial x_s} + \boldsymbol{w}\,\mathrm{div}\boldsymbol{z} = \sum_{s=1}^2 \frac{\partial \boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w})}{\partial x_s}, \tag{5.1}$$

where

$$\boldsymbol{w} = (w_1, \dots, w_4)^T = (\rho, \rho v_1, \rho v_2, E)^T \in I\!R^4, \tag{5.2}$$
$$\boldsymbol{g}_s(\boldsymbol{w}) = \boldsymbol{f}_s(\boldsymbol{w}) - z_s \boldsymbol{w},$$
$$\boldsymbol{f}_s(\boldsymbol{w}) = \left(\rho v_s, \rho v_1 v_s + \delta_{1s}\, p, \rho v_2 v_s + \delta_{2s}\, p, (E+p) v_s\right)^T, \tag{5.3}$$
$$\boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w}) = \left(0, \tau^V_{s1}, \tau^V_{s2}, \tau^V_{s1} v_1 + \tau^V_{s2} v_2 + k\partial\Theta/\partial x_s\right)^T,$$
$$\boldsymbol{R}_s(\boldsymbol{w}, \nabla \boldsymbol{w}) = \sum_{k=1}^2 \boldsymbol{K}_{sk}(\boldsymbol{w}) \frac{\partial \boldsymbol{w}}{\partial x_k}, \quad \boldsymbol{f}_s(\boldsymbol{w}) = \frac{D\boldsymbol{f}_s(\boldsymbol{w})}{D\boldsymbol{w}}\boldsymbol{w}, \tag{5.4}$$
$$\tau^V_{ij} = \lambda\,\mathrm{div}\boldsymbol{v}\,\delta_{ij} + 2\mu\, d_{ij}(\boldsymbol{v}),\ d_{ij}(\boldsymbol{v}) = \left(\partial v_i/\partial x_j + \partial v_j/\partial x_i\right)/2. \tag{5.5}$$

It is completed by the relations $p = (\gamma - 1)(E - \rho|\boldsymbol{v}|^2/2), \Theta = \left(E/\rho - |\boldsymbol{v}|^2/2\right)/c_v$, initial condition $\boldsymbol{w}(\boldsymbol{x}, 0) = \boldsymbol{w}^0(\boldsymbol{x}), \boldsymbol{x} \in \Omega_0$ and boundary conditions: we write $\partial\Omega_t = \Gamma_I \cup \Gamma_O \cup \Gamma_{W_t}$, where Γ_I is inlet, Γ_O outlet and Γ_{W_t} is an impermiable wall. We assume that $\rho|_{\Gamma_I \times (0,T)} = \rho_D, \boldsymbol{v}|_{\Gamma_I \times (0,T)} = \boldsymbol{v}_D = (v_{D1}, v_{D2})^T, \sum_{j=1}^2 \left(\sum_{i=1}^2 \tau^V_{ij} n_i\right) v_j + k\frac{\partial\Theta}{\partial \boldsymbol{n}} = 0$ on $\Gamma_I \times (0,T)$, $\boldsymbol{v}_{\Gamma_{W_t}} = \boldsymbol{z}$, $\frac{\partial\Theta}{\partial \boldsymbol{n}} = 0; \sum_{i=1}^2 \tau^V_{ij} n_i = 0$, $\frac{\partial\Theta}{\partial \boldsymbol{n}} = 0$ $j = 1, 2$ on Γ_{W_t}. The following notation is used: ρ - density, p - pressure, E - total energy, $\boldsymbol{v} = (v_1, v_2)$ - velocity, Θ - absolute temperature, $\gamma > 1$ - Poisson adiabatic constant, $c_v > 0$ - specific heat at constant volume, $\mu > 0, \lambda = -2\mu/3$ - viscosity coefficients, $k > 0$ - heat conduction. The 4×4 matrices $\boldsymbol{K}_{sk}(\boldsymbol{w})$ are defined in [3] and [5].

The compressible flow problem is discretized by the STDGM formulated in a similar way as in Section 3. See, e.g., [3] and [5].

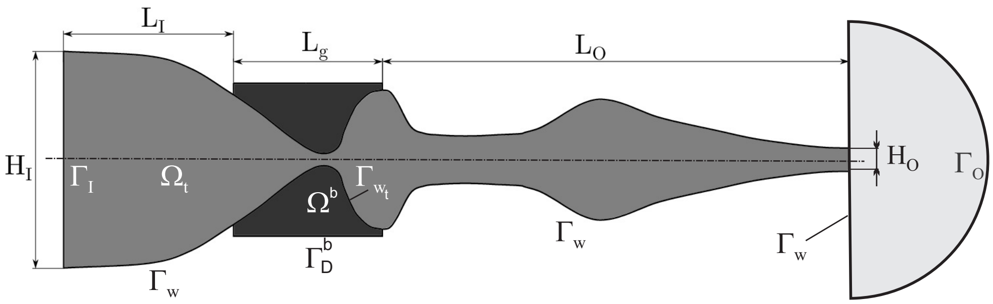

FIG. 6.1. *Geometry of the computational domain Ω_t for flow and Ω^b for the vocal folds at time $t = 0$ and the description of its size: $L_I = 20.0$ mm, $L_g = 17.5$ mm, $L_O = 55.0$ mm, $H_I = 25.5$ mm, $H_O = 2.76$ mm. The semicircle subdomain at the outlet of the channel has the radious 3.0 cm (not shown in the scale)*

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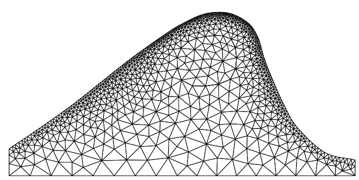

FIG. 6.2. *Model of vocal folds - computational mesh.*

6. Dynamic elasticity problem. The solution of the compressible flow is combined with the simulation of the deformation of an elastic body. It is represented by a bounded domain $\Omega^b \subset I\!R^2$ with boundary $\partial\Omega^b = \Gamma_D^b \cup \Gamma_N^b$. We want to find a displacement function $\boldsymbol{u} : Q_T = \Omega^b \times [0,T] \to I\!R^2$ such that

$$\rho^b \frac{\partial^2 \boldsymbol{u}}{\partial t^2} + c_M^b \rho^b \frac{\partial \boldsymbol{u}}{\partial t} - \mathrm{div}\boldsymbol{P}(\nabla \boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega^b \times [0,T], \tag{6.1}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{in } \Gamma_D^b \times [0,T], \tag{6.2}$$

$$\boldsymbol{P}(\nabla \boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{g}_N \quad \text{in } \Gamma_N^b \times [0,T], \tag{6.3}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \quad \frac{\partial \boldsymbol{u}}{\partial t}(\cdot,0) = \boldsymbol{z}_0 \quad \text{in } \Omega^b. \tag{6.4}$$

Here $\boldsymbol{f}$ - outer volume force, $\rho^b > 0$ - material density $\boldsymbol{P}$ - (Piola-Kirchhoff) stress tensor, c_M^b - structural damping.

In the case of linear elasticity, $\boldsymbol{P}(\nabla \boldsymbol{u}) = \boldsymbol{\sigma}(\boldsymbol{u}) = \lambda^b\ \mathrm{tr}(\boldsymbol{e}(\boldsymbol{u}))\mathbb{I} + 2\mu^b \boldsymbol{e}(\boldsymbol{u})$, $\boldsymbol{e}(\boldsymbol{u}) = (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2$ strain tensor, $\mathrm{tr}(\boldsymbol{e}) = \sum_{i=1}^2 e_{ii} = \mathrm{div}\boldsymbol{u}$, λ^b, μ^b are Lamé parameters.

In the case of nonlinear elasticity, the following quantities are defined: $\vartheta(\boldsymbol{x}) = \boldsymbol{x}+\boldsymbol{u}(\boldsymbol{x})$ - deformation mapping, $\boldsymbol{F} = \nabla\vartheta(\boldsymbol{x})$ - deformation gradient, i.e. the Jacobian matrix of the deformation mapping ϑ, $J = \det\boldsymbol{F} > 0$ - the Jacobian of the deformation, $\boldsymbol{e} \in I\!R^{2\times 2}$ - Green strain tensor with components $\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right)$, $\quad \boldsymbol{E} =$

$(E_{ij})_{i,j=1}^2$, where

$$E_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{e_{ij}-\text{linear part}} + \underbrace{\frac{1}{2}\sum_{k=1}^{2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}}_{E^*_{ij}-\text{nonlinear part}} . \tag{6.5}$$

In our simulations we consider the St. Venant-Kirchhoff material with the Piola-Kirchhoff stress tensor and the second Piola-Kirchhoff stress tensor defined by

$$\boldsymbol{P}(\nabla \boldsymbol{u}) = \boldsymbol{F}\boldsymbol{\Sigma}, \quad \boldsymbol{\Sigma} = \lambda^b \mathrm{tr}(\boldsymbol{E})\boldsymbol{I} + 2\mu^b \boldsymbol{E}. \tag{6.6}$$

See, e.g., [4].

The structural problem is also discretized by the STDGM. In this case the computational domain does not depend on time. The system is split into the systems of first order in time for the deformation velocity $\boldsymbol{y}$ and the displacement $\boldsymbol{u}$:

$$\rho^b \frac{\partial \boldsymbol{y}}{\partial t} + c_M^b \rho^b \boldsymbol{y} - \mathrm{div}\boldsymbol{P}(\nabla \boldsymbol{u}) = \boldsymbol{f}, \quad \frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{y} = 0 \quad \text{in } \Omega^b \times [0,T], \tag{6.7}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{in } \Gamma_D^b \times [0,T], \tag{6.8}$$

$$\boldsymbol{P}(\nabla \boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{g}_N \quad \text{in } \Gamma_N^b \times [0,T], \tag{6.9}$$

$$\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \quad \boldsymbol{y}(\cdot,0) = \boldsymbol{y}_0 \quad \text{in } \Omega^b. \tag{6.10}$$

In the space discretization we construct the partition $\mathcal{T}_{ht}^b$ of $\overline{\Omega}^b$ into a finite number of closed triangles K with mutually disjoint interiors and define the space

$$\mathcal{S}_{hs}^b = \left\{ v \in L^2(\Omega^b); v|_K \in P^s(K), K \in \mathcal{T}_{ht}^b \right\}^2, \tag{6.11}$$

where $s > 0$ is an integer. Further we define $\mathcal{F}_h^b$ – the system of all faces of all elements $K \in \mathcal{T}_{ht}^b$, boundary, "Dirichlet", "Neumann" and inner faces: $\mathcal{F}_h^{bB} = \left\{\Gamma \in \mathcal{F}_h^b; \Gamma \subset \partial\Omega^b\right\}$, $\mathcal{F}_h^{bD} = \left\{\Gamma \in \mathcal{F}_h^b; \Gamma \subset \Gamma_D^b\right\}$, $\mathcal{F}_h^{bN} = \left\{\Gamma \in \mathcal{F}_h^b; \Gamma \subset \Gamma_N^b\right\}$ and $\mathcal{F}_h^{bI} = \mathcal{F}_h^b \backslash \mathcal{F}_h^{bB}$. For each $\Gamma \in \mathcal{F}_h^b$ we define a unit normal vector $\boldsymbol{n}_\Gamma$. We assume that for $\Gamma \in \mathcal{F}_h^{bB}$ the normal $\boldsymbol{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega^b$, $\boldsymbol{\varphi}_\Gamma^{(L)}$, $\boldsymbol{\varphi}_\Gamma^{(R)}$ – traces of $\boldsymbol{\varphi} \in \mathcal{S}_{hs}^b$ on Γ, $\langle\boldsymbol{\varphi}\rangle_\Gamma$ – average of the traces on Γ, $[\boldsymbol{\varphi}]_\Gamma$ – jump of $\boldsymbol{\varphi}$ on Γ.

The discrete problem uses the following forms:

$$a_h^b(\boldsymbol{u},\boldsymbol{\varphi}) = \sum_{K\in\mathcal{T}_{ht}^b} \int_K \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{\varphi})\,\mathrm{d}x - \sum_{\Gamma\in\mathcal{F}_h^{bI}} \int_\Gamma (\langle\boldsymbol{\sigma}(\boldsymbol{u})\rangle \cdot \boldsymbol{n}) \cdot [\boldsymbol{\varphi}]\,\mathrm{d}S, \tag{6.12}$$

for the linear model. The nonlinear elasticity form is defined as

$$a_h^b(\boldsymbol{u},\boldsymbol{v}) = \sum_{K\in\mathcal{T}_{ht}^b} \int_K \boldsymbol{P}(\nabla\boldsymbol{u}) : \nabla\boldsymbol{v}\,\mathrm{d}x - \sum_{\Gamma\in\mathcal{F}_h^{bI}} \int_\Gamma (\langle\boldsymbol{P}(\nabla\boldsymbol{u})\rangle\boldsymbol{n}) \cdot [\boldsymbol{v}]\,\mathrm{d}S \tag{6.13}$$
$$- \sum_{\Gamma\in\mathcal{F}_h^{bD}} \int_\Gamma (\boldsymbol{P}(\nabla\boldsymbol{u})\,\boldsymbol{n}) \cdot \boldsymbol{v}\,\mathrm{d}S$$

Further forms read

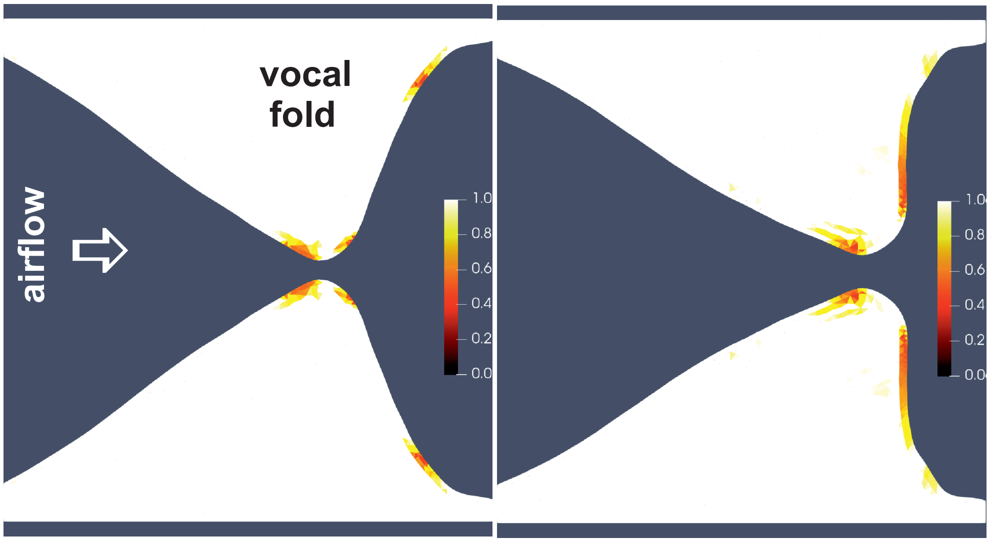

FIG. 6.3. *Detail of the deformation of vocal folds at two different time instants during vocal folds self-oscillations excited by airflow in glottis.*

$$J_h^b(\boldsymbol{u},\boldsymbol{\varphi}) = \sum_{\Gamma\in\mathcal{F}_h^{bI}} \int_\Gamma \frac{C_W^b}{h_\Gamma}[\boldsymbol{u}]\cdot[\boldsymbol{\varphi}]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_h^{bD}} \int_\Gamma \frac{C_W^b}{h_\Gamma}\boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}S, \tag{6.14}$$

$$\ell_h^b(\boldsymbol{\varphi})(t) = \sum_{K\in\mathcal{T}_{ht}^b} \int_K \boldsymbol{f}(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h^{bN}} \int_\Gamma \boldsymbol{g}_N(t)\cdot\boldsymbol{\varphi}\mathrm{d}S \tag{6.15}$$

$$+ \sum_{\Gamma\in\mathcal{F}_h^{bD}} \int_\Gamma \frac{C_W^b}{h_\Gamma}\boldsymbol{u}_D(t)\cdot\boldsymbol{\varphi}\mathrm{d}S,$$

$$A_h^b = a_h^b + J_h^b, \quad (\boldsymbol{u},\boldsymbol{\varphi})_{\Omega^b} = \int_{\Omega^b} \boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}x. \tag{6.16}$$

The STDG approximate solution is sought in the space of piecewise polynomial vector functions $\mathcal{S}_{h\tau}^{b,sq} = (S_{h\tau}^{b,sq})^2$, where

$$S_{h\tau}^{b,sq} = \Big\{v\in L^2(\Omega^b\times(0,T));\, v|_{I_m} = \sum_{i=0}^q t^i\varphi_{i,m}, \tag{6.17}$$
$$\varphi_{i,m}\in S_{hs}^b,\ m=1,\dots,M\Big\}.$$

The approximate solution is defined as a couple $\boldsymbol{u}_{h\tau},\boldsymbol{y}_{h\tau}\in\mathcal{S}_{h\tau}^{b,sq}$ such that

$$\text{a)}\int_{I_m}\Big(\big(\rho^b\frac{\partial\boldsymbol{y}_{h\tau}}{\partial t},\boldsymbol{\varphi}_{h\tau}\big)_{\Omega^b} + c_M^b\left(\rho^b\boldsymbol{y}_{h\tau},\boldsymbol{\varphi}_{h\tau}\right)_{\Omega^b}\Big) + A_h^b(\boldsymbol{u}_{h\tau},\boldsymbol{\varphi}_{h\tau})\Big)\mathrm{d}t \tag{6.18}$$
$$+(\{\boldsymbol{u}_{h\tau}\}_{m-1},\boldsymbol{\varphi}_{h\tau}(t_{m-1}+))_{\Omega^b} = \int_{I_m}\ell^b(\boldsymbol{\varphi}_{h\tau})\mathrm{d}t \quad \forall\boldsymbol{\varphi}_{h\tau}\in\mathcal{S}_{h\tau}^{b,sq},$$

$$\text{b)}\int_{I_m}\Big(\Big(\frac{\partial\boldsymbol{u}_{h\tau}}{\partial t},\boldsymbol{\varphi}_{h\tau}\Big)_{\Omega^b} - (\boldsymbol{y}_{h\tau},\boldsymbol{\varphi}_{h\tau})_{\Omega^b}\Big)\mathrm{d}t \tag{6.19}$$
$$+(\{\boldsymbol{u}_{h\tau}\}_{m-1},\boldsymbol{\varphi}_{h\tau}(t_{m-1}+))_{\Omega^b} = 0 \quad \forall\boldsymbol{\varphi}_{h\tau}\in\mathcal{S}_{h\tau}^{b,sq},$$
$$m=1,\dots,M.$$

The initial states $\boldsymbol{u}_h(0-), \boldsymbol{y}_h(0-) \in \mathcal{S}_{hs}^b$ are defined as $L^2(\Omega^b)$-projections of $\boldsymbol{u}^0, \boldsymbol{y}^0$. The nonlinear elasticity discrete problem is solved by Newton iterations.

The coupling of flow problem and elasticity problem is realized by transmission conditions representing the continuity of the velocity and normal stress on the common boundary $\tilde{\Gamma}_{Wt} = \{\boldsymbol{x} \in I\!R^2;\ \boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}, t),\ \boldsymbol{X} \in \Gamma_N^b\}$ between fluid and structure.
a) linear elasticity:

$$\sum_{j=1}^{2} \sigma_{ij}(\boldsymbol{u}(\boldsymbol{X}))n_j(\boldsymbol{X}) = \sum_{j=1}^{2} \tau_{ij}^f(\boldsymbol{x})n_j(\boldsymbol{X}),\ i = 1, 2,\ \boldsymbol{v}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{u}(\boldsymbol{X}, t)}{\partial t}, \tag{6.20}$$

b) nonlinear elasticity:

$$\boldsymbol{P}(\boldsymbol{F}(\boldsymbol{u}(\boldsymbol{X}, t)))\boldsymbol{n}(\boldsymbol{x}) = \tau^f(\boldsymbol{x}, t)\mathrm{Cof}(\boldsymbol{F}(\boldsymbol{u}(\boldsymbol{X}, t)))\boldsymbol{n}(\boldsymbol{x}),\ \boldsymbol{v}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{u}(\boldsymbol{X}, t)}{\partial t}. \tag{6.21}$$

Here $\tau^f = \{\tau_{ij}^f\}_{i,j=1}^2$ is the stress tensor of the fluid, σ_{ij}^b are components of stress tensor of the structure, $i, j = 1, 2$. $\boldsymbol{n}(\boldsymbol{X}) = (n_1(\boldsymbol{X}), n_2(\boldsymbol{X}))$ – the unit outer normal to the body Ω^b on Γ_N^b at the point $\boldsymbol{X}$. (The first condition in (6.21) is obtained on the basis of results in [4].)

There is a question if it is suitable to use linear or nonlinear model. It is possible to answer it with the aid of testing the behaviour of the Green strain tensor $\boldsymbol{E}$ with components defined in (6.5). The influence of the nonlinear part of the strain tensor is given by the ratio

$$R := \frac{\|\boldsymbol{e}\|}{\|\boldsymbol{E}\|} = \frac{\|\boldsymbol{e}\|}{\|\boldsymbol{e} + \boldsymbol{E}^*\|}. \tag{6.22}$$

If $R \approx 1$, then the nonlinear part of the strain tensor has negligible influence to the computation (the linear elasticity model is sufficient). On the other hand, if $R << 1$, then the nonlinear part strongly takes effect and it is necessary to use a nonlinear elasticity model. This is shown on Figure 6.3, where the case $R \approx 1$ is depicted by white and the case $R \approx 0$ by dark red color. It can be seen, that nonlinear part of the strain tensor takes effect in elements near to the boundary. Therefore to capture correctly deformations of the vocal folds, it is necessary to use a nonlinear model of elasticity.

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