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REGULARIZATION METHODS FOR THE CONSTRUCTION OF PRECONDITIONERS FOR SADDLE POINT PROBLEMS*

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Abstract. For the iterative solution of saddle point problems one needs efficient preconditioners to achieve a fast convergence. Three types of preconditioners are presented which are based on regularization by use of an augmented matrix. They are applicable also for problems with a highly singular pivot block matrix. One of the methods is applicable also for nonsymmetric saddle point problems.

Key words. Saddle-point, regularization, preconditioners.

AMS subject classifications. 15A09, 15A23, 65F08, 65F22

1. Introduction. During many years numerical solution of saddle point problems has been an important research topic, see e.g. [1, 2, 3]. Saddle point problems arise in many applications, such as fluid flow problems with an incompressible fluid. In many cases they are not fully incompressible or for their numerical solution one adds a regularization which corresponds to such a small compressibility. This is also done in structural engineering problems to avoid locking phenomenon, see e.g. [4, 5]. Use of such a regularization gives a greater freedom in the choice of finite element basis functions used for the discretization of the problem. The major purpose of the present paper is to compare three preconditioners for regularized matrices. The methods give strong clustering of the eigenvalues of the preconditioned matrix which results in a fast convergence of the iterative acceleration method. One of the methods is particularly suitable for nonsymmetric saddle point problems and can handle problems with a highly singular pivot block matrix.

The standard form of a discretized saddle point problem is $\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$ where A, of order $n \times n$, is symmetric and positive definite (spd) and B, of order $m \times n$, $m < n$, is assumed to have full rank. Its weakly incompressible or regularized form is $\mathcal{A} = \begin{bmatrix} A & B^T \\ B & -K \end{bmatrix}$, where K is spsd, and if B is rank deficient, it is assumed that K is positive definite on $\mathcal{N}(B)$. This implies that the Schur complement, $S = K + BA^{-1}B^T$ is spd.

As shown e.g. in [6], this matrix has eigenvalues located in two intervals, i.e. with both positive and negative values,

$$[-\lambda_{\max}(S), -\lambda_{\min}(S)/(1 + \frac{\gamma^2}{\mu_1}\lambda_{\max}(S))] \cup [\mu_1, \mu_n + \sigma_m],$$

where $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ and $0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_m$ denote the eigenvalues of A respectively of $BA^{-1}B^T$. Further $\gamma^2 = \varrho(S^{-1/2}BA^{-1}B^TS^{-1/2})$, i.e. the spectral radius.

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In [6] is also shown that the real part of the eigenvalues of $\mathcal{A}' = \begin{bmatrix} A & B^T \\ -B & K \end{bmatrix}$, which are positive, are bounded as

$$\min\{\lambda_{\min}(A), \lambda_{\min}(S)\} \leq \mathrm{Re}\,(\lambda) \leq \max\{\lambda_{\max}(A), \lambda_{\max}(K)\}.$$

If the imaginary part $\mathrm{Im}(\lambda) \neq 0$ then $\|x\| = \|y\|$ if $(x^T, y^T)^T$ is an eigenvector, and $\mathrm{Im}(\lambda) = y_2^T B x_1 - y_1^T B x_2$ if $\|x\| = \|y\| = 1$, where $x = x_1 + i x_2$, $y = y_1 + i y_2$, and x_i, y_i, $i = 1, 2$ are real valued. Further

$$\frac{1}{2}\left(\lambda_{\min}(A) + \lambda_{\min}(K)\right) \leq \mathrm{Re}(\lambda) \leq \frac{1}{2}\left(\lambda_{\max}(A) + \lambda_{\max}(K)\right)$$

To cope with such strong variations of the eigenvalues one must use a good preconditioning for the iterative solution of the given problem. A major aim of this paper is to consider preconditioning methods involving an acceptable solution cost, which lead to real and positive and also well distributed eigenvalues.

As shown e.g. in [6], block diagonal preconditioners can not achieve such a goal. For instance, even with the preconditioner $\mathcal{B} = \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix}$, the eigenvalues of $\mathcal{B}^{-1}\mathcal{A}$ equal unity when the first component of the eigenvector, $x \in \mathcal{N}(B)$ and $y = 0$, but for $x \in \mathcal{N}(B)^\perp$ they are indefinite and for $\mathcal{B}^{-1}\mathcal{A}'$ they are complex, that is,

$$\lambda = \frac{1-c}{2} \pm \sqrt{\left(\frac{1-c}{2}\right)^2 + 1}, \quad \text{respectively} \quad \lambda = \frac{1+c}{2} \pm i\sqrt{1 - \left(\frac{1+c}{2}\right)^2},$$

where c is an eigenvalue of $S^{-1/2} K S^{-1/2}$.

For these reasons, instead of block diagonal we consider preconditioners on block triangular form. To give a hint how they are constructed, we note that $\mathcal{A}$ and $\mathcal{A}'$ can be factorized as

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ B & -S \end{bmatrix} \begin{bmatrix} I & A^{-1}B^T \\ 0 & I \end{bmatrix}, \qquad \mathcal{A}' = \begin{bmatrix} S^{(1)} & B^T \\ 0 & K \end{bmatrix} \begin{bmatrix} I & 0 \\ -K^{-1}B & I \end{bmatrix},$$

where $S^{(1)} = A + B^T K^{-1} B$, if K is spd. We consider also nonsymmetric saddle point problems. The following methods will be analysed:

(i) $\mathcal{B} = \begin{bmatrix} A_0 & 0 \\ B & -S_0 \end{bmatrix}$ to $\mathcal{A} = \begin{bmatrix} A & B^T \\ B & -K \end{bmatrix}$

where A_0 is an spd matrix approximating A such that $\alpha = x^* \widetilde{A} x / x^* x \geq 1$, $\forall x$, where $\widetilde{A} = A_0^{-1/2} A A_0^{-1/2}$ and $S_0 = K + B A_0^{-1} B^T$. Here K may be a zero matrix if B has full rank.

(ii) A block matrix factorized preconditioner,

$$\mathcal{B} = \begin{bmatrix} A_W & B^T \\ 0 & W \end{bmatrix} \begin{bmatrix} I & 0 \\ -W^{-1}B & I \end{bmatrix} = \begin{bmatrix} A & B^T \\ -B & W \end{bmatrix}, \text{ for } \mathcal{A}' = \begin{bmatrix} A & B^T \\ -B & K \end{bmatrix},$$

where $A_W = A + B^T W^{-1} B$, and W is spd somehow approximating K. We consider augmented methods where W has been replaced by $\frac{1}{r}W$.

(iii) $\mathcal{B} = \begin{bmatrix} A_r & 2B^T \\ 0 & -W_r \end{bmatrix}$ for $\begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix}$, to be applied both for symmetric, where $C = B$ and nonsymmetric problems, where $A_r = A + B^T W_r^{-1} C$, $W_r = \frac{1}{r} W$, and r is a normally large real number.

These methods have been partly considered in [6], [7] and [8] but are here presented with more general and condensed proofs. Replacing A with A_W respectively A_r can be seen as regularization methods.

2. Preconditioning methods. We consider now block triangular preconditioners, see e.g. [9, 10] for early references.

2.1. A lower block triangular preconditioner. Following [6, 7], let $\mathcal{B} = \begin{bmatrix} A_0 & 0 \\ B & -S_0 \end{bmatrix}$ be a preconditioner to $\mathcal{A} = \begin{bmatrix} A & B^T \\ B & -K \end{bmatrix}$. Here a more precise analysis will be given. We assume that A_0 is spd and $A_0 \le A$ in the inner product ordering. In some applications A is a mass matrix, for which the construction of A_0 is simple. For instance in porous media modelled by Darcy's flow equations, see e.g. [8], the velocity vector $\mathbf{u} = -K\nabla p$, $\nabla \cdot \mathbf{u} = f$ in a given domain Ω, where p is the pressure variable. For Stokes problem, A is a discretized Laplacian matrix and there exists various methods to approximate it.

By assumptions made, $\alpha \ge 1$ where

$$\alpha = \alpha(x) = x^* \widetilde{A} x / x^* x, \quad \forall x, \quad \widetilde{A} = A_0^{-1/2} A A_0^{-1/2}.$$

Further it is assumed that $K \ge 0$. Then, with $S_0 = K + BA_0^{-1}B^T$, which is spd it follows that

$$0 \le \widetilde{K} := S_0^{-1/2} K S_0^{-1/2} = I - \widetilde{B}\widetilde{B}^T,$$

where $\widetilde{B} = S_0^{-1/2} B A_0^{-1/2}$. Note that $\widetilde{B}\widetilde{B}^T = S_0^{-1/2} B A_0^{-1} B^T S_0^{-1/2}$. Hence $0 \le \beta \le 1$, where $\beta = \beta(x) = x^* \widetilde{B}\widetilde{B}^T x / x^* x$. We assume that

$$(\widetilde{A} - I)x \notin \mathcal{N}(B), \qquad \text{except if} \qquad \widetilde{A}x = x. \tag{2.1}$$

In practice A_0 is chosen such that multiplications with A_0^{-1} can be done without much computational effort. For instance, $A_0 = LL^T$, where L is lower triangular, or similarly LL^T is a factorization of A_0^{-1}, see [11] and [12] for references to the construction of approximate inverses. Methods to solve systems with matrix S_0 will be discussed in Section 4.

For the spectral analyses of $\mathcal{B}^{-1}\mathcal{A}$, we make a congruence transformation of both $\mathcal{B}$ and $\mathcal{A}$ with $\begin{bmatrix} A_0^{-1/2} & 0 \\ 0 & S_0^{-1/2} \end{bmatrix}$, which is equivalent to a similarity transformation of the preconditioned matrix $\mathcal{B}^{-1}\mathcal{A}$. Hence the eigenvalue problem takes the form,

$$\lambda \begin{bmatrix} I & 0 \\ \widetilde{B} & -I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \widetilde{A} & \widetilde{B}^T \\ \widetilde{B} & -\widetilde{K} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \|x\| + \|y\| \ne 0. \tag{2.2}$$

THEOREM 2.1. *Let $A_0 \le A$ be spd and $S_0 = K + BA_0^{-1}B^T$ and let λ be an eigenvalue of $\mathcal{B}^{-1}\mathcal{A}$. Then $\lambda = 1$ if and only if the corresponding eigenvector satisfies $Ax = A_0 x$. For $\lambda \ne 1$, the eigenvalues are contained in the interval* $[1/\lambda_{\max}; \lambda_{\max}]$, *where* $\lambda_{\max} \le \frac{\alpha_{\max}+1}{2} + \sqrt{\left(\frac{\alpha_{\max}+1}{2}\right)^2 - 1}$ *and* $\alpha_{\max} = \sup_x \frac{x^* A x}{x^* A_0 x}$.

Proof. For a proof, see [6, 7]. ☐

Hence the condition number of $\mathcal{B}^{-1}\mathcal{A}$ is bounded by

$$\lambda_{\max}/\lambda_{\min} = \lambda_{\max}^2 < (\alpha_{\max} + 1)^2.$$

It is seen that the choice of matrix A_0, i.e. the value of $\alpha_{\max}$, plays an important role to avoid too large values of $\lambda_{\max}$.

2.2. A preconditioner with a regularized matrix. Consider now methods involving the regularized, i.e. augmented matrix,

$$A_r = A + B^T W_r^{-1} B,$$

where $W_r = \frac{1}{r}W$, W is a chosen spd matrix and r is a, normally large, positive number. Note first that the solution of

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} A_r & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{2.3}$$

i.e. where the constraint $Bx = 0$ holds, have the same solutions. Hence a preconditioned method to compute the solution of (2.3) can be based on this latter matrix. Let then $\mathcal{B}_r = \begin{bmatrix} A_r & 0 \\ B & W_r \end{bmatrix}$ be a preconditioner to $\mathcal{A}_r = \begin{bmatrix} A_r & B^T \\ B & 0 \end{bmatrix}$. It holds

$$\mathcal{B}_r^{-1}\mathcal{A}_r = \begin{bmatrix} A_r^{-1} & 0 \\ -W_r^{-1}BA_r^{-1} & W_r^{-1} \end{bmatrix}\begin{bmatrix} A_r & B^T \\ B & 0 \end{bmatrix} = \begin{bmatrix} I & A_r^{-1}B^T \\ 0 & W_r^{-1}S_r \end{bmatrix}, \tag{2.4}$$

where $S_r = BA_r^{-1}B^T = B(A + B^T W_r^{-1} B)^{-1}B^T$.

Lemma 2.2. *Assume that B has full rank and A and W are spd. Then*
(i) $(B(I + B^T B)^{-1}B^T)^{-1} = I + (BB^T)^{-1}$
(ii) $(B(A + B^T W^{-1} B)^{-1}B^T)^{-1} = W^{-1} + (BA^{-1}B^T)^{-1}$.

Proof. (These relations are well known but since the proof in short for completeness we present it here also.)
Since $B^T(I + BB^T) = (I + B^T B)B^T$, it follows that

$$(I + B^T B)^{-1}B^T = B^T(I + BB^T)^{-1}$$

and multiplying with B and taken inverses, it follows that

$$(B(I + B^T B)^{-1}B^T)^{-1} = (I + BB^T)(BB^T)^{-1} = I + (BB^T)^{-1}.$$

Replacing here B with $W^{-1/2}BA^{-1/2}$ and multiplying from both sides with $W^{-1/2}$, gives (ii). □

Theorem 2.3. *Assume that B has full rank and let $\mathcal{B}_r = \begin{bmatrix} A_r & 0 \\ B & W_r \end{bmatrix}$ be a preconditioner to $\mathcal{A}_r = \begin{bmatrix} A_r & B^T \\ B & 0 \end{bmatrix}$, where $A_r = A + B^T W_r^{-1} B$, $W_r = \frac{1}{r}W$ and W is spd. Then*

$$\mathcal{B}_r^{-1}\mathcal{A}_r = \begin{bmatrix} I & A_r^{-1}B^T \\ 0 & W_r^{-1}S_r \end{bmatrix}.$$

There are n eigenvalues of $\mathcal{B}_r^{-1}\mathcal{A}_r$ equal to unity. The remaining eigenvalues are the eigenvalues of $W_r^{-1}S_r$ which equal $1/(1 + \frac{1}{r\mu})$, where μ is an eigenvalue of $\mu W x = BA^{-1}B^T x$, $x \neq 0$.

Proof. This follows from (2.4). Further Lemma 2.2 shows that

$$S_r^{-1} = (B(A + B^T W_r^{-1} B)^{-1}B^T)^{-1} = W_r^{-1} + (BA^{-1}B^T)^{-1}.$$

Hence

$$S_r^{-1}W_r = I + (BA^{-1}B^T)^{-1}W_r,$$

whose eigenvalues are $1 + \frac{1}{r\mu}$, where μ is positive. $\square$

Assuming that μ is not too small it follows that all eigenvalues cluster at unity when r increases. To get a not too small value of $\mu_{\min}$ one can choose $W = D$ as a diagonal matrix such that

$$\mu D\mathbf{e} = BA^{-1}B^T\mathbf{e},$$

where $e = (1, 1, \cdots, 1)^T$. How to solve systems with A_r will be discussed in Section 4.

That the off-diagonal block matrix part in (2.4) does not harm the convergence of an optimal Krylov space acceleration method follows from the next Lemma.

LEMMA 2.4. *The degree of the minimal polynomial to a matrix of the form* $\begin{bmatrix} I & F \\ 0 & E \end{bmatrix}$, *where E is positive definite and $E - I$ is nonsingular, is independent of F.*

Proof. It holds

$$\begin{bmatrix} I & F \\ 0 & E \end{bmatrix}^k = \begin{bmatrix} I & F(I + \cdots + E^{k-1}) \\ 0 & E^k \end{bmatrix} = \begin{bmatrix} I & F(I-E)^{-1}(I-E^k) \\ 0 & E^k \end{bmatrix}.$$

Hence, if $\sum\limits_{\ell=0}^{q} a_\ell E^\ell = 0$ where $\sum\limits_{\ell=0}^{q} a_\ell = 1$, for some $q \geq 2$, then

$$\sum_{k=0}^{q} a_k \begin{bmatrix} I & F \\ 0 & E \end{bmatrix}^k = \begin{bmatrix} I & \sum\limits_{k=0}^{q} a_k F(I-E)^{-1} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & F(I-E)^{-1} \\ 0 & 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} I & F(I-E)^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & F \\ 0 & E \end{bmatrix} = \begin{bmatrix} I & F(I-E)^{-1} \\ 0 & 0 \end{bmatrix},$$

it follows that

$$\sum_{k=0}^{q+1} a_k \begin{bmatrix} I & F \\ 0 & E \end{bmatrix}^{k+1} - \sum_{k=0}^{q} a_k \begin{bmatrix} I & F \\ 0 & I \end{bmatrix}^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

that is, the degree of the minimal polynomial is $q+1$, i.e. just one degree higher than the minimal polynomial for matrix E. $\square$

Therefore the convergence of a Krylov subspace optimal method needs just one additional iteration due to the presence of the off diagonal block in the preconditioned matrix.

2.3. An upper block triangular preconditioner. Let $\mathcal{A}' = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$ and the preconditioner $\mathcal{B}' = \begin{bmatrix} A_r & B^T \\ 0 & -W_r \end{bmatrix}$, where $A_r = A + B^T W_r^{-1} B$, $W_r = \frac{1}{r} W$ and W is spd. Assume also that A is spsd and $\mathcal{N}(A) \cap \mathcal{N}(B^T) = \{\emptyset\}$, so A_r is spd. The generalized eigenvalue problem takes the form,

$$\lambda \begin{bmatrix} A_r & B^T \\ 0 & -W_r \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \|x\| + \|y\| \neq 0.$$

A congruence transformation with $\begin{bmatrix} A_r^{-1/2} & 0 \\ 0 & W_r^{-1/2} \end{bmatrix}$ on both sides results in

$$\lambda \begin{bmatrix} I & \widehat{B}^T \\ 0 & -I \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix} = \begin{bmatrix} I - \widehat{B}^T \widehat{B} & \widehat{B}^T \\ \widehat{B} & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix},$$

where $\widehat{B} = W_r^{-1/2} B A_r^{-1/2}$ and $\widehat{x} = A_r^{1/2} x$, $\widehat{y} = W_r^{1/2} y$. Hence

$$\lambda \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix} = \begin{bmatrix} I & \widehat{B}^T \\ 0 & -I \end{bmatrix} \begin{bmatrix} I - \widehat{B}^T \widehat{B} & \widehat{B}^T \\ \widehat{B} & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix}, \quad \text{i.e.} \quad \lambda \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix} = \begin{bmatrix} I & \widehat{B}^T \\ -\widehat{B} & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix}.$$

Here the eigenvalues are complex and with both positive and negative real parts. To improve on that we choose the modified preconditioner

$$\mathcal{B} = \begin{bmatrix} A_r & 2B^T \\ 0 & -W_r \end{bmatrix} \qquad \text{to} \qquad \mathcal{A} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}. \tag{2.5}$$

THEOREM 2.5. *The preconditioned matrix $\mathcal{B}$ to $\mathcal{A}$ in (2.5) can be written on the similarity transformed form,*

$$\begin{bmatrix} I & -\widehat{B}^T \\ -\widehat{B} & I \end{bmatrix}^{-1} \begin{bmatrix} I + \widehat{B}\widehat{B}^T & \widehat{B}^T \\ -\widehat{B} & 0 \end{bmatrix} \begin{bmatrix} I & -\widehat{B}^T \\ -\widehat{B} & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \widehat{B}\widehat{B}^T \end{bmatrix} \tag{2.6}$$

which shows that its eigenvalues are real and equal unity with multiplicity n and the remaining eigenvalues equal $1/(1+\frac{1}{r\mu})$. The eigenvectors for the unit eigenvalue equal $[\mathbf{e}_i^T, -(W_r^{-1} B \mathbf{e}_i)^T]^T$, where $\mathbf{e}_i$ are the unit vectors in $\mathbb{R}^m$.

Proof. Here

$$\lambda \mathcal{B} \begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{A} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \|x\| + \|y\| \neq 0,$$

which after transformation leads to

$$\lambda \begin{bmatrix} I & 2\widehat{B}^T \\ 0 & -I \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix} = \begin{bmatrix} I - \widehat{B}^T \widehat{B} & \widehat{B}^T \\ \widehat{B} & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix}.$$

Hence

$$\lambda \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix} = \begin{bmatrix} I & 2\widehat{B}^T \\ 0 & -I \end{bmatrix} \begin{bmatrix} I - \widehat{B}^T \widehat{B} & \widehat{B}^T \\ \widehat{B} & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix}, \quad \text{i.e.} \quad \lambda \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix} = \begin{bmatrix} I + \widehat{B}^T \widehat{B} & \widehat{B}^T \\ -\widehat{B} & 0 \end{bmatrix} \begin{bmatrix} \widehat{x} \\ \widehat{y} \end{bmatrix}.$$

A computation shows that

$$\begin{bmatrix} I + \widehat{B}^T \widehat{B} & \widehat{B}^T \\ -\widehat{B} & 0 \end{bmatrix} \begin{bmatrix} I & -\widehat{B}^T \\ -\widehat{B} & I \end{bmatrix} = \begin{bmatrix} I & -\widehat{B}^T \\ -\widehat{B} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \widehat{B}\widehat{B}^T \end{bmatrix},$$

which implies (2.6). Further it follows that the eigenvectors for the unit eigenvalues equal $(e_i^T, -(W_r^{-1} B e_i)^T)^T$. It holds

$$(\widehat{B}\widehat{B}^T)^{-1} = W_r^{1/2} (B(A + B^T W_r^{-1} B)^{-1} B^T)^{-1} W_r^{1/2} = (\widetilde{B}(I + \widetilde{B}^T \widetilde{B})^{-1} \widetilde{B}^T)^{-1},$$

where $\widetilde{B} = W_r^{-1/2} B A^{-1/2}$. By Lemma 2.4,

$$(\widehat{B}\widehat{B}^T)^{-1} = I + (\widetilde{B}\widetilde{B}^T)^{-1} = I + W_r^{1/2}(BA^{-1}B^T)^{-1}W_r^{1/2}.$$

It follows that the eigenvalues of $\widehat{B}\widehat{B}^T$ equal $1/(1+\frac{1}{r\mu})$, where we recall that $W_r = \frac{1}{r}W$ and μ are the eigenvalues of $\mu W z = BA^{-1}B^T z$, $z \neq 0$. □

As before, it follows that the eigenvalues cluster strongly at unity and the iteration method will need very few iterations. Furthermore, the preconditioning method can be implemented in an efficient low cost way.

The eigenvalues are equal to those for the method in the previous subsection. However, there the preconditioned matrix has an off-diagonal matrix block.

3. A preconditioner for nonsymmetric saddle point problems with a highly singular pivot block matrix. Preconditioners for saddle point problems with (highly) singular pivot block matrix has been considered e.g. [13, 14, 6, 7] and in [15], where also nonsymmetric saddle point problems have also been considered. Nonsymmetric saddle point problems where $\mathcal{A} = \begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix}$, and B and C may not be equal, arise in some applications. Consider for instance a convection diffusion problem, $\nabla \cdot (-\varepsilon \nabla u + \mathbf{w} u) = f$, where $\mathbf{w}$ is a given velocity vector, with provided boundary conditions. If we let $\mathbf{v} = -\varepsilon u + \mathbf{w} u$, the saddle point equation becomes

$$\begin{cases} \mathbf{v} + \varepsilon \nabla u - \mathbf{w} u &= 0 \\ \nabla \cdot \mathbf{v} &= f. \end{cases}$$

Another problem where B and C differ arises in electrolytic cell problems (see [16]).

In some problems, A is singular. As before, we assume that B and C have full rank and that $\mathcal{N}(A) \cap \mathcal{N}(B) = \{\emptyset\}$, $\mathcal{N}(A) \cap \mathcal{N}(C) = \{\emptyset\}$. Then $\mathcal{A}$ is regular, even though A may be highly singular.

Judged by the favourable results in the previous subsection, we let

$$\mathcal{B} = \begin{bmatrix} A_r & 2B^T \\ 0 & -W_r \end{bmatrix}$$

be preconditioner to $\mathcal{A}$, where $A_r = A + B^T W_r^{-1} C$.

THEOREM 3.1. *Under the above stated assumptions, the preconditioned matrix $\mathcal{B}^{-1}\mathcal{A}$ has eigenvalues equal to unity for eigenvectors x of the form $x \notin \mathcal{N}(\mu A - B^T W^{-1} C)$. The nonzero eigenvalues equal $\lambda = \mu/(r + \mu r)$ where μ is an eigenvalue of $\mu A x = B^T W^{-1} C x$, $x \neq 0$. There are at most $m - m_0$ such eigenvalues, where $m_0 = dim \mathcal{N}(A)$.*

Proof. The eigenvalue problem takes the form

$$\lambda \begin{bmatrix} A_r & 2B^T \\ 0 & -W_r \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \|x\| + \|y\| \neq 0.$$

Since $\mathcal{A}$ is regular, it follows that $\lambda \neq 0$. Hence $y = -\lambda^{-1} W_r^{-1} C x$, which after substitution in the first equation, multiplied by λ, gives

$$\lambda^2 A_r x - 2\lambda B^T W_r^{-1} C x = \lambda A x - B^T W_r^{-1} C x,$$

that is,

$$(\lambda^2 - \lambda) A x + (\lambda^2 - 2\lambda + 1) B^T W_r^{-1} C x = 0.$$

It follows that

$$(1-\lambda)(-\lambda A+(1-\lambda)B^TW_r^{-1}C)x=0.$$

Hence either $\lambda=1$ or x is an eigenvector of

$$\frac{\lambda}{1-\lambda}Ax=B^TW_r^{-1}Cx=rB^TW^{-1}Cx.$$

Hence $\frac{1}{r}\frac{\lambda}{1-\lambda}=\mu$, that is, $\lambda=\frac{\mu r}{1+\mu r}$ or $1-\lambda=1/(1+\mu r)$.

If $m_0=\dim\mathcal{N}(A)$, there are at most $m-m_0$ such eigenvalues. The remaining eigenvalues, i.e. $n+m_0$, equal unity. □

REMARK 3.1. *If $Re\,\mu>0$, then the eigenvalues cluster at unity when $r\to\infty$.*

REMARK 3.2. *Indefinite matrix problems where A is indefinite can be solved by use of the augmented matrix method if $r\ge r_0>0$, if r_0 is sufficiently large so that $A+r_0B^TD^{-1}C$ is regular, where $D=W$ or an approximation of W. That is, for $C=B$, it is positive definite for $r\ge r_0$. For further treatments of such problems, see e.g. [17].*

REMARK 3.3. *In all methods, there arise inner systems that are also normally solved by iteration. This can often be done to a quite rough accuracy without increasing the number of outer iterations much. For an early presentation of this, see [18], see also [19].*

4. A projection matrix approach to solve the augmented matrix systems. First we note that the appearance of small eigenvalues μ can be avoided by choosing

$$W=BD^{-1}B^T$$

where D is a diagonal matrix, for instance $D\mathbf{e}=BA^{-1}B^T\mathbf{e}$, $\mathbf{e}=(1,1,\cdots,1)^T$. A simpler choice could be $D\mathbf{e}=A\mathbf{e}$. To solve the arising systems with W one can use a Cholesky or modified incomplete factorization method, see e.g. [12], a multilevel iteration method [18, 19] or a domain decomposition method.

To solve the arising systems with the block matrix A_r in the preconditioner one can use a projection matrix. We consider then the choice

$$W=CD^{-1}B^T, \tag{4.1}$$

and assume that systems with W can be solved efficiently.

Let $P=D^{-1}B^TW^{-1}C$. Note that, besides matrix vector multiplications with sparse matrices, actions of P on vectors involve only solution of systems with matrix W.

Assuming that W satisfies (4.1) exactly or at least is a good approximation, it follows from this definition of W that

$$P^2=D^{-1}B^TW^{-1}CD^{-1}B^TW^{-1}C=D^{-1}B^TW^{-1}C=P.$$

Hence P is a projection matrix satisfying $CP=B$. To utilize this for the solution of systems with A_r we first multiply A_r with D^{-1} to get

$$\widetilde{A}_r=D^{-1}A_r=D^{-1}A+rD^{-1}B^TW^{-1}C.$$

This matrix is multiplied with $(I+rP)^{-1}$ from the right. Note first that since $P^2 = P$, it follows that

$$(I + rP)\left(I - \frac{r}{r+1}P\right) = I + rP - \frac{r}{r+1}P - \frac{r^2}{r+1}P = I,$$

that is,

$$(I + rP)^{-1} = I - \frac{r}{r+1}P.$$

Hence

$$\begin{aligned}\widetilde{A}_r(I + rP)^{-1} &= (D^{-1}A + rD^{-1}B^TW^{-1}B)(I + rP)^{-1} = \\ &= (D^{-1}A - I + I + rP)(I + rP)^{-1} = \\ &= I + (D^{-1}A - I)\left(I - \frac{r}{r+1}P\right).\end{aligned} \tag{4.2}$$

For vectors $x \in \mathcal{N}(B)$ it holds $Px = 0$ and since D is an approximation of A it can be expected that

$$(I + (D^{-1}A - I))x = D^{-1}Ax$$

will not take large values.

For $x \in \mathcal{N}(B)^{\perp}$, then since $P^2x = Px$, it follows from $P(I - P)x = 0$ that is Px is close to x. Hence for such vectors, it holds approximately that

$$\widetilde{A}_r(I + rP)^{-1}x = x + \frac{1}{r+1}(D^{-1}A - I)x,$$

that is a small perturbation of x when r is large. It follows that the preconditioning method leads to a rapid convergence. The major cost is in solving systems with matrix W, which takes place twice during each iteration, once to compute actions of P which takes place in solving systems with A_r, and once to solve for the lower block systems in $\mathcal{B}$. It is possible to use inner iteration solutions for A_r, in which case however more actions of W_r^{-1} are involved.

Saddle point problems arise in many applications, such as for Stokes equation in fluid flow problems. It is also possible to rewrite a second order elliptic problem in a saddle point form involving both the solution and its gradient as unknown variables. As is well known, this leads to higher accuracies for the gradients even for the lowest order of finite element basis functions. Let

$$\mathcal{L}u = -\boldsymbol{\nabla} \cdot (\varepsilon \boldsymbol{\nabla} u) + q^2 u = f \tag{4.3}$$

in a bounded domain Ω with say homogeneous Dirichlet boundary conditions. Here ε is a variable coefficient that can take small values in part of the domain. Further q is typically a main long wave length where a wave equation is modelled by operator $\mathcal{L}$. Let $\mathbf{w} = \boldsymbol{\nabla} u$. Then (4.3) can be rewritten in the block operator form,

$$\left\{\begin{array}{rcl}\mathbf{w} - \boldsymbol{\nabla} u &=& 0\\ \nabla \cdot (\varepsilon \mathbf{w}) - q^2 u &=& -f.\end{array}\right.$$

Assume for simplicity that ε is piecewise constant. After a standard finite element discretization, the corresponding block matrix form is

$$\begin{bmatrix}\widetilde{M} & B^T\\ D_0B & -q^2M\end{bmatrix}\begin{bmatrix}w\\ u\end{bmatrix} = \begin{bmatrix}0\\ f\end{bmatrix} \tag{4.4}$$

where we keep the same notations for the arising vectors. Here $\widetilde{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$ in a 2D space problem, B a discretization of the divergence operator, M is the mass matrix and D_0 is a diagonal matrix with entries corresponding to its value in different elements. To solve (4.4) any of the presented preconditioners can be used.

We scale the system to obtain a nearly symmetric form,

$$\begin{bmatrix} \widetilde{M} & \widehat{B}^T \\ \widehat{B} & -q^2\widehat{M} \end{bmatrix} \begin{bmatrix} w \\ \widehat{u} \end{bmatrix} = \begin{bmatrix} 0 \\ \widehat{f} \end{bmatrix},$$

where $\widehat{M} = D_0^{-1/2} M D_0^{-1/2}$, $\widehat{B} = D_0^{1/2} B$, $\widehat{u} = D_0^{-1/2} u$, $\widehat{f} = D_0^{-1/2} f$. It is symmetric if it is a lumped mass matrix.

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