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DYNAMIC CORRELATION IN A CONVERGENCE MODEL OF INTEREST RATES*

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Abstract. Short rate models are formulated in terms of stochastic differential equations governing the instantaneous interest rate, so called short rate. The bond prices, as well as other derivatives, are then given as a solution to a parabolic partial differential equation with a terminal condition equal to the payoff of the derivative. Convergence models are used to model a situation where a country is going to enter a monetary union and its short rate is affected by the short rate in the monetary union. In addition, Wiener processes which model random shocks in the behaviours of the short rates can be correlated. In this paper we consider a dynamic correlation (i.e., the correlation is a given function of time) in a convergence model with volatilities proportional to powers of the respective short rates. Firstly, we consider a special case with constant volatilities which is analytically tractable. Based on observations made in this case, we propose an approximate analytical solution for the bond prices in the general model and derive order of its accuracy.

Key words. interest rate, convergence mode, dynamic correlation, bond price, approximate analytical solution

AMS subject classifications. 91G30, 35K10, 35A35

1. Introduction.

Short rate models of interest rates are formulated in terms of a stochastic differential equation (one factors models) or a system of them (multifactor models) which govern the behaviour of the instantaneous interest rate, so called short rate. The derivatives of the short rate, i.e. financial securities, which depend on the short rate are then priced by a parabolic partial differential equation, whose terminal condition is the payoff of the derivative. A simple interest rate derivative is a discount bond, which pays a unit of currency at the specified time, called maturity of the bond. Bond prices are used only in the construction of yield curve and are necessary in discounting any future cash flows. Therefore it is important to be able to compute bond prices in various short rate models which has been proposed. The reader can find an overview of short rate models in [1] or [6].

In this paper we deal with convergence models of interest rates. These models capture the situation when a country is going to join a monetary union and its domestic short rate is affected by the short rate in the monetary union. Furthermore, the Wiener processes which model random fluctuations and shocks can be correlated which adds another form of relation between the interest rates in the given country and the monetary union.

A dynamic correlation in financial markets has been studied example in [8] (the correlation of Wiener processes refers to the stochastic differential equations for a stock price and an exchange rate) and [9] (the stochastic differential equations describe stock price and volatility; the model generalizes the Heston model). It is meaningful also in the context of convergence models of interest rates. In the pioneering paper suggesting a convergence model of interest rates [4], the estimated correlation is higher

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when more recent data were used (i.e., data closer to entering the monetary union). This suggests that the correlation is not constant, but it changes in time, and we try to model it by a dynamic correlation approach. Furthermore, we extend the model from [4] by considering more general volatilities based on [2] and [11]. Except for special cases, a closed-form solution for the bond pricing partial differential equations are not available and we approximate them using analytical approximation formulae, for which we derive their order of accuracy.

The paper is organized as follows: In Section 2 we introduce convergence models and incorporate a dynamic correlation into them. In Section 3 we consider a so called Vasicek-type model and simplify the expression for the bond price by a formula which simplifies the computation; in contrast to the exact solution, it does not require an integration and is written in a closed form. Firstly we consider a particular choice of the correlation function when we are able to compute also the exact solution explicitly and we use it for a numerical experiment. Afterwards, we derive the order of accuracy in the general case. In Section 4 we propose an approximation formula for the bond price in the model with general form of volatilities. Finally, we conclude the paper with remarks in Section 5.

2. Convergence models of interest rates. The first convergence model of interest rates based on short rates has been suggested by Corzo and Schwartz in [4], where the domestic short rate r_d and the European short rate r_e evolve according to the following stochastic differential equations:

$$\mathrm{d}r_d = (a + b(r_e - r_d))\mathrm{d}t + \sigma_d \mathrm{d}w_1, \tag{2.1}$$

$$\mathrm{d}r_e = c(d - r_e)\mathrm{d}t + \sigma_e \mathrm{d}w_2, \tag{2.2}$$

where $b, c, \sigma_d, \sigma_e > 0$ and $a, d \in \mathbb{R}$ are constants. Here, w_1, w_2 are Wiener processes and the correlation between their increments $\mathrm{d}w_1$ and $\mathrm{d}w_2$ is a constant $\rho \in (-1, 1)$. The European short rate is modelled by a mean-revering process with a long term limit d and speed of convergence given by c. The domestic short rate reverts to the European short rate with a speed determined by b and a possible minor divergence a. The volatilities σ_d and σ_e are constant. Furthermore, the so called market prices of risk λ_d and λ_e considered in [4] are constant. Here, as well as in all subsequent computations, the unit of time is a year. We remark that although that the model has been proposed to model entering the eurozone, it makes sense also to model interest rates in a country which are affected by interest rates in an another country in a similar way.

This model uses the classical Vasicek [10] one-factor model for the European interest rate which has a closed form solution for the bond price. The price $P(r_d, r_e, \tau)$ of a domestic bond with maturity at time T (which is a parameter here) has the form

$$P(r_d, r_e, \tau) = e^{A(\tau) - D(\tau) r_d - E(\tau) r_e}, \tag{2.3}$$

where the functions $A(\tau), D(\tau), E(\tau)$ are solutions to a system of ordinary differential equations and can be expressed in a closed form, see [4]. A modification, where the European short rate follows the model by Cox, Ingersoll and Ross (CIR hereafer) [5] has been outlined in [4] and a generalization, nesting both previous case, based on the model [2] (CKLS hereafter), has been studied in [11]. In this case, the system of stochastic differential equation in the risk-neutral measure can be written as

$$dr_d = (a_1 + a_2 r_d + a_3 r_e)dt + \sigma_d r_d^{\gamma_d} dw_d, \tag{2.4}$$

$$dr_e = (b_1 + b_2 r_e)dt + \sigma_e r_e^{\gamma_e} dw_e, \tag{2.5}$$

with a constant correlation $cor(dw_d, dw_e) = \rho$. We note that the Vasicek-type model corresponds to $\gamma_d = \gamma_e = 0$ and CIR-type model corresponds to $\gamma_d = \gamma_e = 1/2$. The partial differential equation for the domestic bond price $P = P(\tau, r_d, r_e)$ reads as

$$
\begin{aligned}
&-\frac{\partial P}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial P}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial P}{\partial r_e} + \frac{1}{2}\sigma_d^2 r_d^{2\gamma_d}\frac{\partial^2 P}{\partial r_d^2} \\
&+\frac{1}{2}\sigma_e^2 r_e^{2\gamma_e}\frac{\partial^2 P}{\partial r_e^2} + \rho\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\frac{\partial^2 P}{\partial r_d \partial r_e} - r_d P = 0 \qquad (2.6)
\end{aligned}
$$

for $r_d, r_e > 0$ and time remaining to maturity $\tau \in (0, T)$ with terminal condition $P(r_d, r_e, 0) = 1$ for all $r_d, r_e > 0$. With an exception of the Vasicek-type model and the CIR-type model with zero correlation, the solution does not have a separable form of the form (2.3). In [11], an approximation formula was proposed, for which the error of the logarithms of the bond price has the order $O(\tau^4)$ as $\tau \to 0^+$.

In this paper we incorporate a dynamic correlation into the CKLS type model, i.e, the correlation between the Wiener process increments dw_d and dw_e in (2.4) and (2.5) is a function $\rho(t)$, a deterministic function of time. In this case, the constant ρ in (2.6) changes to $\rho(T-\tau)$, i.e., the partial differential equation for the bond price becomes

$$
\begin{aligned}
&-\frac{\partial P}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial P}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial P}{\partial r_e} + \frac{1}{2}\sigma_d^2 r_d^{2\gamma_d}\frac{\partial^2 P}{\partial r_d^2} \\
&+\frac{1}{2}\sigma_e^2 r_e^{2\gamma_e}\frac{\partial^2 P}{\partial r_e^2} + \rho(T-\tau)\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\frac{\partial^2 P}{\partial r_d \partial r_e} - r_d P = 0 \qquad (2.7)
\end{aligned}
$$

for $r_d, r_e > 0$ and time remaining to maturity $\tau \in (0, T)$ with terminal condition $P(r_d, r_e, 0) = 1$ for all $r_d, r_e > 0$. We note that in the Vasicek-case the range for r_d, r_e is the whole $\mathbb{R}$.

3. Dynamic correlation in Vasicek-type convergence models of interest rates. In this case, the european bond price $P_e(r_e, t, T)$ with maturity T at time t when the European short rate equals r_e, is given by classical Vasicek formula (see [10]). The price $P_d(r_d, r_e, t, T)$ of a domestic bond with maturity at time T (which is a parameter here) is a solution to the partial differential equation

$$
\begin{aligned}
&\frac{\partial P_d}{\partial t} + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial P_d}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial P_d}{\partial r_e} + \frac{1}{2}\sigma_d^2\frac{\partial^2 P_d}{\partial r_d^2} + \frac{1}{2}\sigma_e^2\frac{\partial^2 P_d}{\partial r_e^2} \\
&\qquad +\rho\sigma_d\sigma_e\frac{\partial^2 P_d}{\partial r_d \partial r_e} - r_d P_d = 0
\end{aligned}
$$

for $r_d, r_e \in \mathbb{R}$ and $t \in (0, T)$ with terminal condition $P(r_d, r_e, T, T)$ for all $r_d, r_e \in \mathbb{R}$. After the substitution $\tau = T - t$, i.e. considering time remaining to maturity of the bond, has the form

$$P_d(r_d, r_e, \tau) = A(\tau)e^{-D(\tau)r_d - E(\tau)r_e}$$

where the functions $A(\tau), D(\tau), E(\tau)$ are solutions to a system of ordinary differential equations (cf. [4] in a different parametrization for a constant ρ)

$$\dot{D}(\tau) = 1 + a_2 D(\tau) \qquad (3.1)$$

$$\dot{E}(\tau) = a_3 D(\tau) + b_2 E(\tau) \qquad (3.2)$$

$$
\begin{aligned}
\dot{A}(\tau) &= -a_1 D(\tau) - b_1 E(\tau) + \frac{1}{2}\sigma_d^2 D^2(\tau) + \frac{1}{2}\sigma_e^2 E^2(\tau) \\
&\quad +\rho(T-\tau)\sigma_d\sigma_e D(\tau)E(\tau) \qquad (3.3)
\end{aligned}
$$

with initial conditions $A(0) = 1, D(0) = 0, E(0) = 0$. In the generic case $a_2 \neq b_2, a_2, b_2 \neq 0$ the expicit solution is given by

$$D(\tau) = \frac{-1 + e^{a_2\tau}}{a_2} \tag{3.4}$$

$$E(\tau) = \frac{a_3(a_2 - a_2 e^{b_2\tau} + b_2(-1 + e^{a_2\tau}))}{a_2(a_2 - b_2)b_2} \tag{3.5}$$

$$A(\tau) = \int_0^\tau -a_1 D(s) - b_2 E(s) + \frac{1}{2}\sigma_d^2 D^2(s) + \frac{1}{2}\sigma_e^2 E^2(s) + \rho(T-s)\sigma_d\sigma_e D(s)E(s)ds \tag{3.6}$$

We note that if ρ is a constant function (the original model from [4]), also the expression for $A(\tau)$ can be written explicitly, but we leave it in the integral form for the sake of brevity.

Depending on the choice of the function $\rho(t)$, we may or may not be able to express the integral in (3.6) in a closed form. Therefore, we propose an analytical approximation formula for its computation. Although a single computation of the integral (3.6) may not be too time consuming, in a calibration procedure it has to be computed for all combinations of parameters considered in the process. Therefore, a simplified computation is valuable. Also, this approximation formula will be a base for the approximation proposed in a general CKLS type model without a separable solution in the form (2.3). We derive the order of approximation for the logarithm of the bond price. The first advantage is that it enables us to estimate the relative error in the bond price and the second that we immediately obtain the order of accuracy for the interest rates, since they are given by $-\log P/(T-t)$ where t is the current time, T is time of maturity and P is the corresponding bond price.

The main idea of the approximation lies in substituting the term $\rho(T-s)$ in the integal with $\rho(t^*)$, where t^* is a value independent of the integrating variable s. In that case, we are able to compute the integral in the closed form in the same way as in the original convergence Vasicek model. We remark that this is equivalent to considering the classical model with constant correlation equal to $\rho(t^*)$.

3.1. Approximation in a simple example of the correlation function. We consider a simple example of the correlation function $\rho(t)$, for which we wil be able to compute also the exact solution (which will be used subsequently in numerical experiments) and its Taylor expansion (in order to asses the order of accuracy of our proposed approximation). We take

$$\rho(t) = 1 - c_1 e^{-c_2 t} \tag{3.7}$$

with parameters $c_1 \in (0,1), c_2 > 0$.

We note that this choice of the correlation implies monotonically increasing correlation which converges to 1 (which would be a perfect correlation) exponentially fast. However, at this point we remark that also other choices of the correlation function might be meaningful. Figure 3.1 shows the function from (3.7) as well as two other ones. One of them has a similar exponential trend, but with added oscillations which might reflect the oscillatory character of convergence of the domestic economy to that of eurozone. The last choice of the correlation modifies (3.7) in an another way, it keeps its monotonicity, but the convergence is slower, no longer exponential.

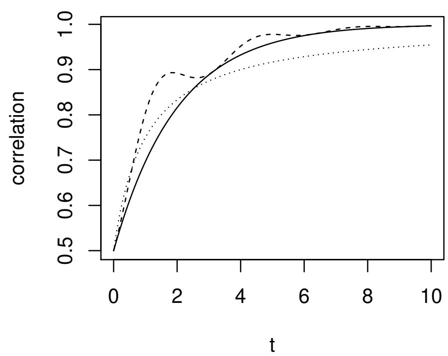

FIG. 3.1. *Three examples of a correlation function:* $\rho(t) = 1 - 0.5e^{-0.5t}$ *(solid line),* $\rho(t) = 1 - 0.25e^{-0.5t}(2 - \sin(t)^2)$ *(dashed line) and* $\rho(t) = (0.5 + t)/(1 + t)$ *(dotted lines).*

If we denote by $P^{ex}(r_d, r_e, \tau)$ the exact solution P_d and by P^{ap} the approximation based on $\rho(t^*)$, we get

$$\log P^{ex} - \log P^{ap} = \int_0^\tau \rho(T-s)\sigma_d\sigma_e D(s)E(s)ds - \int_0^\tau \rho(t^*)\sigma_d\sigma_e D(s)E(s)ds.$$

Computation of the expantions of the integrals yields

$$\log P^{ex} - \log P^{ap} = \frac{1}{8}a_3c_1\sigma_d\sigma_e e^{-c_2T-c_2t^*}(e^{c_2T} - e^{c_2t^*})\tau^4 + O(\tau^5).$$

We see that in order to achieve the highest possible order of accuracy, we should take t^* to be equal to T. In such as case, by computing the next term in the expansion, we get

$$\log P^{ex} - \log P^{ap} = -\frac{1}{10}e^{-c_2T}a_3c_1c_2\sigma_D\sigma_e\tau^5 + O(\tau^6).$$

For the numerical experiments, we take the values of the parameters from [4]. In the original formulation (2.1) and (2.2) they are equal to $a = 0.1877$, $b = 6.0639$, $\sigma_d = 0.0457$, $c = 0.1869$, $d = 0.0346$, $\sigma_e = 0.0198$, $\lambda_e = -0.655$, $\lambda_d = 3.315$, which transforms into $a_1 = 0.0362$, $a_2 = -6.0639$, $a_3 = 6.0639$, $b_1 = 0.0194$, $b_2 = -0.1869$. However, we change the constant correlation to a dynamic one. We take $\rho(t) = 1-(1-\rho_0)e^{-0.2t}$ where $\rho_0 = 0.20$ is the value estimated by [4]. It means that at time $t = 0$, the dynamic correlation equals to ρ_0 and then it increases. The value $c_2 = 0.2$ is chosen for illustration purposes. We take the short rate r_d and r_e equal to values for which both drifts in (2.1) and (2.2) are zero (so it can be seen as a real probability measure equilibrium): $r_e = 0.0346$ and $r_d = 0.0656$. For illustration, we take the time t to be equal to 2. The quantity which we compare for the exact and approximate solution are the interest rates, because they comparison is straightforward from an interpretation point of view.

In Table 3.1, we consider shorter maturities up to one quarter (we recall that the unit of time is a year), since the approximation formula is derived for $\tau \to 0^+$. We see a very good agreement between the exact interest rates and their approximations.

TABLE 3.1
Comparison of the exact and approximate interest rates in the Vasicek-type convergence model with dynamic correlation, shorter maturities up to a quarter

maturity	exact interest rate	approximation	difference
0.250	0.0539969783	0.0539968916	-8.673×10^{-8}
0.225	0.0546750248	0.0546749627	-6.210×10^{-8}
0.200	0.0554303128	0.0554302704	-4.240×10^{-8}
0.175	0.0562714980	0.0562714707	-2.724×10^{-8}
0.150	0.0572082972	0.0572082810	-1.615×10^{-8}
0.125	0.0582516256	0.0582516171	-8.573×10^{-9}
0.100	0.0594137533	0.0594137494	-3.875×10^{-9}
0.075	0.0607084817	0.0607084804	-1.356×10^{-9}
0.050	0.0621513453	0.0621513450	-2.970×10^{-10}
0.025	0.0637598388	0.0637598388	-2.063×10^{-11}

Although the error estimate is for small maturities, the approximation itself is very good also for larger maturities. Again, we take the time equal to $t = 2$ and plot the exact term structure and its approximation in Figure 3.2. Numerically, we compare the interest rates in Table 3.2

Furthermore, from the form of the solution it follows that the difference does not depend of the levels of r_d and r_e (these affect only the exact interest rate and its approximation), but only on the maturity of the bond T and the current time t.

TABLE 3.2
Comparison of the exact and approximate interest rates in the Vasicek-type convergence model with dynamic correlation, longer maturities up to 10 years

maturity	exact interest rate	approximation	difference
1	0.0490498877	0.0490465502	-3.338×10^{-6}
2	0.0522237456	0.0522108289	-1.292×10^{-5}
3	0.0561920268	0.0561667345	-2.529×10^{-5}
4	0.0599488511	0.0599105192	-3.833×10^{-5}
5	0.0633570192	0.0633062432	-5.078×10^{-5}
6	0.0664139472	0.0663520099	-6.194×10^{-5}
7	0.0691477287	0.0690762311	-7.150×10^{-5}
8	0.0715927096	0.0715133399	-7.937×10^{-5}
9	0.0737823874	0.0736967824	-8.560×10^{-5}
10	0.0757472800	0.0756569476	-9.033×10^{-5}

3.2. Appromaxion in the case of a general correlation. We show that by taking $t^* = T$ in the case of a general correlation function $\rho(t)$ we get the same order of accuracy as in the particular case from the previous subsection. In particular:

$$\log P^{ex} - \log P^{ap} = -\frac{1}{10} a_3 \sigma_D \sigma_e \rho'(T) \tau^5 + O(\tau^6).$$

We start with the expression

$$\log P^{ex} - \log P^{ap} = \int_0^\tau \rho(T-s) \sigma_d \sigma_e D(s) E(s) ds - \rho(T) \int_0^\tau \sigma_d \sigma_e D(s) E(s) ds.$$

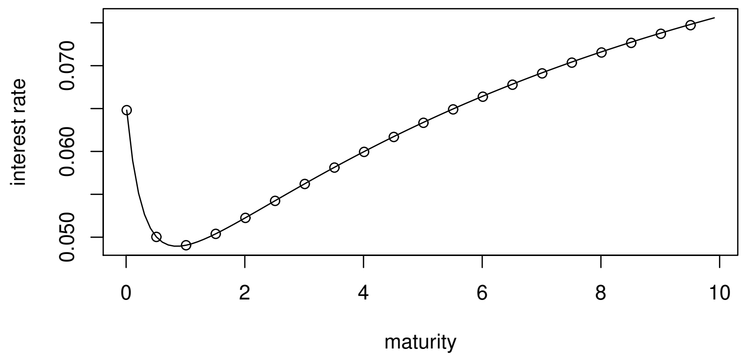

FIG. 3.2. *A graphical comparison of the exact and approximate interest rates in the Vasicek-type convergence model with dynamic correlation, longer maturities up to 10 years*

The functions D and E are known in the closed form, their product $D(s)E(s)$ is of order $O(s^3)$ as $s \to 0$ and its Taylor expansion is given by $D(s)E(s) = \frac{1}{2}a_3 s^3 + O(\tau^4)$ Similarly, we write the expansion of the term $\rho(T-s)$ around $s = 0$: $\rho(T-s) = \rho(T) - \rho'(T)s + O(s^2)$ Substituting these series yields

$$\log P^{ex} - \log P^{ap} = -\sigma_d \sigma_e \int_0^\tau \frac{1}{2} a_3 \rho'(T) s^4 + O(s^5) ds = -\frac{1}{10}\sigma_d \sigma_e a_3 \rho'(T)\tau^5 + O(\tau^6),$$

which we wanted to prove.

4. Dynamic correlation in CKLS-type convergence models of interest rates. Firstly we remark that in the case of the CKLS-type convergence model, even the European interest rate is not known in a closed form, but an approximation is needed. Possible approximations in a closed form can be found in [3], [7]. To price the domestic bond, we need to solve the partial differential equation (2.7). In our proposal of an approximation formula, we use the idea from [7] and [11] where (for a one-factor CKLS model and CKLS-type convergence model with constant correlation respectively), the Vasicek bond price was taken and its constant maturities were replaced by instantaneous maturities from the considered model. In our case, we take a Vasicek-type convergence model as the base and after substituting volatilities we replace the constant correlation by $\rho(T)$, where $\rho(t)$ is the dynamic correlation in our model and T is the maturity of the bond. This yields to the approximation formula in the separated form

$$P^{ap}(r_d, r_e, \tau) = e^{A(r_d, r_e, \tau) - D(\tau) r_d - E(\tau) r_e}, \tag{4.1}$$

where the functions D and E are given by (3.4) and (3.5) respectively; and

$$\begin{aligned} A(r_d, r_e, \tau) = \int_0^\tau & -a_1 D(s) - b_2 E(s) + \frac{1}{2}\sigma_d^2 r_d^{2\gamma_d} D^2(s) + \frac{1}{2}\sigma_e^2 r_e^{2\gamma_e} E^2(s) \\ & + \rho(T)\sigma_d \sigma_e r_d^{\gamma_d} r_e^{\gamma_e} D(s)E(s) ds \end{aligned} \tag{4.2}$$

In what follows, we derive the order of the difference $\log P^{ap} - \log P^{ex}$, where P^{ex} is the exact solution of the equation (2.7).

Firstly, we define $f^{ex} = \log P^{ex}$ and $f^{ap} = \log P^{ap}$. Then, the function f^{ex} satisfies

$$
\begin{aligned}
&-\frac{\partial f^{ex}}{\partial \tau} + (a_1 + a_2 r_d + a_3 r_e)\frac{\partial f^{ex}}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial f^{ex}}{\partial r_e} \\
&+\frac{1}{2}\sigma_d^2 r_d^{2\gamma_d}\left[\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 + \frac{\partial^2 f^{ex}}{\partial r_s^2}\right] + \frac{1}{2}\sigma_d^2 r_e^{2\gamma_e}\left[\left(\frac{\partial f^{ex}}{\partial r_e}\right)^2 + \frac{\partial^2 f^{ex}}{\partial r_s^2}\right] \\
&+\rho(T-\tau)\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\left(\frac{\partial f^{ex}}{\partial r_e}\frac{\partial f^{ex}}{\partial r_e} + \frac{\partial^2 f^{ex}}{\partial r_d \partial r_e}\right) - r_d = 0. \qquad (4.3)
\end{aligned}
$$

If we substitute the function f^{ap} into the left-hand side of (4.3), we obtain a nontrivial right-hand side which we denote by $h(r_d, r_e, \tau)$. Since the function f^{ap} is given in a closed form, we are able to compute its expansion directly. We have

$$
h(r_d, r_e, \tau) = k_3(r_d, r_e)\tau^3 + k_4(r_d, r_e)\tau^4 + O(\tau^5), \qquad (4.4)
$$

where

$$
k_3 = \frac{1}{6}\sigma_d^2\gamma_d r_d^{2\gamma_d - 2}\left(2a_1 r_d + 2a_2 r_d^2 + 2a_3 r_d r_e - r_d^{2\gamma_d} + 2\gamma_d r_d^{2\gamma_d}\sigma_d^2\right), \qquad (4.5)
$$

$$
\begin{aligned}
k_4 = {} & \frac{1}{48} r_e^{-2} r_d^{-2+2\gamma_d}\sigma_d \Big[12 a_2^2\gamma_d r_d^{2+\gamma_d} r_e^2\sigma_d - 16\gamma_d r_d^{1+3\gamma_d} r_e^2\sigma_d^3 \\
& +6a_3 b_1\gamma_e r_d^2 r_e^{1+\gamma_e}\rho(T)\sigma_e + 6a_3 b_2\gamma_e r_d^2 r_e^{2+\gamma_e}\rho(T)\sigma_e + \\
& 6a_3^2\gamma_d r_d r_e^{3+\gamma_e}\rho(T)\sigma_e - 3a_3\gamma_d r_d^{2\gamma_d} r_e^{2+\gamma_e}\rho(T)\sigma_d^2\sigma_e \\
& +3a_3\gamma_d^2 r_d^{2\gamma_d} r_e^{2+\gamma_e}\rho(T)\sigma_d^2\sigma_e + 6a_3\gamma_d\gamma_e r_d^{1+\gamma_d} r_e^{1+2\gamma_e}\rho^2(T)\sigma_d\sigma_e^2 \\
& -3a_3\gamma_e r_d^2 r_e^{3\gamma_e}\rho(T)\sigma_e^3 + 3a_3\gamma_e^2 r_D^2 r_e^{3\gamma_e}\rho(T)\sigma_e^3 \\
& +6a_1\gamma_d r_d r_e^2\left(2a_2 r_d^{\gamma_d}\sigma_d + a_3 r_e^{\gamma_e}\rho(T)\sigma_e\right) \\
& +6a_2\gamma_d r_e^2\left((-1+2\gamma_d) r_d^{3\gamma_d}\sigma_d^3 + a_3 r_d(2r_D^{2\gamma_d} r_e\sigma_d + r_d r_e^{\gamma_e}\rho(T)\sigma)\right)\Big] \\
& -\frac{1}{2}\rho'(T) a_3\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}. \qquad (4.6)
\end{aligned}
$$

Now we define the function $g(r_d, r_e, \tau) = f^{ap} - f^{ex}$; it follows that

$$
\begin{aligned}
&-\frac{\partial g}{\partial \tau} + (a_1 + a_2 r_e + a_3 r_e)\frac{\partial g}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial g}{\partial r_e} \\
&\frac{\sigma_d^2 r_d^{2\gamma_d}}{2}\left[\left(\frac{\partial g}{\partial r_d}\right)^2 + \frac{\partial^2 g}{\partial r_d^2}\right] + \frac{\sigma_e^2 r_d^{2\gamma_e}}{2}\left[\left(\frac{\partial g}{\partial r_d}\right)^2 + \frac{\partial^2 g}{\partial r_e^2}\right] \\
&+\rho(T-\tau)\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\left(\frac{\partial g}{\partial r_d}\frac{\partial g}{\partial r_e} + \frac{\partial^2 g}{\partial r_d \partial r_e}\right) \\
&= h(r_d, r_e, \tau) + \frac{\sigma_d^2 r_d^{2\gamma_d}}{2}\left[\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 + \frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ap}}{\partial r_d}\right] + \frac{\sigma_e^2 r_e^{2\gamma_e}}{2}\left[\left(\frac{\partial f^{ex}}{\partial r_e}\right)^2 + \frac{\partial f^{ex}}{\partial r_e}\frac{\partial f^{ap}}{\partial r_e}\right] \\
&+\rho(T)\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\left(2\frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ap}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ap}}{\partial r_e}\right). \qquad (4.7)
\end{aligned}
$$

We write the function g in the serie form $g(r_d, r_e, \tau) = \sum_{k=\omega}^{\infty} c_k(r_d, r_e)\tau^k$. Firstly we note that for $\tau = 0$, both the exact and the approximative price is equal to 1, and therefore $f^{ex}(r_d, r_e, 0) = 0$ and $f^{ap}(r_d, r_e, 0) = 0$. It means that $\omega > 0$ and the lowest order term on the left hand side of (4.7) is $c_\omega \omega \tau^{\omega-1}$. Therefore we need to find the order of its right hand side.

Since $f^{ex}(r_D, r_e, 0) = 0$, we have $f^{ex} = O(\tau)$ and also $\frac{\partial f^{ex}}{\partial r_d}, \frac{\partial f^{ex}}{\partial r_e}$ are $O(\tau)$. Using the formula for the approximate solution we get that $\frac{\partial f^{ap}}{\partial r_d}$, $\frac{\partial f^{ap}}{\partial r_e}$ are $O(\tau^2)$. Together with information that $h = O(\tau^3)$ we see that the right hand side of the equation (4.7) is at least of order τ^2 (it can have a higher order if the terms at τ^2 cancel). It follows that $\omega - 1 \geq 2$, i.e. $\omega \geq 3$. Therefore we have $f^{ap} - f^{ex} = O(\tau^3)$ and we can make a better estimate of $\frac{\partial f^{ex}}{\partial r_e}$:

$$\frac{\partial f^{ex}}{\partial r_e} = \frac{\partial f^{ap}}{\partial r_e} + O(\tau^3) = O(\tau^2) + O(\tau^3) = O(\tau^2)$$

Now, we make estimates of the terms on the right hand side of (4.7):

$$\left(\frac{\partial f^{ex}}{\partial r_d}\right)^2 - \frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ap}}{\partial r_d} = \frac{\partial f^{ex}}{\partial r_d}\left(\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ap}}{\partial r_d}\right) = O(\tau) \times O(\tau^3) = O(\tau^4), \tag{4.8}$$

similarly

$$\left(\frac{\partial f^{ex}}{\partial r_e}\right)^2 - \frac{\partial f^{ex}}{\partial r_e}\frac{\partial f^{ap}}{\partial r_e} = \frac{\partial f^{ex}}{\partial r_e}\left(\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ap}}{\partial r_e}\right) = O(\tau^2) \times O(\tau^3) = O(\tau^5) \tag{4.9}$$

a finally

$$\begin{aligned} &2\frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ap}}{\partial r_d}\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ex}}{\partial r_d}\frac{\partial f^{ap}}{\partial r_e} \\ = &\frac{\partial f^{ex}}{\partial r_d}\left(\frac{\partial f^{ex}}{\partial r_e} - \frac{\partial f^{ap}}{\partial r_e}\right) + \frac{\partial f^{ex}}{\partial r_e}\left(\frac{\partial f^{ex}}{\partial r_d} - \frac{\partial f^{ap}}{\partial r_d}\right) \\ &O(\tau) \times O(\tau^3) + O(\tau^2) \times O(\tau^3) = O(\tau^4). \end{aligned} \tag{4.10}$$

Since $h = O(\tau^3)$, it now follows that the right hand side of the equation (4.7) is $O(\tau^3)$ and the coefficient at τ^3 in its expansion comes only from the function h, i.e., it equals $k_3(r_d, r_e)$.

It implies that $\omega = 4$
and $-4c_4(r_d, r_e) = k_3(r_d, r_e)$ and hence $c_4(r_d, r_e) = -\frac{1}{4}k_3(r_d, r_e)$.
Therefore, the accuracy of the proposed approximation formula is given by

$$\log P^{ap}(r_d, r_e, \tau) - \log P^{ex}(r_d, r_e, \tau) = c_4(r_d, r_e)\tau^4 + O(\tau^5), \tag{4.11}$$

where

$$c_4 = -\frac{1}{24}\sigma_d^2 \gamma_d r_d^{2\gamma_d - 2}\left(2a_1 r_d + 2a_2 r_d^2 + 2a_3 r_d r_e - r_d^{2\gamma_d}\sigma_d^2 + 2\gamma_d r_d^{2\gamma_d}\sigma_d^2\right).$$

We can see that the lower order term of the error does not include the correlation function. In particular, it is the same as in the case of the constant approximation, when we approximate only volatilities, as has been done in ([11]. Therefore, when using it in the case of a dynamic correlation, the lower order term of the error comes from approximating nonconstant volatilities, and not from approximating the dynamic correlation. We can see the effect of the correlation when we compute the $O(\tau^5)$ term in the expansion of $\log P^{ap}(r_d, r_e, \tau) - \log P^{ex}(r_d, r_e, \tau)$.

In order to do this, we use the estimate $f^{ap} - f^{ex} = O(\tau^4)$ to obtain a more precise estimates (4.8) and (4.10), which are now both $O(\tau^5)$. Together with (4.9) we can see that also the only $O(\tau^4)$ term on the right hand side of the equation (4.7) comes from the function h and therefore it equals k_5. It follows that

$$-5c_5 + (a_1 + a_2 r_3 + a_d r_e)\frac{\partial c_4}{\partial r_d} + (b_1 + b_2 r_e)\frac{\partial c_4}{\partial r_e}$$
$$+\frac{\sigma_d r_d^{2\gamma_d}}{2}\frac{\partial^2 c_4}{\partial r_d^2} + \frac{\sigma_e r_e^{2\gamma_e}}{2}\frac{\partial^2 c_4}{\partial r_e^2} + \rho(T)\sigma_d r_d^{\gamma_d}\sigma_e r_e^{\gamma_e}\frac{\partial^2 c_4}{\partial r_d \partial r_e} = k_4,$$

from which we can express the coefficient c_5 from the expansion

$$\log P^{ap}(r_d, r_e, \tau) - \log P^{ex}(r_d, r_e, \tau) = c_4(r_d, r_e)\tau^4 + c_5(r_d, r_e)\tau^5 + O(\tau^6)$$

and note that it (unlike the leading term c_4) depends on the correlation function throught the terms $\rho(T)$ and $\rho'(T)$.

5. Conclusions. We introduced a dynamic correlation, already considered in other financial application formulated in terms of stochastic differential equations, into convergence models of interest rates. Firstly, we studied a Vasicek-type model, when the computation of the bond price involves a computation of an integral. We simplified this computation by an analytical approximation formula in a closed form. Moreover, this approximation served as an inspiration for the approximate analytical formula for the domestic bond prices in the CKLS-type model where no simple-form solution is available. We derived order of accuracy of the approximation formulae, including the discussion about the error terms, which do and which do not depend on the correlation function. Future work include using these formulae to fit the model to the real data, and determine (among other things) the form of the correlation $\rho(t)$ which seems to be consistent with bond prices and interest rates observed on the market.

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