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CONSTRUCTION OF LOCAL REDUCED SPACES FOR FRIEDRICHS' SYSTEMS VIA RANDOMIZED TRAINING *

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Abstract. This contribution extends the localized training approach, traditionally employed for multiscale problems and parameterized partial differential equations (PDEs) featuring locally heterogeneous coefficients, to the class of linear, positive symmetric operators, known as Friedrichs' operators. Considering a local subdomain with corresponding oversampling domain we prove the compactness of the transfer operator which maps boundary data to solutions on the interior domain. While a Caccioppoli-inequality quantifying the energy decay to the interior holds true for all Friedrichs' systems, showing a compactness result for the graph-spaces hosting the solution is additionally necessary. We discuss the mixed formulation of a convection-diffusion-reaction problem where the necessary compactness result is obtained by the Picard-Weck-Weber theorem. Our numerical results, focusing on a scenario involving heterogeneous diffusion fields with multiple high-conductivity channels, demonstrate the effectiveness of the proposed method.

Key words. multiscale methods, localized training, Friedrichs' systems

AMS subject classifications. 65N22, 65N30, 65N55, 68W20

1. Introduction. Many applications show spatially heterogeneous parameters with local fine-scale structures which are too small to be resolved numerically. Nevertheless, these structures can significantly influence the global solution behavior. Examples of such problems, known as *multiscale problems*, range from fibre-reinforced structures to composite materials and porous media, among others. In addition, we are also interested in *parameterized problems* with locally varying influence of the parameter. If solutions for many different parameter-values are required - for instance in PDE-constrained optimization problems, inverse problems, Monte-Carlo simulations or optimal control problems - conventional techniques such as the Reduced Basis Method [3] prove similarly infeasible without employing localization techniques.

As of now, there exists a wide variety of methods from the multiscale [17, 1, 15, 9], reduced basis [18, 24, 5] or domain decomposition communities [13, 14]. In this contribution we focus on multiscale methods utilizing *local approximation spaces* to incorporate the fine-scale structures. In contrast to methods that separate the solution (locally) into fine- and a coarse-scale contributions, these local spaces are designed to locally approximate the full solution. A global solution is then obtained by solving a globally coupled problem on the coarse scale using the local spaces. Depending on the chosen domain decomposition various coupling conditions are possible, we refer to [6] for an overview of established methods.

One way of constructing the approximation spaces is by means of a *localized training* procedure. Here, the problem is solved on an oversampling domain and subsequently restricted to the interior target domain. One then investigates the behavior of the transfer operator mapping arbitrary boundary values to the solution in the interior domain. Provided that this operator is compact one can then optimally

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approximate its whole range by a few selected vectors, namely its leading left singular vectors. In practice, quasi-optimal range approximations can be obtained by repeatedly applying the operator to 'normal-distributed' boundary conditions [7].

While this method has been proven to be applicable to scalar elliptic [7] and scalar parabolic [23] problems, we extend the idea to the large class of positive symmetric operators known as *Friedrichs' systems* which include scalar and non-scalar elliptic problems, as well as certain operators of hyperbolic or mixed type. For a recent contribution on data-driven model order reduction for Friedrichs' systems we refer the reader to [22]. We show that for Friedrichs' systems a Caccioppoli-inequality holds which quantifies the energy decay of solutions from the oversampling domain to the interior. Provided that the naturally occurring solution spaces admit a compact embedding into $L^2(\Omega)$, we show compactness of the transfer operator. In the last section we apply the developed theory to the elliptic case in its first-order system reformulation. Numerical experiments for a high-conductivity channel problem demonstrate the performance of the method.

2. Linear, positive symmetric PDE-operators. We start by formally defining the class of linear *Friedrichs' operators* [12] as vector-valued differential operators of the form

$$A : C^\infty(\Omega)^m \to L^2(\Omega)^m, \qquad Au \;=\; A_0 u + \sum_{i=1}^{d} A_i \frac{\partial u}{\partial x_i} \tag{2.1}$$

with matrix-valued functions $A_i \in [L^\infty(\Omega)]^{m\times m}$, $\sum_{i=1}^{d} \frac{\partial A_i}{\partial x_i} =: \nabla \cdot A \in [L^\infty(\Omega)]^{m\times m}$. Additionally, we require the following two properties:

1. $A_i = A_i^T$ for all $i = 1, \dots, d$,
2. $A_0 + A_0^T - \nabla \cdot A > 2\varepsilon I_d$ for some $\varepsilon > 0$.

We define the graph space $H(A;\Omega)$ as the space of all $L^2(\Omega)^m$-functions which possess a weak A-derivative, i.e.

$$H(A;\Omega) \;\coloneqq\; \{u \in L^2(\Omega)^m \mid Au \in L^2(\Omega)^m\}. \tag{2.2}$$

A norm on $H(A;\Omega)$ is defined by the graph-norm

$$||u||^2_{H(A)} \;\coloneqq\; ||u||^2_{L^2(\Omega)^m} + ||Au||^2_{L^2(\Omega)^m}. \tag{2.3}$$

One quickly verifies that $H^1(\Omega)^m \subseteq H(A;\Omega) \subseteq L^2(\Omega)^m$. To incorporate boundary conditions we define, following [11], the boundary operator $D : H(A) \to H(A)'$ by

$$(Du)(v) \;\coloneqq\; (Au, v)_{L^2(\Omega)^m} - (u, A^* v)_{L^2(\Omega)^m} \qquad \text{for all } u, v \in H(A). \tag{2.4}$$

For sufficiently smooth A_i one has the representation

$$(Du)(v) \;=\; \int_{\partial\Omega} v^T \mathcal{D} u \,\mathrm{d}s, \qquad \mathcal{D} \coloneqq \sum_{i=1}^{d} n_i A_i$$

where $\vec{n} = (n_1, \dots, n_d)$ denotes the unit outer normal to $\partial\Omega$. This operator D is then paired with a second, non-unique *admissible boundary operator* $M : H(A) \to H(A)'$ which needs to satisfy

1. $(Mu)(u) \;\geq\; 0$ for all $u \in H(A)$,

2. $H(A) = \ker(D - M) \ + \ \ker(D + M^*)$.

Given such an operator M we can define the closed subspace

$$H_0(A) \coloneqq \ker(D - M) \subset H(A)$$

and obtain the following well-posedness result:

THEOREM 2.1 (Well-posedness [11]). *For any $f \in L^2(\Omega)^m$, the problem*

$$\textit{Find } u \in H_0(A): \quad (Au, v)_{L^2(\Omega)^m} = (f, v)_{L^2(\Omega)^m} \qquad \forall v \in L^2(\Omega)^m. \tag{2.5}$$

is well-posed. Its solution u is the unique minimizer of the residual energy

$$\min_{u \in H_0(A)} ||Au - f||_{L^2(\Omega)^m}. \tag{2.6}$$

Instead of working with the non-symmetric problem (2.5) one can thus also solve the corresponding (symmetric) normal equation

$$\text{Find } u \in H_0(A): \quad (Au, Av)_{L^2(\Omega)^m} = (f, Av)_{L^2(\Omega)^m} \qquad \forall v \in H_0(A). \tag{2.7}$$

This approach is also known as first order system least squares, see e.g. [8, 4].

3. Optimal local approximation spaces. For coefficient fields A_i exhibiting fine-scale or locally strongly varying structure, solving (2.5) globally is not feasible. We thus employ an overlapping or non-overlapping domain decomposition $\{\Omega_i\}_{i=1}^{N_V}$ of the domain Ω and aim at approximating the restricted global solution $R_i u \coloneqq u|_{\Omega_i}$ using only local computations.

However, exactly computing $R_i u$ locally is not possible as for a well-posed local problem information about u on the local boundary $\partial\Omega_i$ is required - which is inaccessible without the actual global computation of u. This illustrates the need for a suitable approximation of the local solution via an efficiently computable (localized), small linear subspace. Following [2], one therefore defines oversampling domains $\Omega_i^* \supsetneq \Omega_i$ satisfying

$$\mathbf{dist}(\Omega_i, \partial\Omega_i^*) \ =: \ \delta > 0.$$

Given a function space $\mathcal{B}_i(\partial\Omega_i^*)$ of boundary data on the oversampling boundary, one considers the *transfer operator* $T_i : \mathcal{B}_i(\partial\Omega_i^*) \to X(\Omega_i)$ mapping some function $g \in \mathcal{B}_i$ to the solution of a local problem, restricted to the interior Ω_i. We then aim at approximating its *range* $rg(T_i) \subseteq X(\Omega_i)$ i.e. the space of local solutions for all possible boundary values in $\mathcal{B}_i(\partial\Omega_i^*)$. If T_i can be shown to be compact, the optimal approximation space R_{opt}^k of size k is given by the k leading left singular vectors. Here, optimality is measured in the sense of Kolmogorov, i.e. R_{opt}^k is a minimizer of the Kolmogorov N-width

$$d_N(rg(T_i)) \ \coloneqq \inf_{\substack{R \subset X, \\ dim(R)=N}} \sup_{\varphi \in \mathcal{B}_i} \inf_{\psi \in R} ||T_i\varphi - \psi||_X.$$

The attained minimal value is given by the first neglected singular value σ_{k+1} of T_i.

3.1. Derivation of the local problem formulation. We start the derivation of a local formulation of (2.5) by considering the restricted Friedrichs' operator $A : H(A; \Omega_i^*) \to L^2(\Omega_i^*)^m$. Defining the shared global boundary $\Gamma_i \coloneqq \partial\Omega_i^* \cap \partial\Omega$ and the internal boundary $\Gamma_i^{int} \coloneqq \partial\Omega_i^* \setminus \partial\Omega$ we can also split the boundary operators D_i, M_i as follows

$$(D_i u)(v) = (\mathcal{D}_i^{int} u, v)_{L^2(\Gamma_i^{int})} + (\mathcal{D} u, v)_{L^2(\Gamma_i)},$$
$$(M_i u)(v) = (\mathcal{M}_i^{int} u, v)_{L^2(\Gamma_i^{int})} + (\mathcal{M} u, v)_{L^2(\Gamma_i)}.$$

Note that for consistency reasons we assume the local operators D_i, M_i to coincide with the global operators D, M on the global boundary Γ_i. For a given *admissible boundary function* $\tilde{g} \in \mathcal{B}_i := rg(\mathcal{D}_i^{int} - \mathcal{M}_i^{int})$ we define the subspaces

$$H_0(A;\Omega_i^*) := \{u \in H(A;\Omega_i^*) \mid (\mathcal{D} - \mathcal{M})(u) = 0\},$$
$$H_0(A;\Omega_i^*;\tilde{g}) := \{u \in H_0(A;\Omega_i^*) \mid (\mathcal{D}_i^{int} - \mathcal{M}_i^{int})(u) = \tilde{g}\}.$$

The local weak problem then reads

$$\text{Find } u_i \in H_0(A;\Omega_i^*;\tilde{g}): \quad (Au_i, v)_{L^2(\Omega_i^*)} = (f, v)_{L^2(\Omega_i^*)} \qquad \forall v \in L^2(\Omega_i^*)^m. \tag{3.1}$$

Note, that the restricted global solution $R_i u$ is an element of $H_0(A;\Omega_i^*;\tilde{g})$ if we choose $\tilde{g} := (\mathcal{D}_i^{int} - \mathcal{M}_i^{int})(R_i u)$. This implies that for this specific choice of $\tilde{g}$ the restricted global solution $R_i u$ is recovered as the unique solution of (3.1).

Denoting by $\tilde{\mathcal{S}}_i : \mathcal{B}_i \to H(A;\Omega_i^*)$ the *solution operator* mapping a boundary function $\tilde{g}$ to the solution u_i of (3.1) we define the transfer operator T_i via

$$T_i : \mathcal{B}_i \to H(A;\Omega_i), \qquad T_i(\tilde{g}) := \tilde{\mathcal{S}}_i(\tilde{g})|_{\Omega_i}. \tag{3.2}$$

As $\tilde{\mathcal{S}}_i$ (and thus T_i) is only affine linear due to the right-hand side, we replace it with its linear part $\mathcal{S}_i := \tilde{\mathcal{S}}_i - \tilde{\mathcal{S}}_i(0)$ which is the solution operator of the *shifted* local problem

$$\text{Find } u_i \in H_0(A;\Omega_i^*;\tilde{g}): \quad (Au_i, v) = 0 \qquad \forall v \in L^2(\Omega_i^*)^m. \tag{3.3}$$

Its image, i.e. the set of all possible solutions of (3.3), shall in the following be denoted by

$$\mathcal{H}_i := \mathrm{Im}(\mathcal{S}_i) \quad \subseteq H_0(A;\Omega_i^*).$$

3.2. Compactness of the transfer operator. Showing compactness of T_i usually involves two steps, a compactness argument for the space of possible solutions $\mathcal{H}_i$ and an estimate on the energy decay from the oversampling domain Ω_i^* to the interior Ω_i. The compactness argument heavily relies on the regularizing properties of A which is why we formulate it as an assumption here:

ASSUMPTION 3.1. *$\mathcal{H}_i$ is compactly embedded in $L^2(\Omega_i^*)^m$.*

In section 5 this will be shown to hold for a general convection-diffusion-reaction operator.

PROPOSITION 3.2 (Caccioppoli inequality). *Let $A = A^0 + \sum A^i \partial_{x_i}$ be a Friedrichs operator and $u \in \mathcal{H}_i$ a solution of* (3.3)*. Then, it holds that*

$$||u||_{H(A;\Omega_i)} \leq \left(\max_i ||A^i||_\infty\right) \frac{2}{\mathrm{dist}(\Omega_i, \partial\Omega_i^*)} ||u||_{L^2(\Omega_i^*)}. \tag{3.4}$$

THEOREM 3.3 (Compactness of T_i). *Let Theorem* 3.1 *hold. Then, the transfer operator T_i is compact.*

Proof. Let $(g_k)_{k\in\mathbb{N}} \subseteq \mathcal{B}_i(\partial\Omega_i^*)$ be an arbitrary bounded sequence of admissible boundary values. Continuity of $\mathcal{S}_i$ implies that the local solutions on the oversampling domain $u_k := \mathcal{S}_i(g_k)$ are also bounded in $H(A;\Omega_i^*)$. Theorem 3.1 now guarantees the existence of a subsequence $(u_{k_j})_{j\in\mathbb{N}} \subseteq \mathcal{H}_i$ strongly-convergent in $L^2(\Omega_i^*)^m$. Using the Caccioppoli-inequality (Proposition 3.2) we can then deduce that on the interior domain Ω_i this subsequence even converges in the stronger $H(A;\Omega_i)$-norm which concludes the proof. ∎

4. Quasi-optimal approximation spaces via localized training. Since the computation of the optimal approximation spaces spanned by the singular vectors of T_i is infeasible in practice, we introduce a *localized training* procedure [7] which efficiently generates *quasi-optimal* approximation spaces by repeatedly applying T_i to 'random boundary conditions' $\tilde{g}$.

To that end, let $\mathcal{B}^h \subset \mathcal{B}$ be a discretization of the boundary functions with finite dimension $N_B \in \mathbb{N}$. Given a basis Φ_B of $\mathcal{B}^h$, we define the isomorphism $D_{\Phi_B} : \mathcal{B}^h \to \mathbb{R}^{N_B}$ mapping a function to its coefficient representation in Φ_B. Conversely, we can obtain a 'random boundary function' by sampling a vector $\underline{r} \in \mathbb{R}^{N_B}$ with normal distributed entries $\underline{r}_i \sim \mathcal{N}(0,1)$ and considering $D_{\Phi_B}^{-1}\underline{r} \in \mathcal{B}^h$.

Algorithm 1 is now obtained by combining the repeated application of T_i to random boundary conditions with an a-posteriori error estimator which is given as the projection error of an additional set of random solutions onto the current range approximation, see [7] for details. The error of the obtained range approximation R^n can then a-priori be bounded as follows:

PROPOSITION 4.1 ([7]). *Let R^n be the result of Algorithm 1. Then, for $n \geq 4$ there holds*

$$\mathbb{E}\left(||T_i - P_{R^n} T_i||\right) \;\leq\; \alpha \min_{\substack{k+p=n \\ k,\, p \geq 2}} \left[\left(1 + \sqrt{\frac{k}{p-1}}\right) \sigma_{k+1} + \frac{e\sqrt{n}}{p} \left(\sum_{j>k} \sigma_j^2 \right)^{\frac{1}{2}} \right].$$

Note, that the bound scales approximately as $\sqrt{n}\,\sigma_{n+1}$ and is thus almost optimal.

Algorithm 1 Adaptive randomized range approximation [7]

$B \leftarrow \emptyset$
sample $\underline{r}_i \sim \mathcal{N}(0,1)$
$M \leftarrow \{TD_S^{-1}\underline{r}_1, \ldots, TD_S^{-1}\underline{r}_{n_t}\}$
while $\max_{t \in M} ||t|| \cdot c_{\text{est}} > tol$ **do**
  sample $\underline{r} \sim \mathcal{N}(0,1)$
  $B \leftarrow B \cup (TD_S^{-1}\underline{r})$
  orthonormalize(B)
  $M \leftarrow \left\{ t - P_{\text{span}(B)} t \mid t \in M \right\}$
end while
return $R^n = \text{span}(B)$

5. Application to diffusion problems in mixed form. In this section we apply the developed theory to a classic convection-diffusion-reaction (CDR) problem

$$\begin{cases} -\nabla \cdot (D\nabla u) + \vec{b}\nabla u + cu &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

with $D \in L^\infty(\Omega)^{d\times d}$ symmetric positive definite, $\vec{b} \in H(\text{div}; \Omega)$, $c \in L^\infty(\Omega)$, $c - \frac{1}{2}\nabla \cdot \vec{b} \geq 0$ a.e., $g \in H^{1/2}(\partial\Omega)$.

Localized training using a classic weak formulation has been investigated e.g. in [7]. These results can be considered as a reference for the performance of our approach based on the reformulation as a Friedrichs' system: Introducing the diffusive

flux $\sigma := -D\nabla u$ one obtains the mixed formulation

$$A(\sigma, u) \coloneqq \begin{pmatrix} D^{-1}\sigma + \nabla u \\ \nabla \cdot \sigma + \vec{b}\nabla u + cu \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{5.2}$$

One can verify that A is indeed a Friedrichs' operator with graph space $H(A)$ isomorphic to $H(\mathrm{div};\Omega) \times H^1(\Omega)$ [10]. By choosing the boundary operator

$$\langle M(\sigma, u), (\tau, v)\rangle \coloneqq (\sigma\vec{n}, v)_{L^2(\partial\Omega)} - (u, \tau\vec{n})_{L^2(\partial\Omega)}$$

we obtain classic Dirichlet-boundary conditions, i.e. $H_0(A;\Omega) \cong H(\mathrm{div};\Omega) \times H_0^1(\Omega)$.

5.1. Weak formulation. We consider the shifted local weak formulation (3.3) and the associated transfer operator

$$T_i : \mathcal{B}_i \to H(A;\Omega_i).$$

PROPOSITION 5.1. *Let A be as in (5.2) and $d \leq 3$. Then, T_i is compact.*

Following Theorem 3.3 we need to show that Theorem 3.1 holds for the given operator A. While the full graph space $H(A) \cong H(\mathrm{div};\Omega) \times H^1(\Omega)$ is *not* compact in $L^2(\Omega)^m$ we can further characterize the solutions $u \in \mathcal{H}$ to obtain a compact embedding of the subspace $\mathcal{H} \subsetneq H(A)$ into $L^2(\Omega)^m$:

LEMMA 5.2. *Let $d = 3$, then for $(\sigma, u) \in \mathcal{H}$ one has $D^{-1}\sigma \in H_0(\mathrm{rot};\Omega)$.*

Proof. We have $D^{-1}\sigma = -\nabla u$ in $L^2(\Omega)^d$ and thus $\nabla \times (D^{-1}\sigma) = 0$ as the curl of a gradient field is zero. The tangential trace also vanishes as can be easily seen from the identity

$$\int_{\partial\Omega} \vec{n} \times (D^{-1}\sigma)\,\mathrm{d}s = \int_{\Omega} \nabla \times (D^{-1}\sigma)\,\mathrm{d}x = 0.$$

∎

THEOREM 5.3 (Picard-Weber-Weck [19, 25, 26]). *Let $\varepsilon \in L^\infty(\Omega)^{3\times 3}$ be symmetric and uniformly positive definite and $\Omega \subseteq \mathbb{R}^3$ be a bounded weak Lipschitz-domain. Then, the embedding*

$$H_0(\mathrm{rot};\Omega) \cap \varepsilon^{-1}H(\mathrm{div};\Omega) \to L^2(\Omega)^3$$

is compact.

THEOREM 5.4 (Rellich compactness theorem [20]). *Let $\Omega \subseteq \mathbb{R}^d$ be a open and bounded Lipschitz-domain. Then, the embedding*

$$H^1(\Omega) \to L^2(\Omega)$$

is compact.

The proof of Theorem 5.1 for $d = 3$ now follows as a direct combination of the last three statements: For any bounded sequence $(\sigma_n, u_n)_{n\in\mathbb{N}} \subseteq \mathcal{H}$ we can (due to Theorem 5.2) apply Theorem 5.3 to the sequence $(D^{-1}\sigma_n)_{n\in\mathbb{N}}$ by considering $\varepsilon \coloneqq D^{-1}$. Denoting by $(D^{-1}\sigma_{n_j})_{j\in\mathbb{N}}$ the obtained converging subsequence, boundedness of D implies the convergence of $(\sigma_{n_j})_{j\in\mathbb{N}}$ in $L^2(\Omega)^d$. The claim follows by applying Theorem 5.4 to $(u_{n_j})_{j\in\mathbb{N}}$. In dimension $d = 2$ we have

$$\mathrm{rot}_2(\sigma) \coloneqq \partial_{x_1}\sigma_2 - \partial_{x_2}\sigma_1 \in L^2(\Omega)$$

by an argument similar to Theorem 5.2. Applying Theorem 5.3 to the natural extension $(\sigma_1, \sigma_2, 0) : \Omega \to \mathbb{R}^3$ then yields the desired statement. The case $d = 1$ directly follows from Theorem 5.4 due to the identity $H(\mathrm{div};\Omega) = H^1(\Omega)$.

Test case	Ω	δ	$D(x)$	$\vec{b}$	$c(x)$
Pure diffusion	$[0,1]^2$	various	1.0	$(0,0)^T$	0
Full CDR	$[0,1]^2$	1.0	$\begin{cases} 10^2 & x \in \omega_{HC}, \\ 1.0 & \text{else.} \end{cases}$	$(1,1)^T$	$\begin{cases} 0, & x \in \omega_{HC} \\ 1.0 & \text{else.} \end{cases}$

Table 6.1: Parameters of the test cases

6. Numerical experiments. In this section we will numerically examine the proposed method for the CDR-problem (5.2). An impementation based on DUNE-PDELab[1] and instructions for reproduction are publicly available at [21].

We consider the domain $\Omega = [0,1]^2$ and corresponding oversampling domain $\Omega^* = [-\delta, 1+\delta]^2$, $\delta > 0$ which we approximate using a structured grid Ω_h^* of quadrilaterals with edge length h. For the discretization of the graph-space $H(A) \cong H(\mathrm{div};\Omega) \times H^1$ Lagrange-elements $\mathbb{P}^k(\Omega_h^*)$ of order $k = 1$ for the scalar variable and Raviart-Thomas functions $RT^{k-1}(\Omega_h^*)$ for the flux variable are used. The discrete boundary space $\mathcal{B}$ is chosen as the trace space of $\mathbb{P}^k(\Omega_h^*)$, i.e. $\mathcal{B} := \mathbb{P}^k(\partial\Omega_h^*)$.

6.1. Different oversampling sizes. First, we numerically examine possible choices of the oversampling distance δ. In most publications this distance is chosen as $\delta = \mathrm{diam}(\Omega_i)$ which amounts to an additional layer of coarse grid cells (Figure 6.2). In Figure 6.1 the influence of the oversampling distance δ in a pure diffusion problem is depicted. Here, the size N of the generated basis with the tolerance being fixed, scales approximately like $\mathcal{O}(\delta^{-1})$, reminiscent of the scaling in the Caccioppoli-inequality Theorem 3.2. As the approximation error $||T - P_N T||$ converges exponentially in N we infer the relation

$$||T - P_N T|| \ \in \ \mathcal{O}(\exp(-c\,\delta N)).$$

As the runtime of Algorithm 1 scales approximately quadratically in δ, we stick to $\delta = 1$ for the subsequent experiments which already produces sufficiently small approximation spaces.

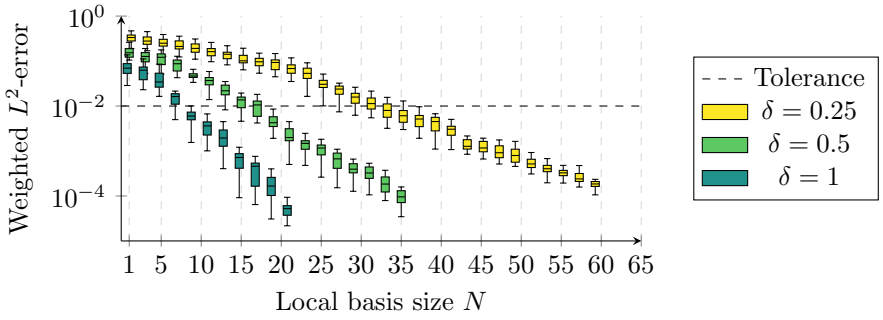


Fig. 6.1: Influence of different oversampling sizes δ in a simple diffusion test case. The same grid width $h = 1/40$ was used in all tests.

6.2. Results for the full CDR problem. We now consider the full CDR problem with diffusion, advection and reaction of comparable magnitude (Table 6.1). In addition, a subdomain $\omega_{HC} \subseteq \Omega^*$ of high-conductivity channels is introduced

[1] `https://www.dune-project.org/modules/dune-pdelab`

where the diffusion tensor D has a significantly larger magnitude. In a first test case the channels do not intersect and run parallel from left to right, whereas the second test case consists of channels in both x- and y-direction (Figure 6.3). In both configurations we evaluate the generated approximation spaces using an additional validation set of 20 solutions. As the flux variable $\sigma \approx -D\nabla u$ has, due to the high diffusivity values, a significantly larger magnitude, we consider the error norm

$$||\sigma, u||^2 \;:=\; ||D^{-1}\sigma||^2_{L^2(\Omega)^d} + ||u||^2_{L^2(\Omega)} \quad \sim \; ||\sigma, u||^2_{L^2(\Omega)^m}.$$

in order to obtain a more uniform convergence of both solution components.

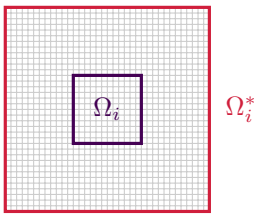


Fig. 6.2: Computational domain.

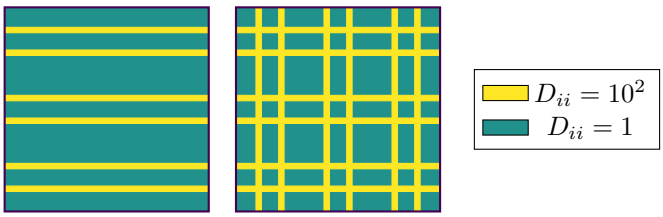
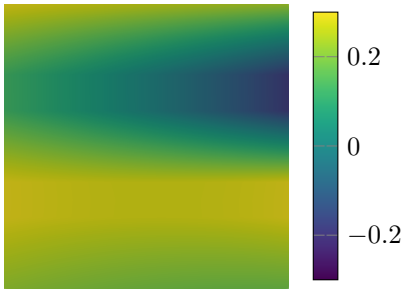
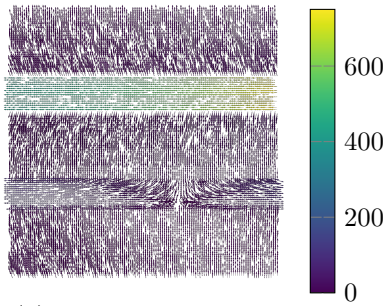


Fig. 6.3: Diagonal entries of the diffusion coefficient D_{ii} for two different test cases.

In the case of parallel channels (Figure 6.4) we first of all note the exponential decrease of the approximation error with increasing size of the computed local basis (Figure 6.5). Furthermore, there is a distinctive improvement in approximation quality at $N = 8$. We suspect that at this point the most energetic modes related to the two channels reaching the interior have all been included in the range approximation leaving only less energetic modes entering through low diffusivity regions.



(a) Restricted solution on Ω



(b) Restricted flux on Ω

Fig. 6.4: Exemplary solution to the local problem, first test case.

The second test case introduces additional channels in the vertical direction creating a lattice-like structure. This results in solutions with lower energy in the interior (Figure 6.6) as the additional connections promote the diffusion of the incoming boundary data. Evaluating the spaces generated by Algorithm 1 subsequently reveals a more rapid decrease of the approximation error (Figure 6.7).

7. Conclusion. In this contribution we evaluated the applicability of localized training methods to PDE-operators of Friedrichs' type. We showed that Caccioppoli-type estimates hold and proved compactness of the transfer operator provided that the operator exhibits sufficient smoothing properties. The method was then applied to a convection-diffusion-reaction problem where we verified the smoothing properties

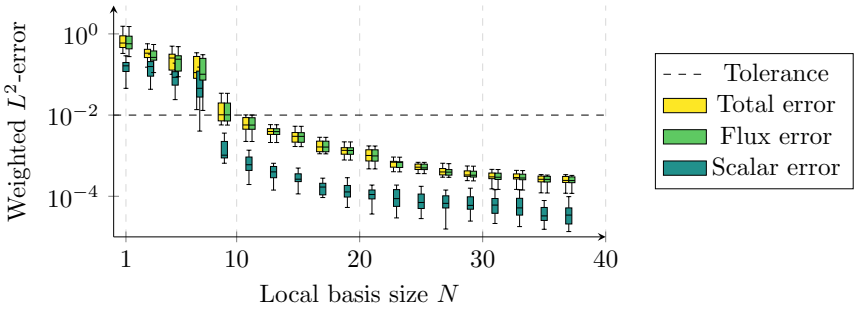
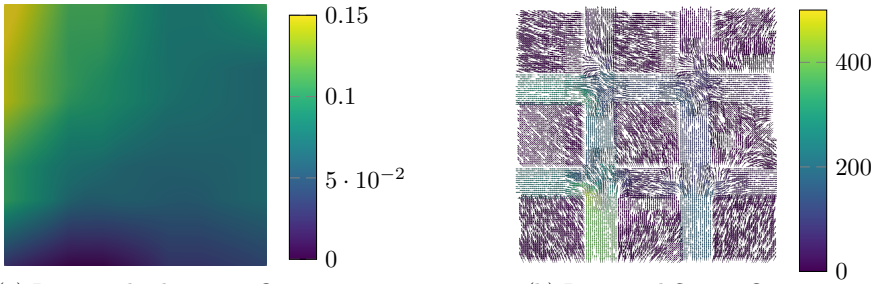


Fig. 6.5: Error in the range approximation, first test case



(a) Restricted solution on Ω (b) Restricted flux on Ω

Fig. 6.6: Exemplary solution to the local problem, second test case

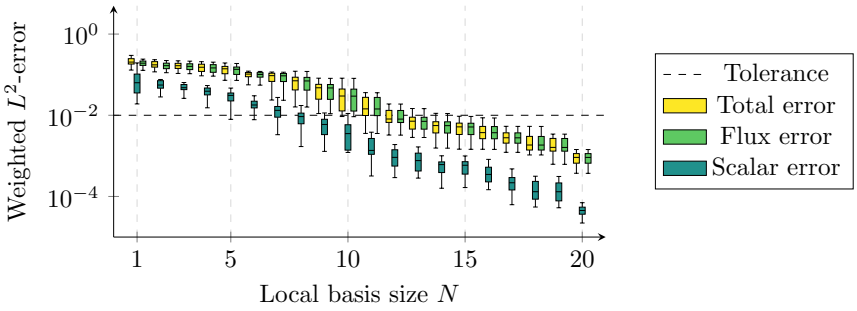


Fig. 6.7: Error in the range approximation, second test case

of the associated first-order Friedrichs' operator in the weak formulation. Employing an algorithm based on randomized numerical linear algebra then produces local spaces that closely approximate the optimal spaces spanned by the left singular vectors of the transfer operator. In a challenging test case featuring multiple high-conductivity channels, we demonstrated the numerical viability of this strategy.

Further research will integrate the generated local spaces into composite methods to solve the globally coupled problem. In order to tackle parametric problems where computing local approximation spaces for many parameters would be required, adaptive approaches using online enrichment strategies provide a promising approach. Notably, recent work proposed using a localized, residual-based error estimator to drive an adaptive algorithm which was then successfully applied to a scalar elliptic problem [16]. Future work will aim at combining this approach with our work.

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