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EVOLUTION OF SPACE CURVES BY PARAMETRIC METHOD WITH NATURAL AND UNIFORM REDISTRIBUTION

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Abstract. Space curve evolution occurs frequently in various domains of science and engineering such as computer graphics, navigation, or vortex motion. This paper focuses on the parametric method for evolving space curves by normal curvature and force. We first introduce the concept of curve evolution and its parametrization. Subsequently, we present a numerical scheme based on method of lines and show several computational studies of the forced curvature flow in space.

Key words. Space curves; curvature flow; parametric method; Frenet frame

AMS subject classifications. 35K57, 35K65, 65N40, 65M08

1. Introduction. In this text we discuss computational results of motion of a closed curve Γ_t, $t \geq 0$ in three dimensional Euclidean space (3D) according to the geometric law:

$$\partial_t \mathbf{X} = \alpha \mathbf{T} + \beta \mathbf{N} + \gamma \mathbf{B} + \mathbf{F}, \tag{1.1}$$

where $\mathbf{T}$ is the unit tangent vector, $\mathbf{N}$ the normal vector, and $\mathbf{B}$ is the binormal vector in the Frenet frame. The scalar velocities α, β, γ are smooth functions of the position vector $\mathbf{X} \in \mathbb{R}^3$, the curvature κ, and of the torsion τ. The term $\mathbf{F}$ is a known external force vector acting on Γ_t in arbitrary direction. We restrict our scope to the motion by curvature in normal direction, i.e. when $\gamma = 0$, $\beta = \kappa$, and α may serve for the redistribution of points along the curve for numerical purposes.

The motion law (1.1) is treated by the direct approach where the evolving curve is parametrized as $\Gamma_t = \{\mathbf{X}(u,t), u \in I, t \geq 0\}$ where $\mathbf{X} : I \times [0,\infty) \to \mathbb{R}^3$ is a smooth mapping, $I = \mathbb{R}/\mathbb{Z} \simeq S^1$ is the interval $I = [0,1]$ isomorphic to the unit circle S^1 corresponding to a curve which is closed.

Dynamics of space curves by a geometric motion law can be identified in many problems in science and engineering. One-dimensional structures can describe defects - dislocations - of the crystalline lattice (voids or interstitial atoms) organized along glide planes (see Hirth and Lothe [10]). The dislocations can move along the glide planes under the external stress field which can lead to the change of the glide plane - the motion becomes three-dimensional (see Devincre *et al.* [6] or Pauš *et al.* [23] or Kolář *et al.* [16]). Applications in image processing are discussed in [29, 22].

Certain class of nano-materials is produced by electrospinning - jetting polymer solutions in high electric fields into ultrafine nanofibers (see Reneker [24], Yarin *et al.* [28], He *et al.* [9]). These structures move freely in space according to (1.1) before they are collected to form the material with desired features.

Linear molecular structures can be recognized inside cells and exhibit specific dynamics in terms of (1.1) in space, which is rather a result of chemical reactions. They can interact with other structures as described in Fierling *et al.* in [8] where

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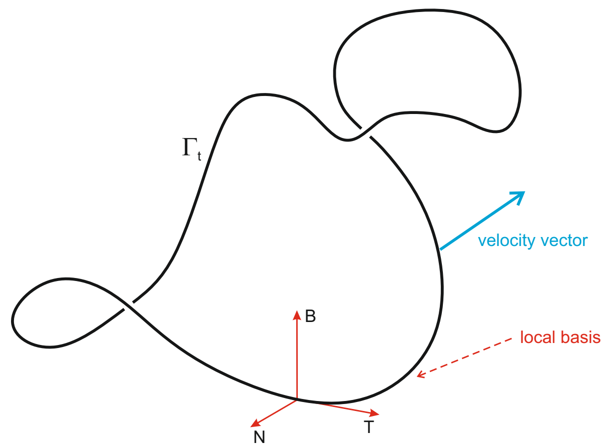


FIG. 2.1. *3D curve and the Frenet frame.*

the deformations and twist of fluid membranes by adhering stiff amphiphilic filaments have been studied.

Theoretical analysis of the motion of space curves can be found in papers by Altschuler and Grayson in [1] and [2]. Recently, this type of motion has been addressed in Jerrard and Smets in [12], Minarčík and Beneš in [21], Beneš et al. in [3]. Particular issues were numerically studied by Ishiwata and Kumazaki in [11].

2. Theoretical Background. Geometric properties of a differentiable space curve are described by the Frenet formulae linking the tangent, normal and bi-normal vectors (see 2.1):

$$\partial_s \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\kappa & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}. \tag{2.1}$$

Due to our selection of the geometric motion by curvature $\beta = \kappa$, we express the normal term as

$$\beta \mathbf{N} = \partial_s \mathbf{T}.$$

Considering the parametrization $\mathbf{X} = \mathbf{X}(u,t)$ of Γ, and the functions in (1.1) as $\alpha = \alpha(u,t)$, $\gamma = 0$, $\mathbf{F} = \mathbf{F}(u,t)$, we have

$$\partial_s \mathbf{T} = \frac{1}{|\partial_u \mathbf{X}|} \partial_u \Big(\frac{\partial_u \mathbf{X}}{|\partial_u \mathbf{X}|} \Big),$$

and express (1.1) as

$$\partial_t \mathbf{X} = \frac{1}{|\partial_u \mathbf{X}|} \partial_u \Big(\frac{\partial_u \mathbf{X}}{|\partial_u \mathbf{X}|} \Big) + \alpha(u,t)\mathbf{T} + \mathbf{F}(u,t). \tag{2.2}$$

The motion law is accompanied by the initial condition

$$\mathbf{X}(u,0) = \mathbf{X}_0(u).$$

We remark that even though (2.2) does not have the bi-normal component, the vector $\mathbf{B}$ is present in the Frenet frame and in our subsequent considerations. The discussed motion law belongs to the class of problems studied in [3]. For abbreviation, we denote $g = |\partial_u \mathbf{X}|$ and $L_t = \int_{\Gamma_t} |\partial_u \mathbf{X}| du$, and we summarize its properties and recall important relations. For future purposes, we also project the force term to the normal, bi-normal and tangential direction

$$v_N = \kappa + \mathbf{F} \cdot \mathbf{N},\ v_B = \mathbf{F} \cdot \mathbf{B},\ v_T = \alpha + \mathbf{F} \cdot \mathbf{T}.$$

We can then recall the following

PROPOSITION 2.1. *For a smooth space curve Γ_t evolving according to (2.2), the following identities hold*

$$\partial_t g = -g\kappa v_N + \partial_u v_T, \tag{2.3}$$

$$\partial_t L_t = -\int_{\Gamma_t} \kappa_\Gamma v_N + \partial_s v_T ds. \tag{2.4}$$

Proof. From the proof presented already in [3], we underline several important facts below. The differentiation is derived from (1.1) with general form of v_T, v_N, and v_B, i.e. including the bi-normal term and the force.

$$\partial_t \partial_u \mathbf{X} \quad = \quad \partial_u(v_N)\mathbf{N} + v_N \partial_u(\mathbf{N}) + \partial_u(v_T)\mathbf{T} + v_T \partial_u(\mathbf{T}) + \partial_u(v_B)\mathbf{B} + v_B \partial_u(\mathbf{B}).$$

The Frenet formulae yield

$$\partial_t \partial_u \mathbf{X} \quad = \quad (-g v_N \kappa + \partial_u v_T)\mathbf{T} + (g v_N \tau + \partial_u v_B)\mathbf{B} + (\partial_u v_N + v_T g\kappa - g\tau v_B)\mathbf{N}.$$

From

$$\partial_t g = \partial_t |\partial_u \mathbf{X}| = \frac{\partial_u \mathbf{X} \cdot \partial_t \partial_u \mathbf{X}}{g}, \tag{2.5}$$

it follows, due to perpendicularity, that

$$\partial_t g = -v_N g\kappa + \partial_u v_T. \tag{2.6}$$

The length decay rate follows from

$$\partial_t L_t \quad = \quad \partial_t \int_{\Gamma_t} ds = \int_0^1 \partial_t g du.$$

□

2.1. The role of redistribution along the curve. For stability of computational algorithms as well as for analytical purposes, the term g should also be bounded from below. This is, in general, not guaranteed for motion law (1.1). We first mention a modified expression for the normal curvature term presented for planar curves in [5], known as the deTurck trick [18] and used e.g. in [22, 14].

As the expression

$$\frac{\partial^2_{uu} \mathbf{X}}{|\partial_u \mathbf{X}|^2}$$

can be decomposed into the normal and tangential part, we have

$$\frac{\partial^2_{uu} \mathbf{X}}{|\partial_u \mathbf{X}|^2} = \frac{1}{|\partial_u \mathbf{X}|} \partial_u \Big(\frac{\partial_u \mathbf{X}}{|\partial_u \mathbf{X}|} \Big) + \frac{((\partial_u \mathbf{X}) \cdot \partial_{uu} \mathbf{X}) \partial_u \mathbf{X}}{|\partial_u \mathbf{X}|^4}. \tag{2.7}$$

The tangential term serves for a natural redistribution along space curves in numerical schemes for space curves as explained below. We then consider the modified motion law

$$\partial_t \mathbf{X} = \frac{\partial^2_{uu}\mathbf{X}}{|\partial_u \mathbf{X}|^2} + \alpha(u,t)\mathbf{T} + \mathbf{F}(u,t), \tag{2.8}$$

$$\mathbf{X}(u,0) = \mathbf{X}_0(u). \tag{2.9}$$

As introduced in literature, an active tool in redistribution along the curve useful under extreme force acting on curves externally is the uniform redistribution (see e.g. [3], [25, 19, 20], [30]). For this purpose, a function of ratio of interest

$$\theta(u,t) = \ln(\frac{g(u,t)}{L_t})$$

is treated. As

$$\partial_t \theta(u,t) = \frac{\partial_t g(u,t)}{g(u,t)} - \frac{\partial_t L_t}{L_t},$$

and $\int_{\Gamma_t} \partial_s v_T ds = 0$ due to periodicity of v_T, the time derivative $\partial_t \theta$ is expressed using Proposition 2.1

$$\partial_t \theta(u,t) = -v_N \kappa + \partial_s \alpha + \frac{\int_{\Gamma_t} v_N \kappa ds}{L_t}.$$

Imposing a requirement $\partial_t \theta(u,t) = 0$ we get

$$\partial_u \alpha = g v_N \kappa - \frac{\int_{\Gamma_t} v_N \kappa ds}{L_t}.$$

For the integration, we assume the conditions of periodicity over S^1 and $\alpha(t,0) = 0$. Uniformity and rate of asymptotical redistribution can be controlled by a parameter $\omega \geq 0$ in a modified expression

$$\partial_u \alpha = g v_N \kappa - \frac{\int_{\Gamma_t} v_N \kappa ds}{L_t} + \omega\big(\frac{L_t}{g} - 1\big), \tag{2.10}$$

as suggested in [19, 20].

2.2. Forced curvature flow. The forced curvature flow, a very common motion in plane (see [15]), along surfaces (see [13]) or along planes in space (see [16]), frequently considers the force given in the direction of the normal vector given by the two-dimensional Frenet frame with both signs of curvature allowed.

Space curves can evolve by an analogue of this force type under limited conditions only, which are given by features of the three-dimensional Frenet frame with non-negative curvature. For a smooth space curve, it may happen that the Frenet frame flips over at some part of it due to zero values of curvature in vicinity of such points (see [7]).

When the force term $\mathbf{F} = \mathbf{F}(t, \mathbf{X})$ is just a smooth function of time and space, it acts on the curve independently of the local orientation of the Frenet frame. However, it is projected to it in the formula for the uniform redistribution (2.10). Such a force has been considered in e.g. [21, 3, 17].

We therefore provide some computational examples related to this issue and open this part of the curve motion for future research.

2.3. Discretization. Motion laws (2.2), (2.7) are discretized by the flowing finite volume method as in [4, 3]. The curve Γ_t is discretized by m nodes $\mathbf{X}_j(t) = \mathbf{X}(t, jh)$, $j = 0, 1, 2, \ldots, m$ with $h = 1/m$. Related quantities expressed or approximated at these nodes are denoted correspondingly. The curvature expression in (2.2) is approximated as

$$\frac{1}{|\partial_u \mathbf{X}|}\partial_u\Big(\frac{\partial_u \mathbf{X}}{|\partial_u \mathbf{X}|}\Big)|_j \approx \mathbf{K}_j = \frac{2}{|\mathbf{X}_{j-1} - \mathbf{X}_{j-1}|}\left(\frac{\mathbf{X}_{j+1} - \mathbf{X}_j}{|\mathbf{X}_{j+1} - \mathbf{X}_j|} - \frac{\mathbf{X}_j - \mathbf{X}_{j-1}}{|\mathbf{X}_j - \mathbf{X}_{j-1}|}\right). \tag{2.11}$$

The deTurck expression for the curvature in (2.7) is approximated as

$$\frac{\partial^2_{uu}\mathbf{X}}{|\partial_u \mathbf{X}|^2}|_j \approx \mathbf{K}^T_j = 4\frac{\mathbf{X}_{j+1} - 2\mathbf{X}_j + \mathbf{X}_{j-1}}{|\mathbf{X}_{j-1} - \mathbf{X}_{j-1}|^2}. \tag{2.12}$$

The tangent vector is approximated as

$$\mathbf{T}|_j \approx \frac{\mathbf{X}_{j+1} - \mathbf{X}_{j-1}}{|\mathbf{X}_{j-1} - \mathbf{X}_{j-1}|}. \tag{2.13}$$

The normal vector explicitly appears in the redistribution formula, and, as mentioned above, may be part of the formula for the force $\mathbf{F}$. One has to be aware of the fact that it is available only for the parts of the curve, where $\kappa \neq 0$, i.e. $\frac{1}{|\partial_u \mathbf{X}|}\partial_u\Big(\frac{\partial_u \mathbf{X}}{|\partial_u \mathbf{X}|}\Big) \neq 0$. Then

$$\mathbf{N}|_j = |\left(\frac{\mathbf{X}_{j+1} - \mathbf{X}_j}{|\mathbf{X}_{j+1} - \mathbf{X}_j|} - \frac{\mathbf{X}_j - \mathbf{X}_{j-1}}{|\mathbf{X}_j - \mathbf{X}_{j-1}|}\right)|^{-1}\left(\frac{\mathbf{X}_{j+1} - \mathbf{X}_j}{|\mathbf{X}_{j+1} - \mathbf{X}_j|} - \frac{\mathbf{X}_j - \mathbf{X}_{j-1}}{|\mathbf{X}_j - \mathbf{X}_{j-1}|}\right)$$

The redistribution formula is approximately solved by denoting

$$\kappa_j = \frac{2}{|\mathbf{X}_{j-1} - \mathbf{X}_{j-1}|}\left|\frac{\mathbf{X}_{j+1} - \mathbf{X}_j}{|\mathbf{X}_{j+1} - \mathbf{X}_j|} - \frac{\mathbf{X}_j - \mathbf{X}_{j-1}}{|\mathbf{X}_j - \mathbf{X}_{j-1}|}\right|,$$

$$g_j = |\mathbf{X}_j - \mathbf{X}_{j-1}|, \quad v_{N,j} = \kappa_j + \mathbf{F}_j \cdot \mathbf{N}|_j,$$

$$I_m = \sum_{j=1}^{m} g_j v_{N,j}\kappa_j, \quad L_m = \sum_{j=1}^{m} g_j,$$

and expressing the summation

$$\alpha_{j+1} = \alpha_j + g_j v_{N,j}\kappa_j - \frac{I_m}{L_m}, \quad \text{for } j = 0, \ldots, m-1, \quad \alpha_0 = 0.$$

We then solve, alternatively, the initial-value problems

$$\begin{aligned} \frac{d\mathbf{X}_j}{dt} &= \mathbf{K}_j + \alpha_j \frac{\mathbf{X}_{j+1} - \mathbf{X}_{j-1}}{|\mathbf{X}_{j-1} - \mathbf{X}_{j-1}|} + \mathbf{F}_j, \quad \text{for } j = 1, \ldots, m \\ \mathbf{X}_m &= \mathbf{X}_0, \\ \mathbf{X}_j|_{t=0} &= \mathbf{X}_0(jh), \quad \text{for } j = 1, \ldots, m \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} \frac{d\mathbf{X}_j}{dt} &= \mathbf{K}^T_j + \alpha_j \frac{\mathbf{X}_{j+1} - \mathbf{X}_{j-1}}{|\mathbf{X}_{j-1} - \mathbf{X}_{j-1}|} + \mathbf{F}_j, \quad \text{for } j = 1, \ldots, m \\ \mathbf{X}_m &= \mathbf{X}_0, \\ \mathbf{X}_j|_{t=0} &= \mathbf{X}_0(jh), \quad \text{for } j = 1, \ldots, m \end{aligned} \tag{2.15}$$

Both systems of ordinary differential equations are numerically solved using the Runge–Kutta–Merson method as in [26, 27, 3].

3. Case studies. We present several qualitative computational examples of space curve dynamics showing curve shortening, effect of redistribution and external force, all obtained by schemes (2.11), (2.15). The schemes are convergent, a detailed numerical study is beyond the limits of this text.

Example 1 - curve shortening of a space curve. The initial curve is parametrized as

$$\begin{aligned} r &= \frac{1}{\sqrt{1+16\cos^2(12\pi u)}}, \\ x(0,u) &= r\cos(2\pi u), \\ y(0,u) &= r\sin(2\pi u), \\ z(0,u) &= r\cos(12\pi u), \quad u \in \langle 0,1 \rangle. \end{aligned}$$

The dynamics has been computed by scheme (2.15) with natural redistribution. Fig 3.1 shows the curve dynamics attaining a planar position and shrinking to a point.

Example 2 - natural and uniform redistribution. The initial curve is parametrized as

$$\begin{aligned} r &= \frac{1}{\sqrt{1+\cos^2(12\pi u)}}, \\ x(0,u) &= r\cos(2\pi u), \\ y(0,u) &= r\sin(2\pi u), \\ z(0,u) &= r\cos(12\pi u), \quad u \in \langle 0,1 \rangle. \end{aligned}$$

The dynamics has been computed by schemes (2.15), and (2.14) with redistribution. Fig 3.2 illustrates the comparison of the natural redistribution and the uniform redistribution.

Example 3 - forced curvature flow. The initial curve is parametrized as

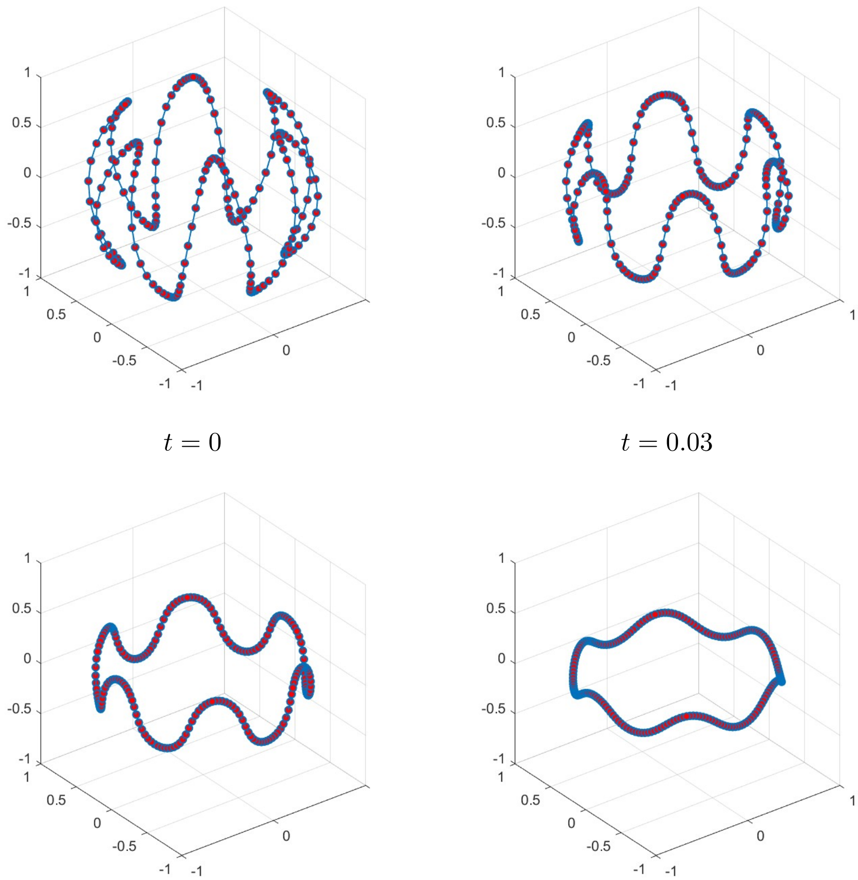
$$\begin{aligned} x(0,u) &= \cos(2\pi u), \\ y(0,u) &= \sin(2\pi u), \\ z(0,u) &= 0.8\sin^2(2\pi u), \quad u \in \langle 0,1 \rangle. \end{aligned}$$

The dynamics has been computed by scheme (2.15) with natural redistribution. The force is $\mathbf{F} = -4\mathbf{N}_n$, where $\mathbf{N}_n$ is the normalized projection of the vector $\mathbf{N}$ of Γ_t to the horizontal plane. Fig 3.3 shows the dynamics under which the curve is brought by the force to a planar shape. However the originally strictly positive curvature becomes zero at some points and the projection of the curve to the plane xy is not convex.

Example 4 - forced curvature flow with nonzero curvature. The initial curve is parametrized as

$$\begin{aligned} x(0,u) &= \cos(2\pi u), \\ y(0,u) &= \sin(2\pi u), \\ z(0,u) &= 0.2\sin^2(2\pi u), \quad u \in \langle 0,1 \rangle. \end{aligned}$$

The dynamics has been computed by scheme (2.15) with natural redistribution. The force is $\mathbf{F} = -1.2\mathbf{N}_n$, where $\mathbf{N}_n$ is the normalized projection of the vector $\mathbf{N}$ of Γ_t to the plane rotated 45 degrees along the y-axis. Fig 3.4 shows the dynamics under which the curve is brought by the force to a planar shape. The originally strictly positive curvature is kept during the evolution.



$t = 0$ $t = 0.03$

$t = 0.06$ $t = 0.09$

FIG. 3.1. **Example 1.** *Curve shortening dynamics computed on the time interval* $[0, 0.09]$ *for the curve discretized to* $m = 200$ *segments.*

4. Conclusion. Computational studies of space curve dynamics were presented. Natural and uniform redistribution were compared. Challenges of forced space curve dynamics were indicated, which motivate future directions of research.

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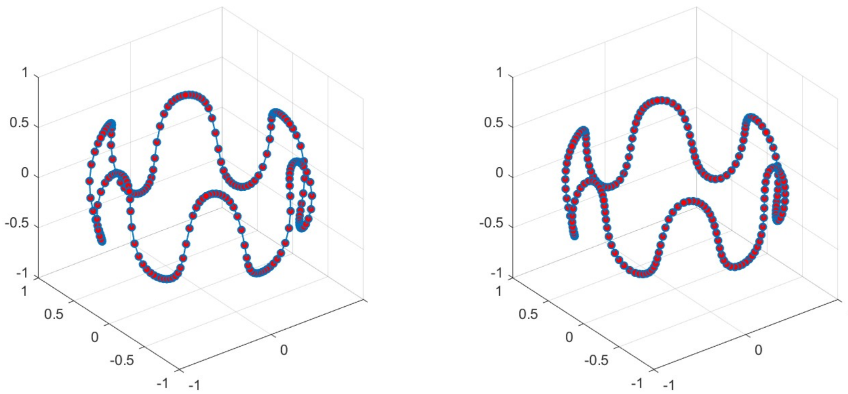

with natural redistribution with uniform redistribution

FIG. 3.2. **Example 2.** *Comparison of numerical solution by (2.15) on the left, and by (2.14) with redistribution on the right for* $m = 200$ *at the time level* $t = 0.030$.

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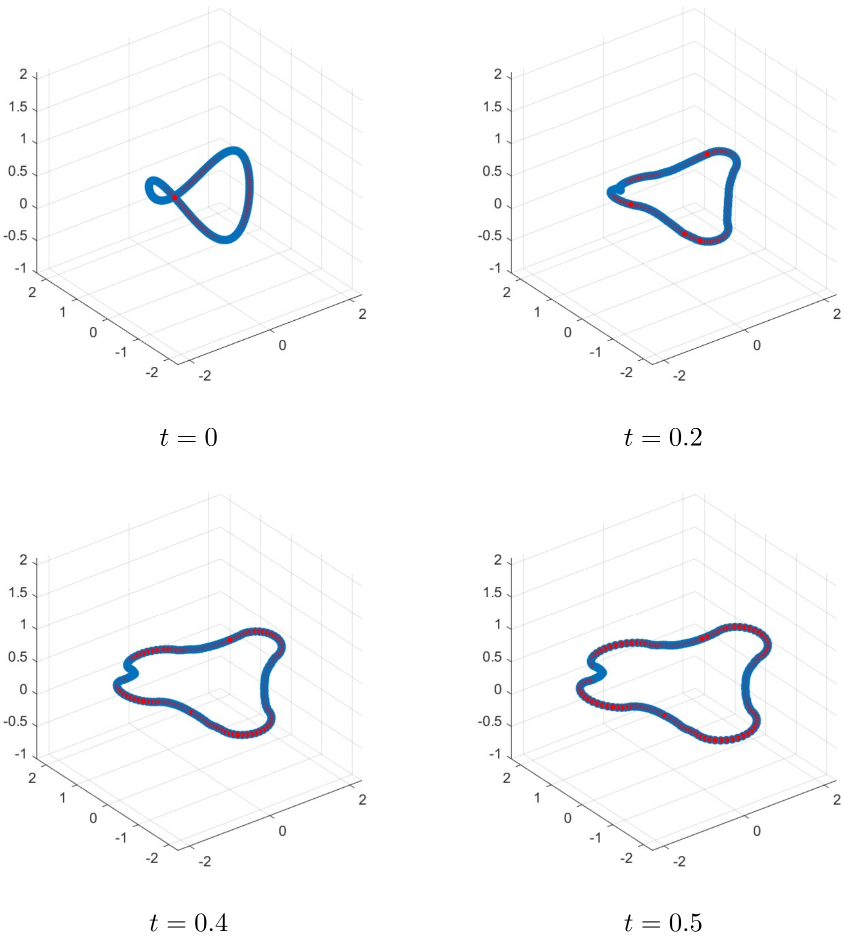

Fig. 3.3. **Example 3.** *Forced curvature flow computed on the time interval* $[0, 0.5]$ *for the curve discretized to* $m = 200$ *segments.*

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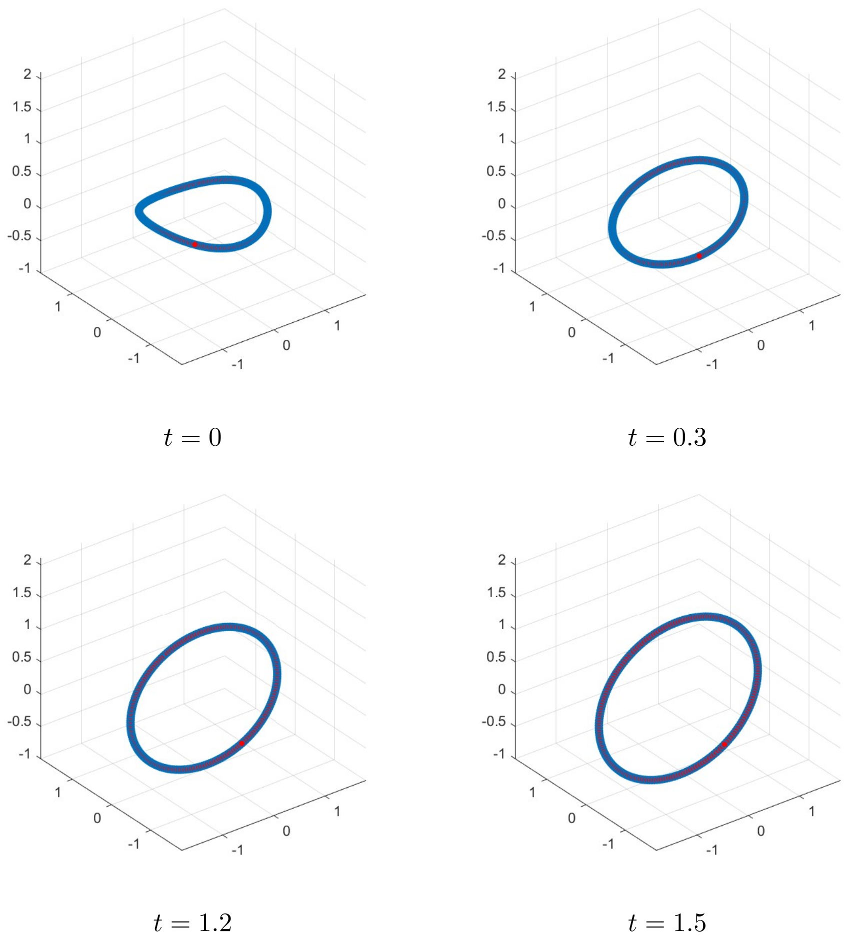

Fig. 3.4. **Example 4.** *Forced curvature flow computed on the time interval* $[0, 1.5]$ *for the curve discretized to* $m = 200$ *segments.*

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