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DERIVATION OF A CURVATURE-DEPENDENT KURAMOTO–SIVASHINSKY EQUATION

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Abstract. A curvature-dependent Kuramoto–Sivashinsky equation is derived as a model of flame spreading motion on a cylinder. We start from a three-phases (gas-solid-gas) reaction-diffusion system defined on a cylinder and reduce it to a two-dimensional system by averaging over the thickness. This two-dimensional model includes curvature dependence diffusion coefficients. Matched asymptotic expansion is conducted on the two-dimensional model to construct a traveling wave solution and determine the interface corresponding to the flame front. Finally, a linear stability analysis of the travelling wave solution and geometric arguments to the motion of the front lead to a curvature-dependent Kuramoto–Sivashinsky equation, which is a generalization of a known result obtained by Kagan & Sivashinsky [3].

Key words. curvature dependence Kuramoto–Sivashinsky equation, matched asymptotic expansion, combustion, dispersion relation

AMS subject classifications. 80A25, 35K57, 35R37

1. Introduction.

The goal of this paper is to derive the following curvature-dependent Kuramoto–Sivashinsky equation:

$$f_t + (\alpha(\kappa) - \alpha_c) f_{xx} + 4 f_{xxxx} + \frac{1}{2} f_x^2 = 0, \tag{1.1}$$

where $\alpha(\kappa)$ is a prescribed function of the curvature of a given accordion folded paper (see Fig. 1.1, below) and α_c is a parameter. Here and hereafter, we put $\mathsf{F}_t = \partial \mathsf{F}/\partial t$, $\mathsf{F}_x = \partial \mathsf{F}/\partial x$, $\mathsf{F}_{xx} = \partial^2 \mathsf{F}/\partial x^2$, and so forth.

It is well known that in solid combustion, the flame spreading rate depends on the shape of the combustible material ([4, 8, 9]). In particular, an experiment reported in [8] involves burning paper folded in an accordion manner, standing vertically, and igniting it from the top down, parallel to the direction of the folds (Fig. 1.1). From this experiment, we can observe that the flame spreading rate depends on the number of folds (i.e., the fold width). These observations suggest the possibility of controlling the flame spreading rate through the shape of the paper, which motivates this study.

This paper aims to derive a combustion model that depends on the curvature of the region. The Kuramoto–Sivashinsky (KS) equation ([1, 2]) is well-known as a mathematical model for gas combustion. It has also been derived by [3] as a model for flame fronts on a thin solid fuel, which is contained in a narrow space between two parallel plates and burns against a forced oxidizing convective flux. The accordion region has angles, making it analytically challenging. Formulating it on a general one-dimensional graph is also technically difficult. Therefore, in this paper, we extend the methodology of [3] to a cylindrical region and investigate how the curvature of the region appears in the KS equation.

This paper is organized as follows. In the next section, we prepare a 3-dimensional reaction-diffusion system on the cylinder, which leads to a three-layer gas-solid-gas

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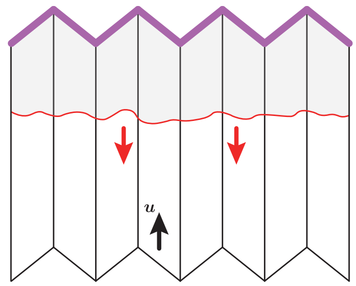

FIG. 1.1. *Schematic figure of the accordion folded paper. The gray (resp. white) color part indicates the burned (resp. unburned, i.e., the paper) region and the interface between them is the combustion flame front (the red curve). The flame front moves downward (red arrows), while the buoyancy flow goes upward (black arrow).*

system (Fig. 2.1). We consider the case where ignition occurs from above, and the flame front propagates downward. In this scenario, the buoyant flow opposes the direction of the frame spread, similar to the setup described in [3]. To reduce the model to a 2-dimensional one, we adopt depth-averaging across the width of each phase and approximation of diffusive-thermal equilibrium in §3. §4 is devoted to constructing a planar traveling wave with matched asymptotic expansions and to derive the dispersion relation, which determines the coefficients of the linear terms of (1.1). In §5, by semi-heuristic argument from a geometric point of view, we derive the curvature-dependent KS equation (1.1) and the coefficient of the nonlinear term of it. Finally, a summary of the presence paper and a future work are given in §6.

2. Reaction-diffusion system on cylinder. Put $O_\pm = R \pm (d_s + d_g)$, $I_\pm = R \pm d_s$ and denote the 3-phases (gas-solid-gas) cylindrical domain (see Fig. 2.1) as the union of $\Omega_{g_1} = \{(x, r, \theta);\, x \in \mathbb{R},\ I_+ < r < O_+,\ -\pi \le \theta < \pi\}$, $\Omega_{g_2} = \{(x, r, \theta);\, x \in \mathbb{R},\ O_- < r < I_-,\ -\pi \le \theta < \pi\}$, $\Omega_s = \{(x, r, \theta);\, x \in \mathbb{R},\ I_- < r < I_+,\ -\pi \le \theta < \pi\}$, $\Gamma_{g_1} = \{(x, r, \theta);\, x \in \mathbb{R},\ 0 \le \theta < 2\pi,\ r = O_+\}$, $\Gamma_{s_1} = \{(x, r, \theta);\, x \in \mathbb{R},\ 0 \le \theta < 2\pi,\ r = I_+\}$, $\Gamma_{s_2} = \{(x, r, \theta);\, x \in \mathbb{R},\ 0 \le \theta < 2\pi,\ r = I_-\}$, and $\Gamma_{g_2} = \{(x, r, \theta);\, x \in \mathbb{R},\ 0 \le \theta < 2\pi,\ r = O_-\}$. The set of governing equations is

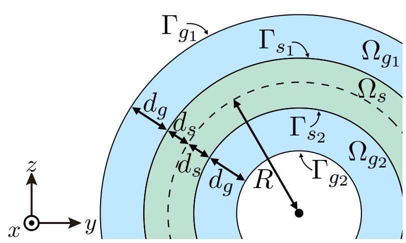

FIG. 2.1. *A schematic picture from the top view of the cylindrical domain.*

$$
\begin{aligned}
\rho_g c_g \left(T_{g_i,t} + \boldsymbol{u} \cdot \nabla_{x,r,\theta} T_{g_i}\right) &= \lambda_g \Delta_{x,r,\theta} T_{g_i}, \quad (x, r, \theta) \in \Omega_{g_i}, \quad t > 0 \quad (i = 1, 2),\\
C_{i,t} + \boldsymbol{u} \cdot \nabla_{x,r,\theta} C_i &= D \Delta_{x,r,\theta} C_i, \quad (x, r, \theta) \in \Omega_{g_i}, \quad t > 0 \quad (i = 1, 2),\\
\rho_s c_s T_{s,t} &= \lambda_s \Delta_{x,r,\theta} T_s, \quad (x, r, \theta) \in \Omega_s, \quad t > 0,
\end{aligned}
$$

where $T_{g_i} = T_{g_i}(x, r, \theta, t)$, $C_i = C_{g_i}(x, r, \theta, t)$, $T_s = T_s(x, r, \theta, t)$, $\boldsymbol{u} = (u, 0, 0)$ $(u > 0)$, $\nabla_{x,r,\theta} = (\partial_x, \cos\theta\, \partial_r - r^{-1} \sin\theta\, \partial_\theta, \sin\theta\, \partial_r + r^{-1} \cos\theta\, \partial_\theta$ and $\Delta_{x,r,\theta} = \partial_x^2 + \partial_r^2 + r^{-1}\partial_r +$

$r^{-2}\partial_\theta^2$. Here, the subscript labels g_i $(i=1,2)$ and s stand for gaseous and solid phases, respectively.

The law of conservation of heat and the mass on Γ_{s_i} $(i=1,2)$ are

$$\begin{aligned} \lambda_g T_{g_1,r} - \lambda_s T_{s,r} &= -QW(T_s, C_s), \qquad \rho_g D C_{1,r} = W(T_s, C_s) \quad \text{on } \Gamma_{s_1}, \\ -\lambda_g T_{g_2,r} + \lambda_s T_{s,r} &= -QW(T_s, C_s), \quad -\rho_g D C_{2,r} = W(T_s, C_s) \quad \text{on } \Gamma_{s_2}, \end{aligned} \tag{2.1}$$

where $W(T,C) = AC\exp(-T_a/T)$ with positive constants A and T_a. We assume the continuity for temperature on Γ_{s_i} as

$$T_{g_i}(x,r,\theta,t) = T_s(x,r,\theta,t) \quad \text{on } \Gamma_{s_i} \quad (i=1,2). \tag{2.2}$$

Furthermore, we denote the values on Γ_{s_i} and Γ_{g_i} $(i=1,2)$ as

$$\begin{aligned} \bar{C}_{s_i}(x,\theta,t) &= C_i(x,r,\theta,t), \quad \bar{T}_{s_i}(x,\theta,t) = T_s(x,r,\theta,t) \quad \text{on } \Gamma_{s_i}, \\ \bar{C}_{g_i}(x,\theta,t) &= C_i(x,r,\theta,t), \quad \bar{T}_{g_i}(x,\theta,t) = T_{g_i}(x,r,\theta,t) \quad \text{on } \Gamma_{g_i}. \end{aligned}$$

We give the adiabatic boundary condition at the walls

$$T_{g_i,r} = 0, \ C_{i,r} = 0 \quad \text{on } \Gamma_{g_i} \quad (i=1,2), \tag{2.3}$$

and the upstream/downstream boundary conditions

$$\begin{aligned} T_{g_i} &= T_u, \quad T_s = T_u, \quad C_i = C_u \quad \text{at } x = -\infty \quad (i=1,2), \\ T_{g_i,x} &= 0, \quad T_{s,x} = 0, \quad C_{i,x} = 0 \quad \text{at } x = \infty \quad (i=1,2). \end{aligned}$$

The unknown functions T and C stand for the temperature and the volumetric mass fractions of the deficient gaseous reactant (oxygen), respectively. In the above system, λ is the thermal conductivity, D is the molecular diffusivity, ρ is the constant density, c is the specific heat, Q is the heat release, W is the reaction rate with the Arrhenius temperature dependence, A is the pre-exponential factor, T_a is the activation temperature, d is the width of the gaseous/solid layer, and $\boldsymbol{u} = (u,0,0)$ is the prescribed constant flow-field. For T_u and C_u, the subscript u means the unburned (initial) state of the system prior to ignition.

3. Reduction to a 2-dimensional system. We first approximate the model to a 2-dimensional, say $\hat{T}_i$ and $\hat{C}_i$, where the hat indicates quantities depth-averaged across the width of each phase. For this purpose, we approximate the transversal r-profiles of the temperature and concentration by second-order polynomials adapted to the conditions (2.2) and (2.3):

$$\begin{aligned} T_{g_i}(x,r,\theta,t) &= \bar{T}_{g_i}(x,\theta,t) + \left(\bar{T}_{s_i}(x,\theta,t) - \bar{T}_{g_i}(x,\theta,t)\right) H_{g_i}^2 \quad (i=1,2), \\ C_i(x,r,\theta,t) &= \bar{C}_{g_i}(x,\theta,t) + \left(\bar{C}_{s_i}(x,\theta,t) - \bar{C}_{g_i}(x,\theta,t)\right) H_{g_i}^2 \quad (i=1,2), \\ T_s(x,r,\theta,t) &= T_s(x,R,\theta,t) + \phi_1(r-R) + \phi_2(r-R)^2, \end{aligned}$$

where $H_{g_1} = 1 + (R + d_s - r)/d_g$, $H_{g_2} = 1 - (R - d_s - r)/d_g$, $\phi_1 = (\bar{T}_{s_1} - \bar{T}_{s_2})/(2d_s)$, and $\phi_2 = (\bar{T}_{s_1} + \bar{T}_{s_2} - 2T_s(x,R,\theta,t))/(2d_s^2)$.

Let

$$\hat{T}_i = \frac{2}{3}\bar{T}_{g_i} + \frac{1}{3}\bar{T}_{s_i}, \quad \hat{C}_i = \frac{2}{3}\bar{C}_{g_i} + \frac{1}{3}\bar{C}_{s_i} \quad (i=1,2),$$

$$\mu_1 = 2\left(\frac{O_+}{d_g}\log\frac{I_+}{O_+} + 2\right), \quad \mu_2 = 2\left(\frac{O_-}{d_g}\log\frac{O_-}{I_-} + 2\right), \quad \mu_3 = 2 - \frac{R}{2d_s}\log\frac{I_+}{I_-},$$

$$\nu_1 = \frac{1}{2d_s}\log\frac{I_+}{I_-}, \quad \nu_2 = \frac{1}{2d_s}\log\frac{I_+}{I_-} - \frac{R}{I_- I_+}, \quad \nu_3 = \frac{R^2}{I_- I_+} - \frac{R}{d_s}\log\frac{I_+}{I_-} + 1.$$

From these expressions, we obtain the r-averaged system as follows:

$$
\begin{aligned}
\rho_g c_g d_g \left(\hat{T}_{1,t} + u\hat{T}_{1,x}\right) &= \lambda_g \Big[d_g \hat{T}_{1,xx} + \frac{d_g}{I_+ O_+} \bar{T}_{g_1,\theta\theta} + \frac{\mu_1}{d_g}(\bar{T}_{s_1} - \bar{T}_{g_1}) \\
&\quad + \frac{1}{d_g}\left(\mu_1 - 2 + \frac{d_g}{I_+}\right)(\bar{T}_{s_1,\theta\theta} - \bar{T}_{g_1,\theta\theta})\Big], \\
\rho_g d_g \left(\hat{C}_{1,t} + u\hat{C}_{1,x}\right) &= D \Big[\rho_g d_g \hat{C}_{1,xx} + \frac{\rho_g d_g}{I_+ O_+} \bar{C}_{g_1,\theta\theta} + \frac{\rho_g \mu_1}{d_g}(\bar{C}_{s_1} - \bar{C}_{g_1}) \\
&\quad + \frac{\rho_g}{d_g}\left(\mu_1 - 2 + \frac{d_g}{I_+}\right)(\bar{C}_{s_1,\theta\theta} - \bar{C}_{g_1,\theta\theta})\Big], \\
\rho_g c_g d_g \left(\hat{T}_{2,t} + u\hat{T}_{2,x}\right) &= \lambda_g \Big[d_g \hat{T}_{2,xx} + \frac{d_g}{I_- O_-} \bar{T}_{g_2,\theta\theta} + \frac{\mu_2}{d_g}(\bar{T}_{s_2} - \bar{T}_{g_2}) \\
&\quad + \frac{1}{d_g}\left(\mu_2 - 2 - \frac{d_g}{I_-}\right)(\bar{T}_{s_2,\theta\theta} - \bar{T}_{g_2,\theta\theta})\Big], \\
\rho_g d_g \left(\hat{C}_{2,t} + u\hat{C}_{2,x}\right) &= D \Big[\rho_g d_g \hat{C}_{2,xx} + \frac{\rho_g d_g}{I_- O_-} \bar{C}_{g_2,\theta\theta} + \frac{\rho_g \mu_2}{d_g}(\bar{C}_{s_2} - \bar{C}_{g_2}) \\
&\quad + \frac{\rho_g}{d_g}\left(\mu_2 - 2 - \frac{d_g}{I_-}\right)(\bar{C}_{s_2,\theta\theta} - \bar{C}_{g_2,\theta\theta})\Big], \\
\rho_s c_s d_s \left(T_{s,t}(R) + \frac{d_s^2}{3}\phi_{2,t}\right) &= \lambda_s \Big[d_s T_{s,xx}(R) + \frac{d_s}{I_- I_+} T_{s,\theta\theta}(R) \\
&\quad + \frac{\mu_3}{d_s}(\bar{T}_{s_1} + \bar{T}_{s_2} - 2T_s(R)) + \frac{d_s^3}{3}\phi_{2,xx} + d_s(\nu_1\phi_1 + \nu_2\phi_{1,\theta\theta} + \nu_3\phi_{2,\theta\theta})\Big].
\end{aligned}
$$

Hence, combining the aboves and (2.1), and using the approximation of diffusive-thermal equilibrium with setting $\bar{T}_{s_i} = \bar{T}_{g_i} = T_s(R) = \bar{T}$, $\bar{C}_{s_i} = \bar{C}_{g_i} = \bar{C}$, we can obtain the two-dimensional reaction-diffusion-advection system for $\bar{T}$ and $\bar{C}$:

$$
\begin{gathered}
\left(\rho_g c_g \bar{d}_g + \rho_s c_s \bar{d}_s\right)\bar{T}_t + \rho_g c_g \bar{d}_g u \bar{T}_x = \left(\lambda_g \bar{d}_g + \lambda_s \bar{d}_s\right)\bar{T}_{xx} + \left(\lambda_g \breve{d}_g + \lambda_s \breve{d}_s\right)\bar{T}_{\sigma\sigma} + QW, \\
\rho_g \bar{d}_g \bar{C}_t + \rho_g \bar{d}_g u \bar{C}_x = D\rho_g \bar{d}_g \bar{C}_{xx} + D\rho_g \breve{d}_g \bar{C}_{\sigma\sigma} - W,
\end{gathered}
\tag{3.1}
$$

where we put $\sigma = R\theta$ and

$$
\bar{d}_g = \frac{\mu_1 + \mu_2}{\mu_1 \mu_2} d_g, \quad \bar{d}_s = \frac{1}{\mu_3} d_s, \quad \breve{d}_g = \left(\frac{R^2}{\mu_1 I_+ O_+} + \frac{R^2}{\mu_2 I_- O_-}\right) d_g, \quad \breve{d}_s = \frac{R^2}{\mu_3 I_- I_+} d_s.
$$

Note that by formally taking the limit for fixed R such as $R \to \infty$, we can transform (3.1) into the system of flat case ((26)–(27) in [3]).

4. Traveling wave solution and dispersion relation equation. We apply the following non-dimensionalization based on [3] to (3.1):

$$
\begin{gathered}
m = \frac{\rho_s c_s \bar{d}_s}{\rho_g c_g \bar{d}_g}, \quad D_\xi = \frac{\lambda_g \bar{d}_g + \lambda_s \bar{d}_s}{\rho_g c_g \bar{d}_g}, \quad D_\eta = \frac{\lambda_g \breve{d}_g + \lambda_s \breve{d}_s}{\rho_g c_g \bar{d}_g}, \quad D_{\mathrm{th}} = \frac{\lambda_g d_g + \lambda_s d_s}{\rho_g c_g d_g}, \\
l_r = \frac{D_{\mathrm{th}}}{u_r}, \quad t_r = \frac{\beta m l_r}{2u_r}, \quad \beta = \frac{T_a(T_b - T_u)}{T_b^2}, \quad T_b = T_u + \frac{QC_u}{c_g}.
\end{gathered}
$$

The scaled spatiotemporal coordinates, temperature, concentrations, flow field, and Lewis number is given by

$$\xi = \frac{x}{l_r},\ \eta = \frac{\sigma}{l_r},\ \tau = \frac{t}{t_r},\ \Theta = \frac{\bar{T} - T_u}{T_b - T_u},\ \Phi = \frac{\bar{C}}{C_u},\ \frac{u}{u_r} = U,\ Le = \frac{D_{\text{th}}}{D}$$

leads to the scaled version of the model (3.1) as follows:

$$\begin{aligned}
\frac{2(1+m)}{\beta m}\Theta_\tau + U\Theta_\xi &= \frac{D_\xi}{D}\frac{1}{Le}\Theta_{\xi\xi} + \frac{D_\eta}{D}\frac{1}{Le}\Theta_{\eta\eta} + \omega(\Phi,\Theta),\\
\frac{2}{\beta m}\Phi_\tau + U\Phi_\xi &= \frac{1}{Le}\left(\Phi_{\xi\xi} + d\Phi_{\eta\eta}\right) - \omega(\Phi,\Theta),
\end{aligned}$$

where

$$d = \frac{\breve{d}_g}{\bar{d}_g},\ \omega(\Phi,\Theta) = \Lambda\Phi\exp\left(\frac{\beta(\Theta-1)}{\gamma(\Theta-1)+1}\right),\ \Lambda = \frac{Al_r}{\rho_g\bar{d}_g u_r}\exp\left(-\frac{T_a}{T_b}\right),\ \gamma = 1 - \frac{T_u}{T_b}.$$

Due to the nonlinearity of the reaction rate ω, the system (3.1) is still difficult for theoretical treatment. Therefore, we turn to the conventional high activation energy limit which replaces the reaction rate term with a concentrated (Dirac delta function) source on a flame front. The strength $\Lambda^{(0)}$ of the source, or equivalently, the magnitude of the jump of the fluxes across the front, will be computed by an asymptotic analysis on the parameter β^{-1} in section 4.2.

We consider the regions both within and outside the boundary layer by matched asymptotic expansions. The difference in the solution's derivatives at the two extremes of the boundary layer then yields jump conditions.

To construct the planar travelling wave and calculate the linear stability, we carry out a so-called near-equidiffusive approximation. The approximation is based on the assumption that the Lewis number deviates slightly from unity such that $Le = 1 - \frac{2\alpha}{\beta}$ (as $\beta \to \infty$) with $\alpha = \mathcal{O}(1)$. We also assume that $\frac{D_\xi}{D} = 1 - \frac{2\alpha_\xi}{\beta}$ and $\frac{D_\eta}{D} = 1 - \frac{2\alpha_\eta}{\beta}$.

We suppose that the equation of the front may be written locally as $\xi = F(\eta,\tau)$, and define the moving coordinate such that $\hat{\xi} = \xi - F(\eta,\tau)$, $F(\eta,\tau) = V\tau + \delta F(\eta,\tau)$. The unknown parameter V corresponds to the planar flame's propagating speed and will be defined later. Then, we have

$$\frac{2(1+m)}{\beta m}(\Theta_\tau - \Theta_{\hat{\xi}}F_\tau) + U\Theta_{\hat{\xi}} = \hat{\Delta}\Theta + \frac{2}{\beta - 2\alpha}\big((\alpha-\alpha_\eta)\hat{\Delta}\Theta + (\alpha_\eta - \alpha_\xi)\Theta_{\hat{\xi}\hat{\xi}}\big) + \Lambda\delta_F,$$

$$\frac{2}{\beta m}(\Phi_\tau - \Phi_{\hat{\xi}}F_\tau) + U\Phi_{\hat{\xi}} = \frac{\beta}{\beta - 2\alpha}\big(d\hat{\Delta}\Phi + (1-d)\Phi_{\hat{\xi}\hat{\xi}}\big) - \Lambda\delta_F$$

and the boundary condition is

$$\Theta(-\infty,\eta,\tau) = 0, \quad \Phi(-\infty,\eta,\tau) = 1, \quad \Theta_{\hat{\xi}}(+\infty,\eta,\tau) = 0, \quad \Phi_{\hat{\xi}}(+\infty,\eta,\tau) = 0.$$

Here, the Laplacian $\hat{\Delta}$ in $(\hat{\xi},\eta,\tau)$ coordinate is expressed as

$$\hat{\Delta}f = (1+F_\eta^2)f_{\hat{\xi}\hat{\xi}} - F_\eta(f_{\hat{\xi}\eta} + f_{\eta\hat{\xi}}) - F_{\eta\eta}f_{\hat{\xi}} + f_{\eta\eta}$$

and $\delta_F(\hat{\xi}) = \sqrt{1+F_\eta^2}\,\delta(\hat{\xi})$ is the surface δ-function.

4.1. Outer problem. We seek outer expansion of the form

$$\Theta = \Theta^{(0)} + 2\sum_{j=1} \frac{\Theta^{(j)}}{\beta^j} \ (\hat{\xi} \neq 0), \quad \Phi = \Phi^{(0)} + 2\sum_{j=1} \frac{\Phi^{(j)}}{\beta^j} \ (\hat{\xi} \neq 0), \quad F = F^{(0)} + \sum_{j=1} \frac{F^{(j)}}{\beta^j},$$

and formulate the outer problem in terms of the leading order temperature $\Theta^{(0)}$, $\Phi^{(0)}$ and the quantity $\Psi^{(j)} := \Theta^{(j)} + \Phi^{(j)}$. We assume $m = \mathcal{O}(\beta)$ and $\Theta, \Phi \in C^2$ in both outer regions. Furthermore, we assume $\max\{d_s, d_g\}/R = o(\beta^{-1/2})$, which leads to $1 - d = o(\beta^{-1})$. Then, from the zeroth order, we have

$$U\Theta^{(0)}_{\hat{\xi}} = \hat{\Delta}^{(0)}\Theta^{(0)}, \quad U\Phi^{(0)}_{\hat{\xi}} = \hat{\Delta}^{(0)}\Phi^{(0)}, \tag{4.1}$$

where we put $\Delta^{(j)} f = (1 + (F^{(j)}_\eta)^2) f_{\hat{\xi}\hat{\xi}} - F^{(j)}_\eta (f_{\hat{\xi}\eta} + f_{\eta\hat{\xi}}) - F^{(j)}_{\eta\eta} f_{\hat{\xi}} + f_{\eta\eta}$. We also obtain the following from the $\mathcal{O}(\beta^{-1})$ terms:

$$\begin{aligned} -\Theta^{(0)}_{\hat{\xi}} F^{(0)}_\tau + \Theta^{(0)}_\tau + U\Psi^{(1)}_{\hat{\xi}} &= \hat{\Delta}^{(0)}\Psi^{(1)} + F^{(0)}_\eta F^{(1)}_\eta \Psi^{(0)}_{\hat{\xi}\hat{\xi}} - F^{(1)}_\eta \Psi^{(0)}_{\hat{\xi}\eta} - \frac{F^{(1)}_{\eta\eta}}{2}\Psi^{(0)}_{\hat{\xi}} \\ &\quad + \alpha\hat{\Delta}^{(0)}\Psi^{(0)} - \alpha_\eta \hat{\Delta}^{(0)}\Theta^{(0)} - (\alpha_\xi - \alpha_\eta)\Theta^{(0)}_{\hat{\xi}\hat{\xi}} \end{aligned} \tag{4.2}$$

with the boundary conditions

$$\begin{aligned} &\Theta^{(0)} = 0, \quad \Phi^{(0)} = 1, \quad \Theta^{(j)} = \Phi^{(j)} = 0 \ (j \geq 1) \quad \text{at } \hat{\xi} = -\infty, \\ &\Theta^{(0)} = 1, \quad \Phi^{(j)} = 0 \ (j \geq 0) \quad \text{for } \hat{\xi} > 0, \\ &\Theta^{(j)}_{\hat{\xi}} = 0, \quad \Phi^{(j)}_{\hat{\xi}} = 0 \ (j \geq 0) \quad \text{at } \hat{\xi} = \infty. \end{aligned}$$

The planar flame front propagating to the negative ξ direction corresponds to the 1-dimensional stationary traveling wave solution to (4.1) and (4.2) with $\delta F \equiv 0$, namely,

$$\Theta^{(0)} = \begin{cases} \exp(U\hat{\xi}) & (\hat{\xi} < 0) \\ 1 & (\hat{\xi} > 0) \end{cases}, \qquad \Phi^{(0)} = \begin{cases} 1 - \exp(U\hat{\xi}) & (\hat{\xi} < 0) \\ 0 & (\hat{\xi} > 0) \end{cases}. \tag{4.3}$$

We note that $\Theta^{(0)} + \Phi^{(0)} \equiv 1$. As remarked in [6, 7], we have $U\Psi^{(0)}_{\hat{\xi}} = \hat{\Delta}^{(0)}\Psi^{(0)}$ from (4.1), and hence, we can set $\Psi^{(0)} \equiv 1$, provided that the initial conditions are chosen to satisfy $\Theta|_{t=0} + \Phi|_{t=0} = 1 + \mathcal{O}(\beta^{-1})$. This assumption implies $\Psi^{(0)}|_{t=0} = 1$, so that $\Psi^{(0)} \equiv 1$ by the boundary condition $\Psi^{(0)} = 1$ at $\hat{\xi} = \pm\infty$. Then the following follows from (4.2):

$$\Psi^{(1)} = \begin{cases} \left[V/U + (\alpha_\xi U - V)\,\hat{\xi}\right] \exp(U\hat{\xi}) & (\hat{\xi} < 0) \\ V/U & (\hat{\xi} > 0) \end{cases}, \tag{4.4}$$

which coincides with the Kagan & Sivashinsky's result [3] with defined $V = U \log U$.

4.2. Inner problem. To derive the jump condition across the flame front, we introduce the stretched coordinate $\zeta = \beta\hat{\xi}$ and seek inner expansions, in the form

$$\Theta = 1 + 2\sum_{j=1} \theta^{(j)}\beta^{-j}, \ \Phi = 2\sum_{j=1} \phi^{(j)}\beta^{-j}, \ F = \sum_{j=0} F^{(j)}\beta^{-j}, \ \Lambda = \beta^2 \sum_{j=0} \Lambda^{(j)}\beta^{-j}.$$

The inner problem to leading order is then given by

$$(1+(F_\eta^{(0)})^2)\theta_{\zeta\zeta}^{(1)} + \Lambda^{(0)}\phi^{(1)}e^{2\theta^{(1)}} = 0, \tag{4.5}$$

$$(1+(F_\eta^{(0)})^2)\phi_{\zeta\zeta}^{(1)} - \Lambda^{(0)}\phi^{(1)}e^{2\theta^{(1)}} = 0. \tag{4.6}$$

From the matching condition $(\theta^{\text{inner}}, \phi^{\text{inner}})(\zeta \to \pm\infty) = (\Theta^{\text{outer}}, \Phi^{\text{outer}})(\hat{\xi} \to \pm 0)$, we have

$$\theta^{(1)} = \Theta^{(1)}(0^+), \quad \phi^{(1)} = 0 \quad \text{as } \zeta \to +\infty \tag{4.7}$$

$$\theta^{(1)} = \Theta^{(1)}(0^-) + \frac{\Theta_{\hat{\xi}}^{(0)}(0^-)}{2}\zeta, \quad \phi^{(1)} = \Phi^{(1)}(0^-) + \frac{\Phi_{\hat{\xi}}^{(0)}(0^-)}{2}\zeta \quad \text{as } \zeta \to -\infty, \tag{4.8}$$

where we put $\mathsf{F}(0^\pm) = \mathsf{F}|_{\hat{\xi}=0^\pm}$. In particular, the zeroth-order matching is insured by requiring $\Theta^{\text{outer}}(0^\pm) = 1$ and $\Phi^{\text{outer}}(0^\pm) = 0$. This implies the continuity of $\Theta^{(0)}$ across the reaction sheet as expressed in $[\![\Theta^{(0)}]\!]_{0^-}^{0^+} = 0$, where $[\![\mathsf{F}]\!]_{0^-}^{0^+} = \mathsf{F}(0^+) - \mathsf{F}(0^-)$ denotes a jump in the value of F across the sheet.

We now put $\psi^{(1)} = \theta^{(1)} + \phi^{(1)}$. Then, $\psi^{(1)}$ satisfies $0 = (1+(F_\eta^{(0)})^2)\psi_{\zeta\zeta}^{(1)}$, which follows from (4.5) and (4.6). We hence solve as $\psi^{(1)} = a\zeta + b$, where $a = 0$ and $b = \Theta^{(1)}(0^+)$ are determined from (4.7). Therefore, by substituting the relation

$$\phi^{(1)}(\zeta) = \Theta^{(1)}(0^+) - \theta^{(1)}(\zeta) \quad \text{as } \zeta \to \infty$$

into (4.5), we obtain a single equation for $\theta^{(1)}$ as follows:

$$(1+(F_\eta^{(0)})^2)\theta_{\zeta\zeta}^{(1)} + \Lambda^{(0)}(\Theta^{(1)}(0^+) - \theta^{(1)}(\zeta))\exp(2\theta^{(1)}) = 0. \tag{4.9}$$

We now multiply (4.9) by $\theta_{\hat{\xi}}^{(1)}$ and integrate with respect to $\hat{\xi}$ from $\hat{\xi} = -\infty$ to $\hat{\xi} = \infty$ to obtain

$$0 = \frac{1+(F_\eta^{(0)})^2}{2}\left[(\theta_\zeta^{(1)})^2\right]_{\zeta=-\infty}^{\zeta=\infty} + \Lambda^{(0)}\int_{\theta^{(1)}|_{\zeta=-\infty}}^{\theta^{(1)}|_{\zeta=\infty}} (\Theta^{(1)}(0^+) - \theta^{(1)}(\zeta))e^{2\theta^{(1)}(\zeta)}\,d\theta^{(1)}(\zeta).$$

Hence, using (4.7) and (4.8), we obtain

$$0 = -\frac{1+(F_\eta^0)^2}{2}\left(\Theta_{\hat{\xi}}^{(0)}(0^-)\right)^2 + \Lambda^{(0)}e^{2\Theta^{(1)}(0^+)}.$$

Therefore, since $\Theta^{(0)} = 1$ ($\hat{\xi} > 0$) and $\Phi^{(j)} = 0$ ($\hat{\xi} > 0$) for $j \geq 0$, we obtain the result

$$[\![\Theta_{\hat{\xi}}^{(0)}]\!]_{0^-}^{0^+} = -\sqrt{\frac{2\Lambda^{(0)}}{1+(F_\eta^{(0)})^2}}\exp\left(\Psi^{(1)}(0^+)\right). \tag{4.10}$$

For the planar traveling wave, the above yields $U = \sqrt{2\Lambda^{(0)}}\exp(V/U)$, and hence, $\Lambda^{(0)} = \frac{1}{2}$ in the case of $V = U\log U$. Hereafter, we put $V = U\log U$.

Summarizing our discussions so far, we can derive the following equation, which represents a closed problem for the zeroth order temperature $\Theta^{(0)}$, the first order enthalpy $\Psi^{(1)}$, and the zeroth order flame position $F^{(0)}$:

$$\begin{aligned} U\Theta_{\hat{\xi}} &= \hat{\Delta}\Theta, \\ -\Theta_{\hat{\xi}}F_\tau + \Theta_\tau + U\Psi_{\hat{\xi}} &= \hat{\Delta}\Psi - \left\{\alpha_\eta\hat{\Delta}\Theta + (\alpha_\xi - \alpha_\eta)\Theta_{\hat{\xi}\hat{\xi}}\right\} \end{aligned} \tag{4.11}$$

applicable for $\hat{\xi} \neq 0$, subject to the boundary conditions

$$\Theta(-\infty,\eta,\tau)=0,\ \Psi(-\infty,\eta,\tau)=0,\ \Theta(\hat{\xi}>F)=1,\ \Psi_{\hat{\xi}}(+\infty,\eta,\tau)=0 \tag{4.12}$$

and the jump conditions

$$[\![\Theta]\!]_{0^-}^{0^+}=0,\quad [\![\Psi]\!]_{0^-}^{0^+}=0, \tag{4.13}$$

$$\sqrt{1+F_\eta^2}\,[\![\Theta_{\hat{\xi}}]\!]_{0^-}^{0^+}=-\exp(\Psi(0^+)), \tag{4.14}$$

$$[\![\Psi_{\hat{\xi}}]\!]_{0^-}^{0^+}=\frac{\alpha_\xi+\alpha_\eta F_\eta^2}{1+F_\eta^2}[\![\Theta_{\hat{\xi}}]\!]_{0^-}^{0^+} \tag{4.15}$$

applicable at $\hat{\xi}=0$. Here, we drop the superscripts. The jump condition (4.13) means the continuity of Θ and Ψ which follows from the matching relation, (4.14) follows readily from (4.10), and (4.15) can be derived by integration of (4.2) from $\hat{\xi}=0^-$ to $\hat{\xi}=0^+$. Note that we used the continuity of $\Theta^{(0)}$, $\Phi^{(0)}$, $\Psi^{(0)}$ and $\Psi^{(1)}$, and hence of their partial derivatives with respect to τ and η to derive (4.15).

4.3. Linear stability analysis. We now calculate the linear stability of the planar travelling (4.3)–(4.4) by considering the perturbation of the form ([6])

$$(\Theta,\Psi,F)=(\Theta^{(0)},\Psi^{(1)},V\tau)+\varepsilon(\theta+f\Theta^{(0)}_{\hat{\xi}},\psi+f\Psi^{(1)}_{\hat{\xi}},f),$$

where ε is a small amplitude of the perturbation. Substituting the above into (4.11) and (4.12), and linearizing on ε gives the problem

$$U\theta_{\hat{\xi}}=\theta_{\hat{\xi}\hat{\xi}}+\theta_{\eta\eta},$$
$$-V\theta_{\hat{\xi}}+\theta_\tau+U\psi_{\hat{\xi}}=\psi_{\hat{\xi}\hat{\xi}}+\psi_{\eta\eta}-(\alpha_\xi\theta_{\hat{\xi}\hat{\xi}}+\alpha_\eta\theta_{\eta\eta})$$

subject to the boundary conditions

$$\theta=0,\quad \psi=0\quad \text{at } \hat{\xi}=-\infty,$$
$$\theta(\hat{\xi}>0)=0,\quad \psi_{\hat{\xi}}=0\quad \text{at } \hat{\xi}=\infty$$

and the jump conditions

$$[\![\theta]\!]=2Uf,\qquad [\![\psi]\!]=\alpha_\xi[\![\theta]\!], \tag{4.16}$$

$$[\![\theta_{\hat{\xi}}]\!]=U(2Uf-\psi(0^+)), \tag{4.17}$$

$$[\![\psi_{\hat{\xi}}]\!]=\alpha_\xi[\![\theta_{\hat{\xi}}]\!]+2U(\alpha_\xi U-V)f \tag{4.18}$$

at $\hat{\xi}=0$. It should be noted that the jump conditions have been approximated by using Taylor expansions for ε.

We seek nontrivial solutions such as $(\theta,\psi,f)=e^{\omega\tau+ik\eta}(\hat{\theta}(\hat{\xi}),\hat{\psi}(\hat{\xi}),1)$, where $\omega\in\mathbb{C}$ is a growth rate of the amplitude, and k is a wavenumber. By using (4.16) and (4.17) we have

$$\hat{\theta}=\begin{cases}A\exp(\lambda_+\hat{\xi}) & (\hat{\xi}<0)\\ 0 & (\hat{\xi}>0)\end{cases},\quad \hat{\psi}=\begin{cases}\left[B+AP\dfrac{(\lambda_+-\lambda_-)\hat{\xi}-1}{(\lambda_+-\lambda_-)^2}\right]\exp(\lambda_+\hat{\xi}) & (\hat{\xi}<0)\\ C\exp(\lambda_-\hat{\xi}) & (\hat{\xi}>0)\end{cases},$$

where

$$A = -2U, \quad B = 2\left\{U(1-\alpha_\xi) - \lambda_+ - \frac{UP}{(\lambda_+ - \lambda_-)^2}\right\}, \quad C = 2(U - \lambda_+),$$
$$\lambda_\pm = \frac{U \pm \sqrt{U^2 + 4k^2}}{2}, \quad P = \alpha_\xi \lambda_+^2 - V\lambda_+ + \omega - k^2\alpha_\eta.$$

The jump condition (4.18) leads, after simple manipulations, to the dispersion relation

$$\omega = k^2(\alpha_\eta - \alpha_\xi) - (\alpha_\xi U - V)\lambda_- + \left(1 - \frac{\lambda_+}{U}\right)(\lambda_+ - \lambda_-)^2.$$

In the long-wavelength limit ($k \ll 1$) and near the stability threshold such as $k \sim \sqrt{\alpha_\eta - \alpha_c} \sim R^{-1/2}$ where $\alpha_c = 1 + \log U$, the expansion above can be written as

$$\omega = (\alpha_\eta - \alpha_c)k^2 - \frac{4k^4}{U^2}, \quad k \ll 1, \tag{4.19}$$

and it implies the following linear evolution equation for the flame front $f(\eta, \tau)$:

$$f_\tau + (\alpha_\eta - \alpha_c)f_{\eta\eta} - \frac{4}{U^2}f_{\eta\eta\eta\eta} = 0. \tag{4.20}$$

We note that the dispersion relation (4.19) and the equation (4.20) reduce to the dispersion relation derived by [3] and the linear KS equation when $\alpha_\eta = \alpha_\xi = \alpha$, i.e., $R \to \infty$, respectively.

It indicates that the time evolution of perturbations along the η direction depends on α_η rather than α_ξ, and the effect of the curvature of the cylinder appears in the second-order derivative terms.

5. Nonlinear effect. The following argument extends Kuramoto's idea [5] on a plane to a surface. The system (4.11) is invariant under the transformation $\eta \mapsto -\eta$, and hence, this invariance is inherited by an evolution equation that the perturbation f should satisfy, say $f_t = \mathcal{F}(f, f_\eta, f_{\eta\eta}, \dots)$. Furthermore, since the perturbation f is along the traveling wave solution with velocity V, the evolution equation $f_t = \mathcal{F}$ should be invariant when the transformations $f \mapsto f + c_0$ and $t \mapsto t - V^{-1}c_0$ are acted simultaneously. Therefore, $\mathcal{F}$ have to take the following form:

$$\mathcal{F} = a_2 f_{\eta\eta} + a_4 f_{\eta\eta\eta\eta} + a_6 f_{\eta\eta\eta\eta\eta\eta} + \cdots + b_1(f_\eta)^2 + b_2(f_{\eta\eta})^2 + b_3 f_\eta f_{\eta\eta\eta} + \cdots .$$

That is, $\mathcal{F}$ consists only of terms where the total order of derivatives is even, and it does not include any terms involving f.

Scaling with $a_2 = \varepsilon$ ($|\varepsilon| \ll 1$) such that $f = \varepsilon\tilde{f}$, $\tilde{\eta} = \varepsilon^{1/2}\eta$ and $\tilde{\tau} = \varepsilon^2\tau$, we can choose the leading terms of $\mathcal{F}$ as follows:

$$f_\tau = a_2 f_{\eta\eta} + a_4 f_{\eta\eta\eta\eta} + b_1(f_\eta)^2, \tag{5.1}$$

where we drop tildes. Combining the above with (4.20), we have $a_2 = -(\alpha_\eta - \alpha_c)$ and $a_4 = 4/U^2$.

The coefficient b_1 is determined semi-heuristically by constructing a special solution. (5.1) has a special solution $f = \nu\eta + b_1\nu^2\tau$, which corresponds to a traveling wave with slope $\varphi = \arctan\nu$. Then, we find that $b_1 = -U/2 + \mathcal{O}(\nu^2)$ for $|\nu| \ll 1$ provided that the traveling speed perpendicular to the line $\xi = f$ is determined by the relation $(V + b_1\nu^2)\cos\varphi = U_n \log U_n$, where $U_n = \boldsymbol{U}\cdot\boldsymbol{n}$ and $\boldsymbol{n}$ is the unit normal to the interface directed towards the reacted material. Meanwhile, if the traveling speed perpendicular to the line $\xi = f$ is given by V, then we have $b_1 = -V/2$. Thus, the coefficient of the nonlinear term differs depending on geometric considerations; therefore, we put b_1 as a parameter V_0.

By using the scaling $\hat{\eta} = U\eta$, $\hat{t} = U^2\tau$, $\hat{f} = V_0 f$ and dropping hats, we end up with the curvature-dependent KS equation (1.1).

6. Concluding remarks. Starting from the three-dimensional model of the cylinder, we reduced it to a two-dimensional model by averaging thickness. Using the assumption of diffusive-thermal equilibrium, the strong temperature dependence of the reaction rate, and the strong disparity between the densities of the solid and gaseous phases, we derive the traveling wave's dispersion relation. We note that the dispersion relation equation accounting for the curvature effect is novel. From the model's symmetric properties, we determine the possible terms of the evolution equation, and from a geometric point of view, we heuristically determine the coefficient of the nonlinear term.

In a forthcoming paper, we intend to apply (1.1) to simulate combustion phenomena on an accordion folded paper and analyze the flame spreading rate's dependence on the combustible material's shape.

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