Proceedings of ALGORITMY 2024
pp. 199–203

# OPTIMAL CONVERGENCE RESULTS FOR FINITE ELEMENTS ON EXTREMELY DEFORMED MESHES*

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**Abstract.** This short notes provides an optimal analysis of finite element convergence on meshes containing a so-called band of caps. These structures consist of a zig-zag arrangement of 'degenerating' triangles which violate the classical maximum angle condition. Previously a necessary condition on the geometry of such a structure was given by Kučera to ensure convergence of finite elements. Here we prove that the condition is also sufficient, providing an optimal analysis of this special case of meshes. This result provides only the second known optimal analysis of finite element convergence on special meshes containing elements violating the maximum angle condition.

**Key words.** Finite element method, maximum angle condition, error estimates

**AMS subject classifications.** 65N30, 65N15, 65N50

**1. Introduction.** Much work has been devoted over the past 60 years to develop various error estimates for the finite element method for a wide range of problems. It may therefore seem surprising that the simplest basic question remains unanswered to this day: What is a necessary and sufficient condition on triangular meshes for piecewise linear finite elements to converge? Even in the simplest of all settings – Poisson's problem and estimates in the corresponding $H^1(\Omega)$ energy norm, this is still an open problem.

The basic textbook result is that if the meshes satisfy the *minimum angle condition*, then finite elements will exhibit optimal $O(h)$ convergence in the energy norm. This condition requires that all angles of all elements in the mesh(es) are uniformly bounded away from zero. A slightly more advanced result is that $O(h)$ convergence occurs under the more general *maximum angle condition*, which requires that the maximal angles of all triangles are uniformly bounded away from $\pi$. This sufficient condition was generally assumed to also be necessary – this confusion was caused by the misleading title "The maximum angle condition is essential" from the original paper [1]. The title refers to a counterexample provided in the paper, where finite elements do not converge on a special mesh consisting only of 'bad' elements. As it turns out, the maximum angle condition is not necessary for $O(h)$ convergence of the finite element method, cf. [3]. Since then, another counterexample was analyzed in the paper [6], where a single structure, a so-called band of caps, contained in the mesh destroys finite element convergence. The analysis leads to conditions on the proportions and geometry of the band of caps that is necessary for $O(h)$ convergence, and more generally $O(h^\alpha)$ convergence for some $\alpha \in [0, 1]$.

The purpose of this short note is to show that the condition on the band of caps derived in [6] is optimal, i.e. both necessary and sufficient for $O(h^\alpha)$ convergence. Although the question of a general necessary and sufficient condition for the convergence of the finite element method still remains open, at least there is a second special case that can be analyzed optimally.

*This research is supported by the European Research Council (project X-MESH, ERC-2022-SyG-101071255).

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**2. Finite element method.** In this work we will be focused on Poisson's problem in $\mathbb{R}^2$. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with Lipschitz boundary $\partial\Omega$. We solve the problem

$$-\Delta u = f \text{ on } \Omega, \quad u|_{\partial\Omega} = 0 \tag{2.1}$$

with the weak form: Find $u \in H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = (f, v), \quad \forall v \in H^1_0(\Omega), \tag{2.2}$$

where $H^1_0(\Omega)$ is the standard Sobolev space of functions with square integrable derivatives and a zero trace on $\partial\Omega$, while $(f, v) = \int_\Omega f v \,\mathrm{d}S$ denotes the $L^2$ scalar product.

In order to define the finite element method, we consider a conforming triangulation $\mathcal{T}_h$ of $\Omega$, i.e. a partition into triangles (elements) with mutually disjoint interiors such that the intersection of two neighboring elements is either a single vertex or a whole edge. Here $h$ denotes the length of the longest edge in the triangulation. This partition defines the piecewise linear finite element space

$$V_h = \{v_h \in C(\overline{\Omega}); v_h|_K \in P^1(K) \text{ for all } K \in \mathcal{T}_h\}, \tag{2.3}$$

where $P^1(K)$ is the space of linear functions on the triangular element $K \in \mathcal{T}_h$.

The finite element method is then defined as follows: Find $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \,\mathrm{d}x = (f, v_h), \quad \forall v_h \in V_h. \tag{2.4}$$

It is desirable to obtain estimates for the error $u - u_h$. To this end, Céa's lemma, cf. [2], gives us an estimate in the $H^1(\Omega)$-seminorm:

$$|u - u_h|_{H^1(\Omega)} = \inf_{v_h \in V_h} |u - v_h|_{H^1(\Omega)}. \tag{2.5}$$

We note that for other problems, one can expect an inequality in (2.5) and a some problem-dependent constant in the upper bound.

Standard finite element estimates are typically derived by taking the piecewise linear Lagrange interpolation $\Pi_h u$ as $v_h$ in (2.5). This is defined element-wise: on each element $K \in \mathcal{T}_h$, the function $\Pi_h u|_K = \Pi_K u \in P^1(K)$ coincides with $u$ at the vertices of $K$. Such a locally defined function naturally gives a globally continuous piecewise linear function in $V_h$.

For triangles, there is an optimal estimate for the error $u - \Pi_K u$ in seminorms in the general Sobolev space $W^{1,p}(\Omega)$. We will need this estimate only in the special case of $p = \infty$. Consider an arbitrary triangle $K \subset \mathbb{R}^2$. Denote the length of its longest edge as $h_K$ and its height perpendicular to this edge as $\overline{h}_K$. Finally, define $R_K$ as the circumradius of $K$, i.e. the radius of the circumscribed circle to $K$. We have the following optimal estimate, cf. [4], [5].

Lemma 2.1 (Circumradius estimate). *Let $K \subset \mathbb{R}^2$ be an arbitrary triangle. Let $u \in W^{2,p}(K)$, $1 \le p \le \infty$, and let $\Pi_K u$ be the linear Lagrange interpolation of $u$ on $K$. Then there exists a constant $C_c$ independent of $u, K$ such that*

$$|u - \Pi_K u|_{W^{1,p}(K)} \le C_c R_K |u|_{W^{2,p}(K)} \le C_c \frac{h_K^2}{\overline{h}_K} |u|_{W^{2,p}(K)}. \tag{2.6}$$

One is especially interested in optimal convergence results of the order $O(h)$ in the $H^1(\Omega)$-seminorm, via (2.5). A sufficient (but not necessary!) condition for this to happen is when $R_K \leq \tilde{C}h$ for all $K \in \mathcal{T}_h$ with some constant $\tilde{C}$ independent of $h$. Geometrically, this is equivalent to satisfying the *maximum angle condition.* This condition requires that all maximal angles $\alpha_K$ of all triangles $K \in \mathcal{T}_h$ are smaller than some $\alpha_0 < \pi$. Then we have the following element-wise estimate, which can then be applied in (2.5).

LEMMA 2.2 (Maximum-angle condition). *Let $K \subset \mathbb{R}^2$ be a triangle satisfying the maximum angle condition: $\alpha_K \leq \alpha_0 < \pi$ for some fixed $\alpha_0$. Let $u \in H^2(K)$ and let $\Pi_K u$ be the linear Lagrange interpolation of $u$ on $K$. Then there exists a constant $C_I$ independent of $u$ (but depending on the maximum angle $\alpha_0$) such that*

$$|u - \Pi_K u|_{H^1(K)} \leq C_I h |u|_{H^2(K)}. \tag{2.7}$$

*Moreover, if $\alpha_K \leq \alpha_0 < \pi$ for all $K \in \mathcal{T}_h$, we have*

$$|u - u_h|_{H^1(\Omega)} \leq C_I h |u|_{H^2(\Omega)}. \tag{2.8}$$

The maximum angle condition has a long and complicated history, being discovered independently by several groups, eg. [1]. In [3] it was proven that this condition is not necessary for $O(h)$ convergence. In fact $\mathcal{T}_h$ can contain many 'bad' triangles violating the maximum angle condition while still exhibiting optimal $O(h)$ convergence. In other words, the finite element method can converge optimally even when the Lagrange interpolation error goes to infinity. This is especially important when we have a sequence of meshes obtained e.g. by refinement and let $h \to 0$. In this situation one usually considers a set of triangulations $\mathcal{T}_h$, $h \in (0, h_0)$ for some $h_0 > 0$.

Apart from the paper [3], paper [6] has dealt with sufficient as well as necessary conditions for $O(h)$ convergence, or more generally, for $O(h^\alpha)$ estimates with $0 \leq \alpha \leq 1$. Specifically, the so-called *band of caps* has been identified as the basic (but not only) villain preventing optimal convergence of the finite element method. The band of caps consists of triangles in a zigzag pattern, where all of the elements violate the maximum angle condition. Specifically, we shall consider such a band of length $L$ and height $\overline{h}$ consisting of identical isosceles triangles with diameters $h$, cf. Figure 2.1. We assume that every $\mathcal{T}_h$ we consider contains one such band, while all other elements satisfy the maximum angle condition.

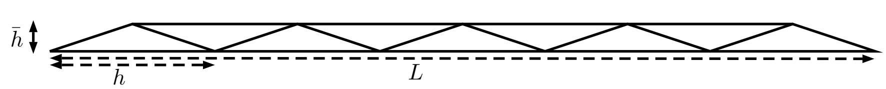


FIG. 2.1. *Band of caps of length $L$ and height $\overline{h}$.*

In [6], the following result is proved as a special case of the main theorem of the paper dealing with a band of general elements (cf. estimate (64) in the cited paper).

THEOREM 2.3. *Let $L \geq C_L h^{2\alpha/5}$ where $C_L$ is a sufficiently large constant and let $\alpha \in [0, 1]$. Then a necessary condition for the estimate*

$$|u - u_h|_{H^1(\Omega)} \leq \hat{C} h^\alpha \tag{2.9}$$

*to hold with some $\hat{C}$ independent of $h$, is*

$$\overline{h} \geq \tilde{C} h^{4-2\alpha} L \tag{2.10}$$

*for some $\tilde{C} > 0$.*

In the special case of a band of caps of length $L \sim 1$, the condition says that for $O(h)$ convergence of the finite element method, we must necessarily have $\overline{h} \geq \tilde{C} h^2$ for some $\tilde{C} > 0$. And for (even arbitrarily slow) convergence of the finite element method, i.e. the limiting case of $\alpha = 0$, we must necessarily have $\overline{h} \geq \tilde{C} h^4$ for some $\tilde{C} > 0$. In the next section, we will show that these conditions are both necessary and sufficient.

**3. Optimal error estimate for a band of caps.** In the following, $C$ will be a generic constant independent of $u, h$. First, we prove that the condition $\overline{h} \geq \tilde{C} h^2$ is also sufficient for $O(h)$ convergence of finite elements on meshes containing a band of caps.

THEOREM 3.1. *Let $u \in W^{2,\infty}(\Omega)$. Let $\mathcal{T}_h$ contain a band of caps $\mathcal{B}$ of length $L \sim 1$ and height $\overline{h}$, while all other elements in $\mathcal{T}_h$ satisfy the maximum angle condition with some $\alpha_0$. Let there exist $\tilde{C} > 0$ such that*

$$\overline{h} \geq \tilde{C} h^2. \tag{3.1}$$

*Then there exists a constant $C$ independent of $h$ and $u$, such that*

$$|u - u_h|_{H^1(\Omega)} \leq C h |u|_{W^{2,\infty}(\Omega)}. \tag{3.2}$$

*Proof.* From Céa's lemma we have

$$|u - u_h|^2_{H^1(\Omega)} \leq |u - \Pi_h u|^2_{H^1(\Omega)} = |u - \Pi_h u|^2_{H^1(\Omega \setminus \mathcal{B})} + |u - \Pi_h u|^2_{H^1(\mathcal{B})}, \tag{3.3}$$

due to additivity of integrals. The first term in (3.3) uses standard estimates (all elements of $\Omega \setminus \mathcal{B}$ satisfy the maximum angle condition):

$$|u - \Pi_h u|^2_{H^1(\Omega \setminus \mathcal{B})} \leq C h^2 |u|^2_{H^2(\Omega)} \leq C h^2 |\Omega| |u|^2_{W^{2,\infty}(\Omega)}. \tag{3.4}$$

The second term in (3.3) is estimated using the circumradius estimate (2.6):

$$\begin{aligned} |u - \Pi_h u|^2_{H^1(\mathcal{B})} &= \int_{\mathcal{B}} |\nabla u - \nabla \Pi_h u|^2 \, \mathrm{d}x \leq |u - \Pi_h u|^2_{W^{1,\infty}(\mathcal{B})} |\mathcal{B}| \\ &\leq C \left( \frac{h^2}{\overline{h}} \right)^2 |u|^2_{W^{2,\infty}(\mathcal{B})} |\mathcal{B}| \leq C \frac{h^4}{\overline{h}} L |u|^2_{W^{2,\infty}(\Omega)}, \end{aligned} \tag{3.5}$$

since $|\mathcal{B}| \leq \overline{h} L$. Using the assumption that $\overline{h} \geq \tilde{C} h^2$ in the right-hand side of (3.5), we get

$$|u - \Pi_h u|^2_{H^1(\mathcal{B})} \leq C \frac{h^4}{\tilde{C} h^2} L |u|^2_{W^{2,\infty}(\Omega)} = C h^2 L |u|^2_{W^{2,\infty}(\Omega)}. \tag{3.6}$$

Combining estimates (3.3), (3.4), and (3.6), and taking the square root gives us the desired estimate. ☐

If we are interested in obtaining $O(h^\alpha)$ estimates for general $L$ (which could depend on $h$), we have the following:

THEOREM 3.2. *Let $u \in W^{2,\infty}(\Omega)$ and let $\alpha \in [0,1]$. Let $\mathcal{T}_h$ be as in Theorem 3.1 and let there exist $\tilde{C} > 0$ such that*

$$\overline{h} \geq \tilde{C} h^{4-2\alpha} L. \tag{3.7}$$

*Then there exists a constant $C$ independent of $h, u$, such that*

$$|u - u_h|_{H^1(\Omega)} \leq C h^{\alpha} |u|_{W^{2,\infty}(\Omega)}. \tag{3.8}$$

*Proof.* We estimate the right-hand side of (3.5) using the assumption on $\overline{h}$, obtaining

$$|u - \Pi_h u|^2_{H^1(\mathcal{B})} \leq C \frac{h^4}{\tilde{C} h^{4-2\alpha} L} L |u|^2_{W^{2,\infty}(\Omega)} = C h^{2\alpha} |u|^2_{W^{2,\infty}(\Omega)}. \tag{3.9}$$

Combining estimates (3.3), (3.4), and (3.9), and taking the square root gives us the desired estimate. □

As we can see, the necessary condition (2.10) and sufficient condition (3.7) coincide, therefore the condition is *both necessary and sufficient*. The question arises, what is a general necessary and sufficient condition on $\mathcal{T}_h$ to ensure $O(h)$ convergence, or any other convergence rate. This question remains open, however we have been able to close the gap between necessary and sufficient conditions in the special case of a mesh containing a band of caps. One other optimality result exists in the literature, the optimal analysis of the Babuška-Aziz counterexample from [7]. This counterexample consists of a triangulation $\mathcal{T}_h$ where the entire domain $\Omega$ is filled with bands of caps only.

**Acknowledgments.** The author thanks Jean-François Remacle, Nicolas Moës, Jonathan Lambrechts and Antoine Quiriny for their kind support within the X-MESH project.

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