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FUZZY FRANKOT-CHELLAPPA METHOD FOR SURFACE NORMAL INTEGRATION

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Abstract. In this paper, we propose the Fuzzy formulation of the classic Frankot-Chellappa method by which surfaces can be reconstructed using normal vectors. In the Fuzzy formulation, the surface normal vectors may be uncertain or ambiguous. The underlying model yields a Fuzzy Poisson partial differential equation, where it is imperative to give meaningful representations of Fuzzy derivatives. The solution of the resulting Fuzzy model is approached numerically. To this end, a fuzzy formulation for the discrete sine transform method is explored, which results in a fast, accurate and robust method for surface reconstruction. In experiments we consider specifically the robustness with respect to noisy surface normal vectors.

Key words. Surface normal integration, Frankot-Chellappa method, Fuzzy derivatives, Fuzzy partition, Fuzzy Poisson equation

AMS subject classifications. 97P80, 65Q99, 65N99

1. Introduction.

The integration of surface normals for the purpose of computing the shape of the corresponding surface in 3D space is a classic problem in computer vision [10]. Many methods have been proposed with the aim to devise an approach that combines accuracy, robustness with respect to noise in the data and computational efficiency, see e.g. [3, 4, 6, 7, 9, 14] as well as the survey paper [11]. It appears evident that even nowadays it is still a challenging task to devise a method that is highly accurate and offers at the same time robustness and computational efficiency.

When it comes to computational efficiency, the classic method of Frankot and Chellappa [7] is still among the most powerful methods. It relies on a finite difference approximation of a Poisson equation which naturally arises in the problem formulation of surface normal integration. The crucial part of the Frankot-Chellappa algorithm is the use of a discrete sine transform for dealing with the surface normal data in the Poisson model. However, it is well-known that the classic Frankot-Chellappa method does not incorporate a mechanism that deals with noisy data, and also accuracy issues may arise in the vicinity of steep surface gradients.

Fuzzy concepts apply human reasoning ability to knowledge-based systems. When the assumptions of the problem have uncertainty, one consider a fuzzy interpretation of parameters or data. Due to uncertainty e.g. due to noise in image acquisition systems, in many aspects of image processing fuzzy processing may be desirable.

As indicated, the underlying problem formulation in the classic method from Frankot and Chellappa a Poisson equation needs to be solved, for which the discrete sine transform can be explored.

In this research, a fuzzy formulation of the classic Frankot-Chellappa model and algorithm for surface normal reconstruction is presented. It turns out that the main methodology from the classic scheme, namely exploring the discrete sine transform, can be transferred to the fuzzy formulation in terms of the fuzzy sine transform. Our fuzzy extension appears to be fast, accurate, and nearly robust to noisy data.

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The technique of fuzzy transform (F-transform for short) has been introduced by I. Perfilieva et al in [12, 13]. Similarly to the classic sine transform it can be cast in two ways, as a direct or an inverse transform. The authors of that work have proved that the inverse F-transform has good approximation properties and is relatively easy to use. As for the fuzzy formulation of the Poisson equation, fuzzy partial differential equations (PDEs) have been introduced in the past. In 2018, the fuzzy Poisson equation with Dirichlet boundary conditions has been discussed, proving uniqueness and stability of a solution in [8]. We explore the use of this recent concept in the current paper. As for a potential benefit of our developments, we study the possible enhanced robustness of the fuzzy formulation to noise. This may be considered to be sometimes a more delicate issue with the classic method that does not assume uncertain data.

2. Fuzzy concepts. The purpose of this section is to present the necessary fuzzy notions and concepts for use with this paper.

2.1. Fuzzy Numbers and Fuzzy Partition. We write $A(x)$, a number in $[0,1]$, for the fuzzy membership function evaluated at x. An α-cut of $\tilde{A}$, written $[\tilde{A}]^{\alpha}$, is defined as $\{x \in X \mid A(x) \geq \alpha\}$, for $0 < \alpha \leq 1$. The Triangular Fuzzy Number (TFN) $\tilde{A}$ is defined by three numbers $a_1 < a_2 < a_3$, where the graph of $A(x)$, the membership function of the fuzzy number $\tilde{A}$, is a triangle with the base on the interval $[a_1, a_3]$ and vertex at $x = a_2$. We specify $\tilde{A}$ as $(a_1/a_2/a_3)$.

By cosidering the membership function of $\tilde{A}$ is defined as:

$$A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2} & \text{if } a_2 \leq x \leq a_3 \\ 0 & \text{otherwise} \end{cases}$$

Given two TFNs $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$, their arithmetic addition $\tilde{C} = \tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ is a TFN:

$$C(x) = \begin{cases} \frac{x-(a_1+b_1)}{(a_2+b_2)-(a_1+b_1)} & \text{if } (a_1+b_1) \leq x \leq (a_2+b_2) \\ \frac{(a_3+b_3)-x}{(a_3+b_3)-(a_2+b_2)} & \text{if } (a_2+b_2) \leq x \leq (a_3+b_3) \\ 0 & \text{otherwise} \end{cases}$$

Also the gH-difference $\tilde{A} \ominus_{gH} \tilde{B} = \tilde{E} \quad \Leftrightarrow \quad \begin{cases} \tilde{A} = \tilde{B} \oplus \tilde{E} \\ \tilde{A} = \tilde{B} \oplus (-)\tilde{E} \end{cases}$, $\tilde{E}$ is a TFN.

Let us consider the fuzzy partition [12]: choose an interval $[a, b]$ as a universe, and assume that a function f is given at points $p_0, ..., p_{l-1} \in [a, b]$.

Below, we recall the definition of a fuzzy partition. Let $a = x_0 < ... < x_n = b$, $n \geq 3$ be fixed nodes within $[a, b]$. Fuzzy sets $A_1, ..., A_{n-1}$ identified with their membership functions $A_1, ..., A_{n-1}$, defined on $[a, b]$, establish a fuzzy partition of $[a, b]$ if they fulfill the following conditions for $k = 1, ..., n-1$:

1) $A_k : [a, b] \to [0, 1], A_k(x_k) = 1;$

2) $A_k(x) = 0 \;\; if \; x \notin (x_{k-1}, x_{k+1}), \;\; k = 1, ..., n-1;$

3) $A_k(x)$ *is continuous*;

4) $A_k(x)$ *strictly increase on* $[x_{k-1}, x_k],\ k = 1, ..., n-1;$
and strictly decrease on $[x_k, x_{k+1}],\ k = 1, ..., n-1;$

5) $\sum_{k=1}^{n} A_k(x) = 1,\ \forall x \in [x_1, x_{n-1}]$.

The membership functions $A_1, ..., A_{n-1}$ are called basic functions.

We say that the fuzzy partition given by $A_1, ..., A_{n-1}$, is an h-uniform fuzzy partition if the nodes $x_k = a + hk,\ k = 0, ..., n$ are equidistant, $h = (b-a)/n$ and two additional properties are met:

6) $A_k(x_k - x) = A_k(x_k + x),\ x \in [0, h],\ k = 1, ..., n-1;$

7) $A_k(x) = A_{k-1}(x-h),\ A_{k+1}(x) = A_k(x-h),\ \forall k = 2, ..., n-1, x \in [x_{k-1}, x_{k+1}]$.

2.2. The F-transform. Consider the discrete F-transform [12]: the fuzzy sets $A_1, ..., A_{n-1}$ establish a fuzzy partition of $[a, b]$ and $f : P \to R$ is a discrete real valued function defined on the set $P = \{p_0, ..., p_{l-1}\}$ where $P \subseteq [a, b]$.

The following vector of real numbers $F_n[f] = [F_1, ..., F_{n-1}]$ is the (direct) discrete F-transform of f w.r.t. $A_1, ..., A_{n-1}$ where the k-th component F_k is defined by

$$F_k = \frac{\sum_{j=0}^{l-1} A_k(p_j) f(p_j)}{\sum_{j=0}^{l-1} A_k(p_j)}, \qquad k = 1, ..., n-1. \tag{2.1}$$

The inverse discrete F-transform reads as:

$$f_{F,n}(p_j) = \sum_{k=1}^{n-1} F_k A_k(p_j), \qquad j = 0, ..., l-1. \tag{2.2}$$

For h-uniform fuzzy partition $A_1, ..., A_{n-1}$ of $[a, b]$, there exists an even function $A : [-h, h] \to [0, 1]$:

$$A_k(x) = A(x - x_k) = A(x_k - x),\ \forall k = 1, ..., n-1,\ x \in (x_{k-1}, x_{k+1})$$

The points $p_0, ..., p_{l-1}$ are equidistant in the interval $[a, b]$ and moreover $p_j = a + jh/m;\ j = 0, ..., l-1,$, where m and l are connected by the following equality: $l = nm + 1$. Thus chosing points $p_0, ..., p_{l-1}$ it is assured that the nodes $x_0, ..., x_n$ are among them, i.e. for each $k = 0, ..., n$, there exists j such that $x_k = p_j$.

Similarity to the case of a function of one variable we can have F-Transform in 3D. Let a function f be given at nodes $(p_i, q_j) \in [a, b] \times [c, d],\ i = 1, ..., M, j = 1, ..., N$, and $A_1, ..., A_m$, $B_1, ..., B_n$ where $m \leq M$, $n \leq N$, be basic functions which form fuzzy partitions of $[a, b]$ and $[c, d]$, respectively. Suppose that sets P and Q of these nodes are sufficiently dense with respect to the chosen partitions.

We say that the $m \times n$-matrix of real numbers $F_{mn}[f] = (F_{kl})$ is the discrete F-transform of f with respect to $A_1, ..., A_m$ and $B_1, ..., B_n$ if

$$F_{kl} = \frac{\sum_{i=1}^{M} \sum_{j=1}^{N} A_k(p_i) B_l(q_j) f(p_i, q_j)}{\sum_{i=1}^{M} \sum_{j=1}^{N} A_k(p_i) B_l(q_j)} \tag{2.3}$$

holds for all $k = 1, ..., m,\ l = 1, ..., n$.

The elements F_{kl} , $k = 1, ..., m, l = 1, ..., n$, are called components of the F-transform.

Let $A_1, ..., A_m$ and $B_1, ..., B_n$ be basic functions which form fuzzy partitions of $[a, b]$ and $[c, d]$ respectively. Let f be a function from $C([a, b] \times [c, d])$ and $F_{mn}[f]$ be the F-transform of f with respect to $A_1, ..., A_m$ and $B_1, ..., B_n$. Then the function

$$f^F_{mn}(p_i, q_j) = \sum_{k=1}^{m} \sum_{l=1}^{n} F_{kl} A_k(p_i) B_l(q_j) \tag{2.4}$$

holds for all $i = 1, ..., M,\ j = 1, ..., N$.

2.3. Fuzzy Partial Derivatives. A fuzzy-valued function f of two variables is a rule that assigns to each ordered pair of real numbers, (x, y), in a set D a unique fuzzy number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

We show the parametric representation of the fuzzy-valued function $f : D \to F$ by $f(x, y; \alpha) = [f^-(x, y; \alpha), f^+(x, y; \alpha)]$, for all $(x, y) \in D$ and $\alpha \in [0, 1]$.

Let $(x_0, y_0) \in D$. Then the first generalized Hukuhara partial derivatives ($[gH-p]$-derivatives for short) of a fuzzy-valued function $f(x, y) : D \to F$ at (x_0, t_0) with respect to variables x, y are the functions $f_{x_{gH}}(x_0, y_0)$ and $f_{y_{gH}}(x_0, y_0)$ given by

$$f_{x_{gH}}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) \ominus_{gH} f(x_0, y_0)}{h}$$

$$f_{y_{gH}}(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) \ominus_{gH} f(x_0, y_0)}{k}$$

provided that $f_{x_{gH}}(x_0, y_0)$ and $f_{y_{gH}}(x_0, y_0)$ in F where $\ominus_{gH}$ is the gH-difference.

3. Classic Frankot-Chellappa Surface Normal Integration. Let us briefly recall the classic Frankot-Chellappa method. For a given normal field $\vec{n}(u, v) = [n_1(u, v), n_2(u, v), n_3(u, v)]^\top$ with $(u, v) \in \mathbb{R}^2$ and $\|\vec{n}(u, v)\| = 1$ the surface $\mathbf{x}(u, v) = [x(u, v), y(u, v), z(u, v)]^\top$ is sought, such that the vector $\vec{n}(u_0, v_0)$ is orthogonal to the surface $\mathbf{x}$ at the point (u_0, v_0). Therefore

$$\begin{pmatrix} \partial_u z(u, v) \\ \partial_v z(u, v) \end{pmatrix} = \nabla z = \begin{pmatrix} -\frac{n_1(u,v)}{n_3(u,v)} \\ -\frac{n_2(u,v)}{n_3(u,v)} \end{pmatrix} = \begin{pmatrix} p(u, v) \\ q(u, v) \end{pmatrix} = g(u, v)$$

Now, in order to numerically approximate a solution of this equation, one may try to find a function z such that the distance between ∇z and g in the display space of the surface is small. The necessary condition for z being a minimizer of the distance $\|\nabla z(u, v) - g(u, v)\|_2$ over all $(u, v) \in \Omega = [0, m] \times [0, n]$ can be written as $\Delta z = \partial_u p + \partial_v q$ that is a Poisson PDE. For approximation of partial derivatives, one may consider

$$\frac{z_{u+1,v} + z_{u-1,v} + z_{u,v+1} + z_{u,v-1} - 4z_{u,v}}{h^2} = f_{u,v}$$

$$f_{u,v} = \frac{p_{u+1,v} - p_{u-1,v}}{2h} + \frac{q_{u,v+1} - q_{u,v-1}}{2h}$$

Now by using the inverse discrete sine transformation one may obtain

$$\overline{z}_{k,l} = \frac{h^2 \overline{f}_{k,l}}{2\left(\cos\left(\frac{\pi k}{m}\right) + \cos\left(\frac{\pi l}{n}\right) - 2\right)}$$

Where $\overline{f}$ and $\overline{z}$ are the discrete sine transform of f and z and

$$\overline{f}_{k,l} = \sum_{u=1}^{m-1} \sum_{v=1}^{n-1} f_{u,v} \sin(\pi \frac{ku}{m}) \sin(\pi \frac{lv}{n})$$

.

4. Fuzzy Poisson Equation. Numerous problems in industry lead to an equation with fuzzy partial differential equation in the following form:

$$\Delta\widetilde{z}(u,v) = \tilde{f}(u,v) \tag{4.1}$$

in which Ω is a regular area, $\Delta\widetilde{z}(u,v) = \frac{\partial^2 \tilde{z}}{\partial u^2} \oplus \frac{\partial^2 \tilde{z}}{\partial v^2}$ and $\tilde{f}(u,v)$ is a fuzzy known function. Since $\tilde{f}(u,v)$ is a fuzzy function and coefficients in $\Delta\widetilde{z}(u,v)$ are fuzzy, then according to the extension principle $\tilde{z}(u,v)$ is a fuzzy function too.

For the first-order derivatives based on u, v we have:

$$\frac{\partial \tilde{z}}{\partial u} = \frac{\tilde{z}_{u,v} \ominus_{gH} \tilde{z}_{u-1,v}}{h} \quad , \quad \frac{\partial \tilde{z}}{\partial v} = \frac{\tilde{z}_{u,v} \ominus_{gH} \tilde{z}_{u,v-1}}{h}$$

Here we assume the $\tilde{A}$ is a symmetric triangular fuzzy number, and we write $\tilde{A} = (a_1/a_2/a_3)$, when $a_2 - a_1 = a_3 - a_2 = 0.01$.

Now for a normal field $\vec{\tilde{n}}(u,v) = [\widetilde{n}_1(u,v), \widetilde{n}_2(u,v), \widetilde{n}_3(u,v)]^\top$ with $(u,v) \in \mathbb{R}^2$, we assume $\widetilde{n}_1(u,v)$ as a symmetric triangular fuzzy number with vertex $n_1(u,v)$, also for $\widetilde{n}_2(u,v)$ with vertex $n_2(u,v)$, and $\widetilde{n}_3(u,v)$ with vertex $n_3(u,v)$. (we can suppose the fuzziness between 0 and 1 here is supposed 0.01)

We now consider the underlying PDE as a fuzzy partial differential equation:

$$\Delta\widetilde{z} = \frac{\partial \tilde{p}}{\partial u} \oplus \frac{\partial \tilde{q}}{\partial v} \tag{4.2}$$

By replacing fuzzy second-order derivative in definitions in previous sections and fuzzy Poisson equation $\Delta\widetilde{z} = \widetilde{f}$, in a similar way as M. Abdi et al. in [1] has obtained by discrete partial difference expressions, we will have:

$$\frac{\partial^2 \tilde{z}}{\partial u^2} = \frac{1}{h^2}(\widetilde{z}_{u+1,v} \ominus_{gH} 2\widetilde{z}_{u,v} \oplus \widetilde{z}_{u-1,v})$$

$$\frac{\partial^2 \tilde{z}}{\partial v^2} = \frac{\widetilde{z}_{u,v+1} \ominus_{gH} 2\widetilde{z}_{u,v} \oplus \widetilde{z}_{u,v-1}}{h^2}$$

$$\frac{\partial \tilde{p}}{\partial u} = \frac{\widetilde{p}_{u+1,v} \ominus_{gH} \widetilde{p}_{u-1,v}}{2h} \quad , \quad \frac{\partial \tilde{q}}{\partial v} = \frac{\widetilde{q}_{u,v+1} \ominus_{gH} \widetilde{q}_{u,v-1}}{2h}$$

By entering the derivatives in the Poisson equation, we reach its discrete formula:

$$\frac{1}{h^2}(\widetilde{z}_{u+1,v} \oplus \widetilde{z}_{u-1,v} \oplus \widetilde{z}_{u,v+1} \oplus \widetilde{z}_{u,v-1} \ominus_{gH} 4\widetilde{z}_{u,v}) =$$

$$\frac{\widetilde{p}_{u+1,v} \ominus_{gH} \widetilde{p}_{u-1,v}}{2h} \oplus \frac{\widetilde{q}_{u,v+1} \ominus_{gH} \widetilde{q}_{u,v-1}}{2h}$$

Then

$$\widetilde{z}_{u+1,v} \oplus \widetilde{z}_{u-1,v} \oplus \widetilde{z}_{u,v+1} \oplus \widetilde{z}_{u,v-1} \ominus_{gH} 4\widetilde{z}_{u,v} = h^2 \widetilde{f}_{u,v} \tag{4.3}$$

5. Summary of Fuzzy Frankot-Chellappa Method. In the following we only consider problems with Dirichlet boundary conditions and write the right-hand side more generally as a function $\widetilde{f}$, so that we are looking for a solution of $\widetilde{\Delta z} = \widetilde{f}$. For this we use the inverse discrete sine transformation $\mathcal{F}_s\{\widetilde{f}_{u,v}\} = \hat{\tilde{f}}_{k,l}$ and $\mathcal{F}_s^{-1}\{\hat{\tilde{f}}_{k,l}\} = \widetilde{f}_{u,v}$:

$$\widetilde{f}_{u,v} = \frac{4}{mn}\sum_{k=1}^{m-1}\sum_{l=1}^{n-1}\hat{\tilde{f}}_{k,l} \odot (\sin(\pi\frac{ku}{m})\sin(\pi\frac{lv}{n})) \tag{5.1}$$

First we solve it only with a homogeneous Dirichlet condition $\widetilde{z}_{u,v} = 0$ for $(u,v) \in \partial\Omega = \{0,m\}\times\{0,n\}$, where $\Omega = [0,m]\times[0,n]$, in order to later determine the solution for problems with an inheterogeneous Dirichlet situation using the knowledge gained.

We again insert the inverse transform from (5.1) into the discrete formula (4.3), leaving h undetermined:

$$\begin{aligned}\frac{4}{mn}\sum_{k=1}^{m-1}\sum_{l=1}^{n-1}\frac{\hat{\tilde{z}}_{k,l}}{h^2}\Big[&\left(\sin\left(\frac{\pi k(u+1)}{m}\right)+\sin\left(\frac{\pi k(u-1)}{m}\right)\right)\sin\left(\frac{\pi lv}{n}\right)\\ &+\sin\left(\frac{\pi ku}{m}\right)\left(\sin\left(\frac{\pi l(v+1)}{n}\right)+\sin\left(\frac{\pi l(v-1)}{n}\right)\right)\\ &-4\sin\left(\frac{\pi ku}{m}\right)\sin\left(\frac{\pi lv}{n}\right)\Big]\ominus_{gH}\hat{\tilde{f}}_{k,l}\sin\left(\frac{\pi ku}{m}\right)\sin\left(\frac{\pi lv}{n}\right)=0.\end{aligned}$$

Then, we have

$$\begin{aligned}&\frac{2\hat{\tilde{z}}_{k,l}}{h^2}\Big[\sin\left(\frac{\pi ku}{m}\right)\cos\left(\frac{\pi k}{m}\right)\sin\left(\frac{\pi lv}{n}\right)+\sin\left(\frac{\pi ku}{m}\right)\sin\left(\frac{\pi lv}{n}\right)\cos\left(\frac{\pi l}{n}\right)\\ &-2\sin\left(\frac{\pi ku}{m}\right)\sin\left(\frac{\pi lv}{n}\right)\Big]=\hat{\tilde{f}}_{k,l}\sin\left(\frac{\pi ku}{m}\right)\sin\left(\frac{\pi lv}{n}\right)\end{aligned}$$

If u and k are not in $\{0,m\}$ and v and l are not elements of $\{0,n\}$, we have

$$\begin{aligned}&\frac{2\hat{\tilde{z}}_{k,l}}{h^2}\left(\cos\left(\frac{\pi k}{m}\right)+\cos\left(\frac{\pi l}{n}\right)-2\right)=\hat{\tilde{f}}_{k,l}\\ \Leftrightarrow\quad &\hat{\tilde{z}}_{k,l}=\frac{h^2\hat{\tilde{f}}_{k,l}}{2\left(\cos\left(\frac{\pi k}{m}\right)+\cos\left(\frac{\pi l}{n}\right)-2\right)}\end{aligned} \tag{5.2}$$

Where $\hat{\tilde{f}}$ and $\hat{\tilde{z}}$ are the discrete fuzzy sine transform of $\tilde{f}$ and $\tilde{z}$ and

$$\hat{\tilde{f}}_{k,l} = \sum_{u=1}^{m-1}\sum_{v=1}^{n-1}\widetilde{f}_{u,v} \odot (\sin(\pi\frac{ku}{m})\sin(\pi\frac{lv}{n}))$$

.

6. Experimental Results. In this section, we investigate our method with using fuzzy concepts. We consider experiments for surface reconstruction using normal vectors, some of which are uncertain normal vectors or ambiguous, and observe if this leads to better results in terms of accuracy or noise suppression due to noisy data.

6.1. Comparison in Terms of Accuracy. Now, to compare the responses obtained from the classical and fuzzy methods, we first use de-fuzzification, then we calculate the difference of all points in the classical solution and compare the de-fuzzified solution with the exact solution.

For every $(u,v) \in \partial\Omega = \{0,m\} \times \{0,n\}$, where $\Omega = [0,m] \times [0,n]$ we use $RMSE$ for $z_{u,v}$ that is obtained by sine discrete transform and $m(\tilde{z}_{u,v})$, the de-fuzzified solution from $\tilde{z}_{u,v}$, is obtained by fuzzy sine discrete transform. The de-fuzzification method that is used here is $m(\tilde{z}) = 0.5(z^- + z^+)$.

In Table.1 we compare exact function, discrete sine transformation approximation and the obtained result from de-fuzzification of fuzzy sine discrete transformation approximation (with $\alpha = 0.3$) for four cases. In Fig.6.1 we compare exact function, discrete sine transformation approximation and α-cut of fuzzy discrete sine transformation approximation ($\alpha = 0.3$) for three cases of the above examples that is done on the rectangle $[-1,1]^2$:

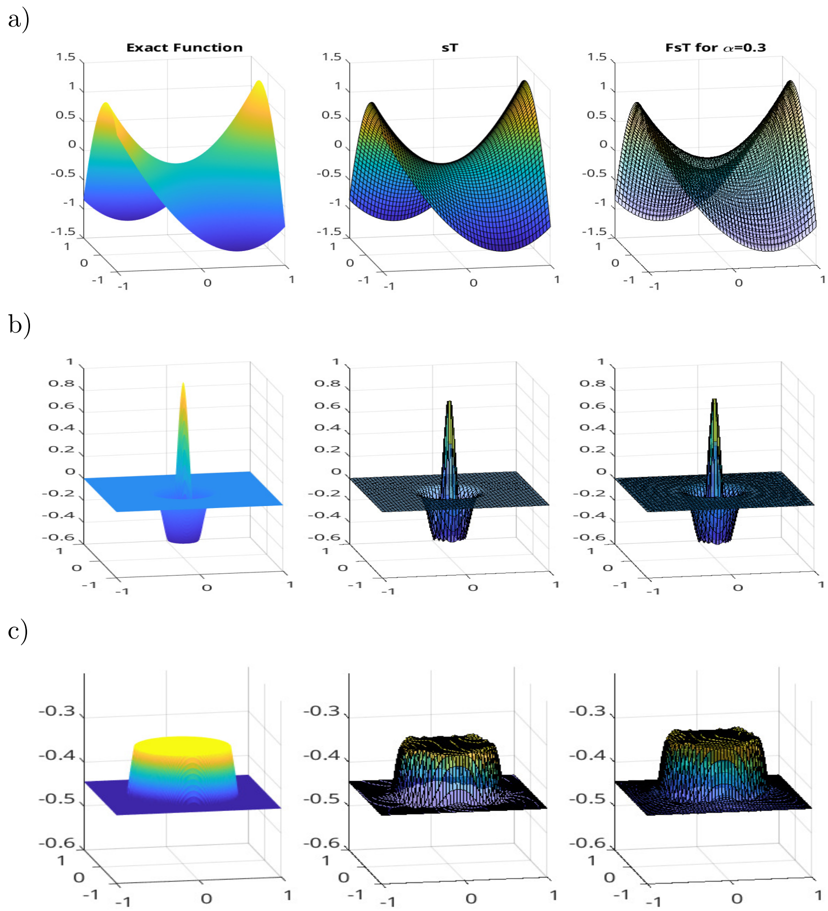

FIG. 6.1. *Comparison of exact function, discrete sine transformation approximation (sT) and α-cut components of the fuzzy sine discrete transformation approximation (FsT), with $\alpha = 0.3$ for $z(u,v)$: a) $\sin(uv) + u^2 - v^2$, b) $(1 - \frac{u^2}{\sigma^2} - \frac{v^2}{\sigma^2})e^{-\frac{u^2+v^2}{2\sigma^2}}$, $\sigma = .1$, c) $-u^2 - v^2$, $\frac{3}{9} < u^2 + v^2 < \frac{4}{9}$. We confirm here that the accuracy of the fuzzy approximation is equivalent to the classical approximation $D = d$.*

6.2. Comparison with respect to Noise Suppression. Since the operation on the function is linear, it makes sense to consider to fuzzify the elements of normal vectors at the beginning of the operation, and then de-fuzzify at the end, so the calculation error is almost the same in the two approximations.

A possible advantage by fuzzy construction of our method is in its application at specific points without given data (i.e. holes in domain where data is given). Also integration with fuzzy components may potentially remove noise better than classic method. Which we confirm in Fig.6.2 where we have solutions with noise for cases in Fig.6.1.

To be more precise, it has been proved in [13] that the inverse F-transform of a noisy function is almost equal to the inverse F-transform of the original function. Which makes us conjecture that our method has good denoising capabilities.

We consider a noise, represented by a function $s(u,v)$ and $z(u,v) + s(u,v)$ is the representation of the noised function z. On the basis of linearity of the direct F-transform, this noise can be removed if its regular components of the direct F-transform are equal to zero. Here we consider to apply both F-transform (direct and inverse) to a function to remove a certain noise. Indeed, the inverse F-transform can be considered as a special fuzzy identity filter which can be utilized.

Let a discretization for the surface with $m \times n$ size be represented a function of two variables $z : [1,m] \times [1,n] \to [0,1]$ defined at nodes $(u,v) : u = 1,...,m;\ v = 1,...,n$. The F-transform (2.3) for $z(u,v)$ is as follows:

$$Z_{kl} = \frac{\sum_{v=1}^{n} \sum_{u=1}^{m} A_k(u) B_l(v) z(u,v)}{\sum_{v=1}^{n} \sum_{u=1}^{m} A_k(u) B_l(v)}$$

Where $A_1,...,A_r$,$B_1,...,B_s$,$r \leq m$, $s \leq n$, are basic functions which form fuzzy partitions of $[1,m]$ and $[1,n]$, respectively. A reconstruction of the image z, being described by $F_{rs}[z] = (Z_{kl})$ with respect to $A_1,...,A_r$ and $B_1,...,B_s$, can then be computed by the inverse F-transform (2.4) adapted to the domain $[1,m] \times [1,n]$:

$$z^F_{rs}(u,v) = \sum_{k=1}^{m} \sum_{l=1}^{n} Z_{kl} A_k(u) B_l(v)$$

which holds for all $u = 1,...,m,\ v = 1,...,n$. Also, we know $z^F_{rs}(u,v) = (z+s)^F_{rs}(u,v)$ that has improved in [13], where the noise $s(u,v)$ be continuous functions on $[1,m] \times [1,n]$. We can see in Fig.6.3 that F-transform performs as an effective filter.

We also have constructive suggestions for specific situations that occur in reality, for example, a part of the procedure cannot be reconstructed due to insufficient information. In this way, to make the components of the normal vector of points with unknown normal, the corresponding directions are used , $\widetilde{p}_{u,v} = \widetilde{p}_{u-1,v}$ and $\widetilde{q}_{u,v} = \widetilde{q}_{u,v-1}$. In fact, in every point (u,v) in area with uncertain normal vector for the $\widetilde{p}$ of the unknown point, we use the information of the closest point in the x-direction, and for the $\widetilde{q}$, we use the information of the closest point in the y-direction. Then $f_{u,v}$ get changed as follow:

$$\widetilde{f}_{u,v} = \frac{\widetilde{p}_{u,v} \ominus_{gH} \widetilde{p}_{u-2,v}}{2h} + \frac{\widetilde{q}_{u,v} \ominus_{gH} \widetilde{q}_{u,v-2}}{2h} \tag{6.1}$$

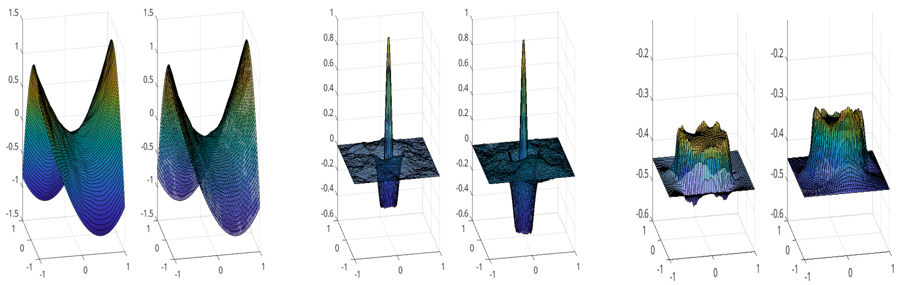

FIG. 6.2. *The solution with noise for three cases in Fig.1. As it shows, fuzzy approximation is better to suppress noise than classic approximation since $D < d$.*

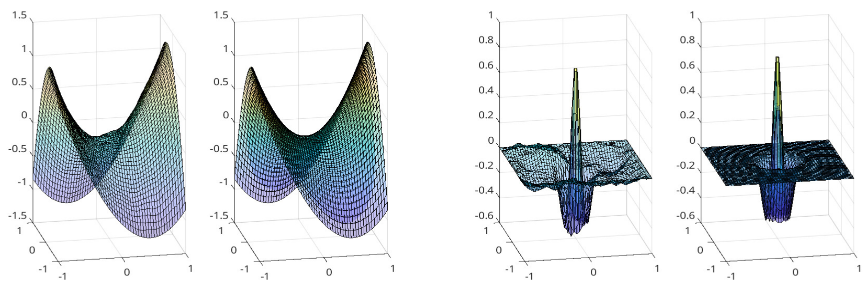

FIG. 6.3. *Removing noise with "F-transform" in order to obtain $z^F_{mn}(u,v) = (z+s)^F_{mn}(u,v)$. As it shows, fuzzy approximation is better to suppress noise than classic approximation since $D(z) < d(z)$.*

7. Conclusion. In this paper, the Fuzzy Poisson equation with Dirichlet boundary conditions was investigated as part of a fuzzy Frankot-Chellappa method. We show experimentally, backed up by some theoretical considerations, that our fuzzy Frankot-Chellappa method reconstructs surfaces, using normal vectors of which some may be considered uncertain or ambiguous, with reasonable results. In terms of accuracy, for smooth surfaces our fuzzy method gives equivalent results than the classic method, while in terms of noise suppression the use of fuzzy concepts appears to be beneficial.

To achieve these results, some concepts such as a fuzzy-valued vector function, fuzzy operators, and a fuzzy sine discrete transform were studied. Consequently, the fuzzy solution of the fuzzy Poisson equation was obtained by our applying an fuzzy extension $\widetilde{z}_{u,v}$ of $z_{u,v}$.

For future research, other types of fuzzy numbers may be used. We may consider also using fuzzy distance between fuzzy normal vectors to can obtain missed normal vectors for approximating related surface with missing data.

Table 7.1
Compare the classical and fuzzy methods with $d = RMSE(z)$ and $D = RMSE(m(\tilde{z}))$

Exact function z	d	D	d(noise 0.5)	D(noise 0.5)
$\sin(uv) + u^2 - v^2$	0.5704e-06	0.5704e-06	1.9234e-02	4.4535e-03
$(1 - \frac{u^2}{\sigma^2} - \frac{v^2}{\sigma^2})e^{-\frac{u^2+v^2}{2\sigma^2}}$	1.1119e-06	1.1119e-06	2.6299e-03	1.6738e-03
$e^{(u+0.25v)}$	4.0103e-06	4.0103e-06	3.1116e-03	4.4851e-04
$-u^2 - v^2$	8.3435e-05	8.3435e-05	4.4281e-04	7.3650e-05

REFERENCES

[1] M. Abdi, T. Allahviranloo, *Fuzzy finite difference method for solving fuzzy Poisson equation*, Journal of Intelligent and Fuzzy Systems., 37 (2019), pp. 5281–5296.

[2] T. Allahviranloo, *Difference methods for fuzzy partial differential equations*, Comput. Methods Appl. Math., 2(3) (2006), pp. 233–242.

[3] M. Bähr, M. Breuss, *An Improved Eikonal Method for Surface Normal Integration*, Proceeding of the 37th German Conference on Pattern Recognition. Aachen, Germany (2015).

[4] M. Bähr, M. Breuss, Y. Queau, A.S. Boroujerdi J.D. Durou, LU-*Fast and accurate surface normal integration on non-rectangular domains*, Comput. Vis. Media., 3 (2017), pp. 107–129.

[5] J.D. Durou, J.F. Aujol, F. Courteille, *Integrating the normal field of a surface in the presence of discontinuities*, Energy Minimization Methods Comput. Vis. Pattern Recogn., 5681 (2009), pp. 261–273.

[6] D. Dubois, H. Prade, *Operations on fuzzy numbers*, International Journal of Systems Science., 9(6) (1978), pp. 613–626.

[7] R.T. Frankot, R. Chellappa, *A method for enforcing integrability in shape from shading algorithms*, IEEE Trans. Pattern Anal. Mach. Intell., 10(4) (1988), pp. 576–593.

[8] R. Ghasemi Moghaddam, T. Allahviranloo, *On the fuzzy Poisson equation*, Fuzzy Sets Syst., 347 (2018), pp. 105–128.

[9] M. Harker, P. O' Leary, *Least squares surface reconstruction from measured gradient fields*, In: Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition., (2008).

[10] B.K.P. Horn, *Robot Vision*, McGraw-Hill Book, (1986)

[11] Y. Queau, J-D. Durou, J-F. Aujol, *Normal Integration: A survey*, J Math Imaging Vis., 60 (2018), pp. 576–593.

[12] I. Perfilieva, *Fuzzy transforms: Theory and applications*, Fuzzy Sets Syst., 157 (2006), pp. 993–1023.

[13] I. Perfilieva, P. Valášek, *Fuzzy Transforms in Removing Noise*, in Computational Intelligence, Theory and Applications, International Conference 8th Fuzzy Days, Dortmund, Germany, (2004).

[14] T. Simchony, R.T. Frankot, *Direct analytical methods for solving poisson equations in computer vision problems*, IEEE Trans. Pattern Anal. Mach. Intell., 12(5) (1990), pp. 435–446.