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A NEWTON-SCHEME FRAMEWORK FOR MULTISCALE METHODS FOR NONLINEAR ELLIPTIC HOMOGENIZATION PROBLEMS*

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Abstract. In this contribution, we present a very general framework for formulating multiscale methods for nonlinear elliptic homogenization problems. The framework is based on a very general coupling of one macroscopic equation with several localized fine-scale problems. In particular, we recover the Heterogeneous Multiscale Method (HMM), the Multiscale Finite Element Method (MsFEM) and the Variational Multiscale Method (VMM) from the framework. In order to solve the arising equations, we also present a solution algorithm that is based on a Newton scheme with damping.

Key words. multiscale methods, nonlinear problems, implementation framework

AMS subject classifications. 35J60, 68W25, 74Q15

1. Introduction.

In this work, we are concerned with solving nonlinear elliptic multiscale problems of the following type:

$$-\nabla \cdot A^\epsilon(\cdot, \nabla u^\epsilon) = f \quad \text{in } \Omega.$$

Here, f is a source term, $A^\epsilon(x, \cdot)$ is a nonlinear function and $A^\epsilon(\cdot, \xi)$ is rapidly oscillating. These oscillations create a fine-scale microstructure. The parameter ϵ can be seen as an indicator for the speed of the oscillations whose gradients are expected to scale with $\frac{1}{\epsilon}$. If we try to solve the problem with standard methods such as Finite Elements, the computational grid must resolve the microstructure. But this implies a tremendous computational demand. In practice, we need alternative strategies that decrease the computational demand. Strategies that are designed for this purpose are generally called *multiscale methods*.

One example for such a method is the Heterogenous Multiscale Method (HMM), which was introduced by E and Engquist [4] and extended by several contributions. The idea is to reconstruct the fine-scale behavior only in a number of cells around quadrature points and to communicate the gained information to a macroscopic equation, which only uses a local average of this information. A good survey on different realizations of the HMM was given by Abdulle [1]. Error control for Heterogenous Multiscale Methods was presented by Ohlberger and Henning [20, 7, 8, 9] and Abdulle and Nonnemacher [2]. Recent works that discuss HMM-formulations for nonlinear elliptic problems are for instance [9, 10] and [3].

Another example for a multiscale method is the Multiscale Finite Element Method (MsFEM) developed by Hou and Wu [12]. Here, a set of multiscale basis functions is constructed by adding fine-scale features to the original basis functions. Then, the original problem is posed and solved in the space that is spanned by the multiscale

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basis. Typically, this space is low dimensional. An overview in Multiscale Finite Element Methods is given in the book by Efendiev and Hou [6]. Nonlinear problems were for instance treated by Efendiev, Hou and Ginting [5].

The last multiscale method that we want to mention is the Variational Multiscale Method (VMM) initially introduced in the works of Hughes *et al.* [13, 14]. This approach is based on a splitting of the original solution space into a direct sum of a coarse scale-space and a fine-scale space. This yields a coarse-scale equation and a fine-scale equation, where the fine-scale equation is solved in dependence of the residual of the solution of the coarse-scale equation. The method was extended to an Adaptive Variational Multiscale Method by Larson and Målqvist [15, 16, 17, 18]. For nonlinear scenarios, we refer to Nordbotten [19].

Among these various contributions in the field of multiscale methods for nonlinear problems, there is a computational issue that has only been addressed for the HMM, but not for multiscale methods in general. This issue is due to the type of the treatable nonlinearities which are either of the type $A^\epsilon(\cdot, u^\epsilon)\nabla u^\epsilon$ (then the fine-scale problems are linear and the macro-problem can be solved straightforwardly with a Newton's method) or of the type $A^\epsilon(\cdot, \nabla u^\epsilon)\nabla u^\epsilon$ (the equation can be linearized with the iteration $A^\epsilon(\cdot, \nabla u^\epsilon_{(n)})\nabla u^\epsilon_{(n+1)}$). A general nonlinearity in the gradient of u^ϵ yields a computational difficulty that we will address in Section 3.

In this work, we present a very general framework for formulating multiscale methods for nonlinear elliptic problems. We show how to recapture existing methods from the framework (such as HMM, MsFEM and VMM) and we present a corresponding solution strategy that is based on a damped Newton scheme. In particular, by applying the framework to the typical setting of the MsFEM, this is the first explicit formulation of an MsFEM for the above mentioned type of nonlinear elliptic problems. Furthermore, depending on the micro-structure of $A^\epsilon(\cdot, \xi)$ the framework can be used to formulate a method specifically constructed for a certain scenario. In a numerical experiment, the proposed solution algorithm is validated by computing the solution of a nonlinear homogenization problem.

2. Multiscale methods for nonlinear elliptic problems. In this section we first introduce the setting in that we are working and then we motivate and present the framework for multiscale methods for nonlinear elliptic problems.

2.1. Setting and definitions. Let V denote the solution space, then we are looking for $u^\epsilon \in V$ with

$$\int_\Omega A^\epsilon(\cdot, \nabla u^\epsilon) \cdot \nabla \Phi = f(\Phi) \qquad \forall \Phi \in V. \tag{2.1}$$

Here, Ω denotes the computational domain, $f \in V'$ some source term and $A^\epsilon : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ a function which is differentiable in the second variable, i.e. $A^\epsilon(x, \cdot) \in \left(C^1(\mathbb{R}^n)\right)^n$ a.e. in x. For generality, we do not specify A^ϵ any further and we do we make any assumptions on the type of the nonlinearity. We only presume solvability of the arising equations and ellipticity of the Jacobian matrix of $A(x, \cdot)$, i.e. there exists some $\alpha > 0$ with

$$(D_\xi A^\epsilon)(x, \xi)\eta \cdot \eta \geq \alpha |\eta|^2 \qquad \forall (x, \xi) \in \Omega \times \mathbb{R}^n.$$

All these assumptions are for instance fulfilled, if A^ϵ is a monotone operator.

2.2. Motivation. Let us sketch the idea behind the multiscale method framework that we are dealing with. The following approach is often used for motivating the Variational Multiscale Method (VMM).

Let $\mathcal{V}$ denote a sufficiently accurate finite dimensional subspace of V, such that $\inf_{v\in\mathcal{V}} \|u^\epsilon - v\|_V \leq \text{TOL}$. Let us also assume that we have a splitting $\mathcal{V} = \mathcal{V}^c \oplus \mathcal{V}^f$, where $\mathcal{V}^c$ denotes a coarse-scale space (i.e. a space in which we can approximate the averaged behaviour of the solution) and a fine-scale space (i.e. a space in which we can approximate 'details' such as the fast oscillations of the solution). Let us consider a corresponding Galerkin approximation of the original problem with:

$$\text{find } \mathcal{U}^\epsilon \in \mathcal{V}: \qquad \int_\Omega A^\epsilon(\cdot, \nabla \mathcal{U}^\epsilon) \cdot \nabla \bar{\Phi} = f(\bar{\Phi}) \qquad \forall \bar{\Phi} \in \mathcal{V}.$$

We rewrite the equation in terms of the splitting $\mathcal{V} = \mathcal{V}^c \oplus \mathcal{V}^f$ to obtain the following coupled problem:

$$\begin{aligned} \text{find } \mathcal{U}^\epsilon \in \mathcal{V}: \qquad & \int_\Omega A^\epsilon(\cdot, \nabla \mathcal{U}^\epsilon) \cdot \nabla \Phi = f(\Phi) \qquad \forall \Phi \in \mathcal{V}^c, \\ & \int_\Omega A^\epsilon(\cdot, \nabla \mathcal{U}^\epsilon) \cdot \nabla \phi = f(\phi) \qquad \forall \phi \in \mathcal{V}^f, \end{aligned} \tag{2.2}$$

Let us define a nonlinear corrector operator $\mathcal{Q}: \mathcal{V}^c \to \mathcal{V}^f$ by:

$$\text{for } \Phi \in \mathcal{V}^c, \ \mathcal{Q}(\Phi) \in \mathcal{V}^f \text{ solves}: \quad \int_\Omega A^\epsilon(\cdot, \nabla \Phi + \nabla \mathcal{Q}(\Phi)) \cdot \nabla \phi = f(\phi) \qquad \forall \phi \in \mathcal{V}^f.$$

Defining $\mathcal{R}(\Phi) := \mathcal{Q}(\Phi) + \Phi$, we can rewrite (2.2) by

$$\text{find } u^c \in \mathcal{V}^c: \qquad \int_\Omega A^\epsilon(\cdot, \nabla \mathcal{R}(u^c)) \cdot \nabla \Phi = f(\Phi) \qquad \forall \Phi \in \mathcal{V}^c, \tag{2.3}$$

$$\int_\Omega A^\epsilon(\cdot, \nabla \mathcal{R}(u^c)) \cdot \nabla \phi = f(\phi) \qquad \forall \phi \in \mathcal{V}^f, \tag{2.4}$$

since $\mathcal{U}^\epsilon = \mathcal{R}(u^c)$. We call $\mathcal{R}$ the reconstruction operator. Once $\mathcal{R}$ is determined, we only need to treat the coarse-scale equation (2.3), which is generally cheap to solve. Typically we use a quadrature rule (with local averaging) to approximate the two equations above. For instance, if $\mathcal{P}$ denotes a partition of Ω into elements D, then (2.3) can be re-written as

$$\sum_{D\in\mathcal{P}} |D| \fint_D A^\epsilon(\cdot, \nabla \mathcal{R}(u^c)) \cdot \nabla \Phi = f(\Phi).$$

If we restrict the fine scale equation (2.4) to the local parts D (to decrease the computational effort by solving them independently from each other), we might get

$$\int_D A^\epsilon(\cdot, \nabla \mathcal{R}(u^c)) \cdot \nabla \phi = f_D(\phi)$$

for all localized fine scale functions $\phi \in \mathcal{V}^f$ and with some localization f_D of f and with an appropriate boundary condition on D. Often, $f_D(\phi)$ is replaced by 0 since it is expected to remain small.

2.3. Multiscale Methods. In the spirit of the above motivation, we now state a general formulation of a framework for multiscale methods for elliptic problems. This framework is proposed to determine the effective macroscopic behavior of the solution of problem (2.1).

DEFINITION 2.1. *Let $\Omega \subset \mathbb{R}^n$ denote a bounded domain with dimension n and let $\mathcal{P}$ denote a non-overlapping partition of Ω with elements D (e.g. a simple triangulation or a more sophisticated collection of large patches). A quadrature rule on D is given by $\{(q_{D,i}, x_{D,i}) | i \in I_D\}$ where $q_{D,i}$ denote the weights and $x_{D,i}$ denote the quadrature points. I_D is an associated index set. An appropriate open environment of $x_{D,i}$ is given by $M_{D,i}$. Note that the sets $M_{D,i}$ are not necessarily disjoint. We define the discrete coarse-scale operator by:*

$$\mathcal{A}(u, \Phi) := \sum_{D \in \mathcal{P}} \sum_{i \in I_D} q_{D,i} \fint_{M_{D,i}} A^\epsilon \left(\cdot, \nabla \mathcal{R}_{D,i}(u)\right) \cdot \nabla \Phi$$

and we solve

$$\text{find } u^c \in \mathcal{V}^c : \qquad \mathcal{A}(u^c, \Phi) = f(\Phi) \qquad \forall \Phi \in \mathcal{V}^c,$$

where $\mathcal{V}^c$ is an appropriate discrete coarse-scale space. For $\Phi \in \mathcal{V}^c$, the local reconstruction $\mathcal{R}_{D,i}(\Phi)$ with $\mathcal{R}_{D,i}(\Phi) - \Phi \in \mathcal{V}^f(O_{D,i})$ is defined as the solution of:

$$\int_{O_{D,i}} A^\epsilon \left(x, \nabla \mathcal{R}_{D,i}(\Phi)(x)\right) \cdot \nabla \phi(x) \; dx = F_{O_{D,i}}(\phi) \quad \forall \phi \in \mathcal{V}^f(O_{D,i}).$$

Here $O_{D,i} \supseteq M_{D,i}$ is an oversampling set (to erase the effect of a possibly wrong boundary condition); $\mathcal{V}^f(O_{D,i})$ a suitable fine-scale space containing an appropriate boundary condition for $O_{D,i}$ and $F_{O_{D,i}}$ a suitable right hand side for these local problems.

Next, we give three examples for explicit realizations of multiscale methods that fit into this framework. For more details on the explicit methods (HMM, MsFEM, VMM), we refer to the literature stated in the introduction.

HMM: Let $\mathcal{T}_H$ denote a regular simplicial partition of Ω with elements T (i.e. $\mathcal{P} = \mathcal{T}_H$). The barycenter of T is denoted by x_T (i.e. x_T is the quadrature point). For small parameters $\delta_2 \geq \delta_1 \geq \epsilon$, $Y_{T,\delta_i} := \{x_T + \delta_i y | \; y \in (0,1)^n\}$ defines a cubic environment of x_T (i.e. '$M_{D,i} = Y_{T,\delta_1}$' and e.g. '$O_{D,i} = Y_{T,\delta_2}$'). If the discrete coarse scale space $V^c = V_H(\Omega)$ is given by

$$V_H(\Omega) := \{\Phi_H \in \mathring{H}^1(\Omega) \cap C^0(\Omega) \;\; |\Phi_{H|T} \in \mathbb{P}^1(T) \;\; \forall T \in \mathcal{T}_H\}, \tag{2.5}$$

the framework yields the following realization of the HMM:

$$\text{find } u_H \in V_H(\Omega) : \sum_{T \in \mathcal{T}_H} |T| \fint_{Y_{T,\delta_1}} A^\epsilon \left(\cdot, \nabla \mathcal{R}_T(u_H)\right) \cdot \nabla \Phi_H = f(\Phi_H) \quad \forall \Phi_H \in V_H(\Omega).$$

For the local problems we obtain that $\mathcal{R}_T(\Phi_H) \in \Phi_H + W_h(Y_{T,\delta_2})$ solves

$$\int_{Y_{T,\delta_2}} A^\epsilon(\cdot, \nabla \mathcal{R}_T(\Phi_H)) \cdot \nabla \phi_h = 0 \quad \forall \phi_h \in W_h(Y_{T,\delta_2}),$$

where $W_h(Y_{T,\delta_2})$ is a discrete function space with a periodic boundary condition on Y_{T,δ_2}. Here, $F_{O_{D,i}}$ is chosen to be equal to zero, since the values of a fine-scale function ϕ_h remain typically small in comparison to the values of the gradient $\nabla\phi_h$.

MsFEM: Again, let $\mathcal{T}_H$ denote a regular simplicial partition of Ω with elements T and let $V_H(\Omega)$ be given by (2.5). For the MsFEM, we use T itself as an environment of the quadrature point and we get from the framework:

$$\text{find } u_H \in V_H(\Omega): \quad \sum_{T\in\mathcal{T}_H} |T| \fint_T A^\epsilon(\cdot, \nabla\mathcal{R}_T(u_H)) \cdot \nabla\Phi_H = f(\Phi_H) \quad \forall \Phi_H \in V_H(\Omega).$$

For the local problems we introduce a simplicial environment $\tilde{T} \supset T$ for the oversampling. With this, we obtain that $\mathcal{R}_T(\Phi_H) \in \Phi_H + W_h(\tilde{T})$ solves

$$\int_{\tilde{T}} A^\epsilon(\cdot, \nabla\mathcal{R}_T(\Phi_H)) \cdot \nabla\phi_h = 0 \quad \forall \phi_h \in W_h(\tilde{T}),$$

where $W_h(\tilde{T})$ can be a discrete function space with a homogenous Dirichlet boundary condition on $\tilde{T}$.

VMM: The situation for the VMM is similar to the preceding situations and as already indicated in the motivation in Section 2.2. Typically, we start with a coarse-scale discrete function space $\mathcal{V}^c$. We refine the underlying computational grid several times until we get a 'high-resolution' discrete function space $\mathcal{V}$. We introduce a projection $\pi : \mathcal{V} \to \mathcal{V}^c$, so that $\mathcal{V}_c = \{v \in \mathcal{V} | (1-\pi)v = 0\}$. Defining $\mathcal{V}^f := \{v \in \mathcal{V} | \pi(v) = 0\}$, we obtain the desired splitting $\mathcal{V} = \mathcal{V}^c \oplus \mathcal{V}^f$. At this point (and with $F_{O_{D,i}}(\phi) = f(\phi)$), we obtain the same situation as in the equations (2.3) and (2.4). There are various possibilities for a localization strategy. Typically, a partition $\mathcal{P}$ in the VMM-case consists of large blocks, where each of them contains several elements of the coarse grid.

3. The Newton-scheme framework. In Section 2.3, we introduced a general framework for formulating multiscale methods such as HMM, MsFEM and VMM. However, there is still a missing piece in the framework. On the one hand, it is clear how to solve the nonlinear local problems with standard schemes like Netwon's method. On the other hand, it is not clear how to solve the nonlinear coarse-scale problem. The difficulty is that we apply the nonlinear function A^ϵ to the nonlinear reconstruction operator $\mathcal{R}_{D,i}$. Using Newton's method for the macro problem therefore means that we need some information on the Fréchet derivative of $\mathcal{R}_{D,i}$. In the following we derive an algorithm for computing the desired information about $\mathcal{R}_{D,i}$. This approach is a generalization to the particular case for the HMM as proposed by Henning and Ohlberger [10].

We make use of the notation introduced in Definition 2.1. Let us furthermore define the local correction operator $\mathcal{Q}_{D,i}$ by $\mathcal{Q}_{D,i}(\Phi) := \mathcal{R}_{D,i}(\Phi) - \Phi$. By $\{\Phi_j | \ 1 \le j \le J\}$ we denote a basis of the coarse space $\mathcal{V}^c$. The coarse-scale problem reads:

$$\text{find } u^c \in \mathcal{V}^c \text{ with}: \quad \mathcal{A}(u^c, \Phi_j) - f(\Phi_j) = 0 \qquad \forall \Phi_j, \ 1 \le j \le J.$$

We are looking for $\bar{\alpha} \in \mathbb{R}^J$ with $G(\bar{\alpha}) = 0$, where $G : \mathbb{R}^J \to \mathbb{R}^J$ is defined by

$$(G(\alpha))_k := \left(\sum_{D \in \mathcal{P}} \sum_{i \in I_D} q_{D,i} \fint_{M_{D,i}} A^\epsilon(\cdot, \sum_{j=1}^J \alpha_j \nabla \Phi_j + \nabla \mathcal{Q}_{D,i}(\sum_{j=1}^J \alpha_j \Phi_j)) \cdot \nabla \Phi_k \right) - f(\Phi_k). \tag{3.1}$$

When $\bar{\alpha}$ is computed, we get u^c by the relation $u^c = \sum_{j=1}^J \bar{\alpha}_j \Phi_j$. Next, we apply Newton's method for solving the nonlinear algebraic equation $G(\bar{\alpha}) = 0$. If $D_\alpha G$ denotes the Jacobian matrix of G, we get the following iteration scheme:

$$\alpha^{(n+1)} := \alpha^{(n)} + \triangle \alpha^{(n)},$$

where $\triangle \alpha^{(n)}$solves:

$$D_\alpha G(\alpha^{(n)}) \triangle \alpha^{(n)} = -G(\alpha^{(n)}).$$

It remains to compute the components of $D_\alpha G$, i.e. to compute $\frac{\mathrm{d}}{\mathrm{d}\alpha_l}(G(\alpha))_k$. Since

$$\begin{aligned} &D_{\alpha_l} A^\epsilon(\cdot, \sum_{j=1}^J \alpha_j \nabla \Phi_j + \nabla \mathcal{Q}_{D,i}(\sum_{j=1}^J \alpha_j \Phi_j)) \\ &= D_\xi A^\epsilon(\cdot, \sum_{j=1}^J \alpha_j \nabla \Phi_j + \nabla \mathcal{Q}_{D,i}(\sum_{j=1}^J \alpha_j \Phi_j))(\nabla \Phi_l + \nabla(D_{\alpha_l}(\mathcal{Q}_{D,i}(\sum_{j=1}^J \alpha_j \Phi_j)))), \end{aligned}$$

we observe that we have to determine $D_{\alpha_l}(\mathcal{Q}_{D,i}(\sum_{j=1}^J \alpha_j \Phi_j))$ which depends on the effect of the Fréchet derivative of the correction operator. To express this unknown, we make use of the local problems for which we have:

$$\begin{aligned} 0 &= D_{\alpha_l} \int_{O_{D,i}} A^\epsilon(\cdot, \sum_{j=1}^J \alpha_j \nabla \Phi_j + \nabla \mathcal{Q}_{D,i}(\sum_{j=1}^J \alpha_j \Phi_j)) \cdot \nabla_y \phi \\ &= \int_{O_{D,i}} (D_\xi A^\epsilon)(\cdot, \sum_{j=1}^J \alpha_j \nabla \Phi_j + \nabla \mathcal{Q}_{D,i}(\sum_{j=1}^J \alpha_j \Phi_j)) \, (\nabla \Phi_l + \nabla(D_{\alpha_l}(\mathcal{Q}_{D,i}(\sum_{j=1}^J \alpha_j \Phi_j)))) \cdot \nabla \phi. \end{aligned}$$

Since this equation holds for every $\phi \in \mathcal{V}^f(O_{D,i})$, we almost have a characterization of $(D_{\alpha_k}(\mathcal{Q}_{D,i}(\sum_{j=1}^J \alpha_j \Phi_j)))$, it only remains to specify the solution space (or the boundary condition) for this term. If we have a periodic- or homogenous Dirichlet-boundary condition for $\mathcal{Q}_{D,i}(\Phi)$ (as for HMM, MsFEM and VMM), it is clear that we have to use the same for $(D_{\alpha_k}(\mathcal{Q}_{D,i}(\Phi)))$. In more general scenarios, we should also use $\mathcal{V}^f(O_{D,i})$ as the solution space for $(D_{\alpha_k}(\mathcal{Q}_{D,i}(\Phi)))$, since the effect of a possibly wrong boundary condition is still minimized by oversampling. Finally, $(D_{\alpha_k}(\mathcal{Q}_{D,i}(\sum_{j=1}^J \alpha_j \Phi_j)))$ can be computed and the remaining procedure is straightforward: we assemble $G(\alpha^{(n)})$ and solve the linear system $D_\alpha G(\alpha^{(n)}) \triangle \alpha^{(n)} = -G(\alpha^{(n)})$. However, if we only use a standard Newton scheme, we can not guarantee global convergence of the method but only local convergence. To overcome this, we use a damped Newton Method for the final algorithm. In order to construct a sequence $a_n := |G(\alpha^{(n)})|_2$ that is monotone decreasing, we determine a suitable damping factor λ_n such that $\alpha^{(n+1)} := \alpha^{(n)} + \lambda_n \triangle \alpha^{(n)}$ and $|G(\alpha^{(n)})|_2 > |G(\alpha^{(n+1)})|_2$. The subsequent *damped Newton algorithm for multiscale methods* is a detailed summary of the whole strategy above. We use the notation introduced in Definition 2.1. Furthermore, $(G(\alpha))_k$ is defined as in (3.1).

Algorithm: multiscaleDampedNewton(*abstol*, *reltol*, $\alpha^{(0)}$)

Set $\alpha^{(n)} := \alpha^{(0)}$.
Set $u_c^{(n)} := \sum_{j=1}^{J} \alpha_j^{(n)} \Phi_j$.
Set $tol := |G(\alpha^{(0)})|_2 \cdot reltol + abstol$.

while $|G(\alpha^{(n)})|_2 > tol$ **do**
  Set $u_c^{(n)} := \sum_{j=1}^{J} \alpha_j^{(n)} \Phi_j$.
  foreach $O_{D,i}$ **do**
    Compute $\mathcal{R}_{D,i}(u_c^{(n)})$ with $\mathcal{R}_{D,i}(u_c^{(n)}) - u_c^{(n)} \in \mathcal{V}^f(O_{D,i})$ and

$$\int_{O_{D,i}} A^\epsilon \left(x, \nabla \mathcal{R}_{D,i}(u_c^{(n)})(x)\right) \cdot \nabla \phi(x) \; dx = F_{O_{D,i}}(\phi) \quad \forall \phi \in \mathcal{V}^f(O_{D,i}).$$

    foreach *coarse-scale basis function* Φ_l **do**
      Compute $D_{\mathcal{Q}_{D,i}}(\Phi_l, u_c^{(n)}) \in \mathcal{V}^f(O_{D,i})$ with

$$0 = \int_{O_{D,i}} (D_\xi A^\epsilon)(\cdot, \nabla \mathcal{R}_{D,i}(u_c^{(n)})) \, (\nabla \Phi_l + \nabla D_{\mathcal{Q}_{D,i}}(\Phi_l, u_c^{(n)}) \cdot \nabla \phi.$$

      $\forall \phi \in \mathcal{V}^f(O_{D,i})$.
    end
  end
  Define the entries of the stiffness matrix $M^{(n)}$ by:

$$M_{kl}^{(n)} := \sum_{D \in \mathcal{P}} \sum_{i \in I_D} q_{D,i} \fint_{M_{D,i}} (D_\xi A^\epsilon) \left(\cdot, \nabla \mathcal{R}_{D,i}(u_c^{(n)})\right) (\nabla \Phi_l + \nabla D_{\mathcal{Q}_{D,i}}(\Phi_l, u_c^{(n)})) \cdot \nabla \Phi_k$$

  Define the entries of the right hand side by:

$$F_k^{(n)} := f(\Phi_k) - \sum_{D \in \mathcal{P}} \sum_{i \in I_D} q_{D,i} \fint_{M_{D,i}} A^\epsilon(\cdot, \nabla \mathcal{R}_{D,i}(u_c^{(n)})) \cdot \nabla \Phi_k.$$

  Find $(\triangle \alpha)^{(n+1)} \in \mathbb{R}^J$, with

$$M^{(n)} (\triangle \alpha)^{(n+1)} = F^{(n)}.$$

  Set $\lambda_n := 1$.
  Set $\alpha^{(n+1)} := \alpha^{(n)} + \lambda_n \triangle \alpha^{(n)}$.
  while $G(\alpha^{(n+1)}) \geq G(\alpha^{(n)})$ **do**
    Set $\lambda_n := \frac{1}{2} \lambda_n$
    Set $\alpha^{(n+1)} := \alpha^{(n)} + \lambda_n \triangle \alpha^{(n)}$.
  end
  Set $\alpha^{(n)} := \alpha^{(n+1)}$.
  Set $tol := |G(\alpha^{(n)})|_2 \cdot reltol + abstol$.
end

Set $u^c := \sum_{j=1}^{J} \alpha_j^{(n)} \Phi_j$.

In the algorithm $D_{\mathcal{Q}_{D,i}}(\Phi_l, u_c^{(n)}) \in \mathcal{V}^f(O_{D,i})$ needs to be computed for any $O_{D,i}$ and any coarse-scale basis function Φ^j. Note that we typically only need to consider the combinations with $\operatorname{supp}\{\Phi^j\} \cap O_{D,i} \neq \emptyset$.

4. Numerical experiment. In this section, we use the proposed framework for multiscale methods to determine the effective macroscopic properties of the solution of the following nonlinear elliptic problem:

Model problem. *Let us define* $\Omega := (0,1)^2 \subset \mathbb{R}^2$ *and* $\epsilon := 10^{-5}$. *Find* u^ϵ *with:*

$$\begin{aligned} -\nabla \cdot A^\epsilon(\cdot, \nabla u^\epsilon) &= f \quad \text{in } \Omega, \\ u^\epsilon &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The source is given by

$$f(x) = -\sum_{i,j=1, i\neq j}^{2} 2(x_i - x_j^2) - 12(2x_i - 1)^2(x_j^2 - x_j)^3$$

and the nonlinear diffusion operator is given by

$$A^\epsilon(x,\xi) := \begin{pmatrix} \xi_1 + (2 + \sin(2\pi \frac{x_1+x_2}{\epsilon}))\xi_1^3 - d_{12}^\epsilon(x) \\ \xi_2 + (2 + \sin(2\pi \frac{x_1+x_2}{\epsilon}))\xi_2^3 - d_{21}^\epsilon(x) \end{pmatrix},$$

where

$$\begin{aligned} h_{ij}^\epsilon(x) &:= \Big(3(2x_i - 1)(x_j^2 - x_j) + 3(x_i + x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon})\Big) \\ &\quad \cdot(2x_i - 1)(x_j^2 - x_j)(x_i + x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon}); \\ g_{ij}^\epsilon(x) &:= (2 + \sin(2\pi\frac{x_i + x_j}{\epsilon}))\left(h_{ij}^\epsilon(x) + \Big((x_i + x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon})\Big)^3\right); \\ d_{ij}^\epsilon(x) &:= (x_i + x_j)\cos(2\pi\frac{x_i}{\epsilon})\sin(2\pi\frac{x_j}{\epsilon}) + \sin(2\pi\frac{x_i + x_j}{\epsilon})(2x_i - 1)^3(x_j^2 - x_j)^3 + g_{ij}^\epsilon(x). \end{aligned}$$

The solution u^ϵ of this problem has an asymptotic expansion $u^\epsilon(x) = u_0(x) + \epsilon u_1(x, \frac{x}{\epsilon})$ (c.f. Hoang [11]) where the exact homogenized solution $u_0 \in \mathring{H}^1(\Omega)$ is given by:

$$u_0(x) = -(x_1^2 - x_1)(x_2^2 - x_2).$$

u_0 can be characterized as the strong L^2-limit of u^ϵ. For a general homogenization theory for monotone operators, we refer to the work of Wall [21].

Now, we use the framework to compute an approximation of the effective part u_0. Due to the extremely fine micro-structure and due to the fact that we know the period ϵ, it is reasonable to use the framework in the HMM-setting as stated in Subsection 2.3. So let $V_H(\Omega)$ denote the discrete coarse scale space defined in (2.5). $u_H \in V_H(\Omega)$ denotes the HMM-approximation. For the cell-size δ (in $Y_{T,\delta}$) we use $\delta = \epsilon$. It remains to specify the discrete fine-scale space $W_h(Y_{T,\delta})$. Here we use a standard Lagrange finite element space based on a periodic triangulation of $Y_{T,\delta}$ and with piecewise polynomials of degree 1. For the mesh size h, we choose $h := \delta H$. This coupling of micro mesh size and macro mesh size yields an optimal order of convergence for the L^2-error $\|u_H - u_0\|_{L^2(\Omega)}$.

H	h	$\frac{\|u_H - u_0\|_{L^2(\Omega)}}{\|u_0\|_{L^2(\Omega)}}$
2^{-2}	$\epsilon 2^{-2}$	0.1892
2^{-3}	$\epsilon 2^{-3}$	0.0359
2^{-4}	$\epsilon 2^{-4}$	0.0089
2^{-5}	$\epsilon 2^{-5}$	0.0024
2^{-6}	$\epsilon 2^{-6}$	0.0006

$(H,h) \to (\frac{H}{2}, \frac{h}{2})$	EOC(e_H)
$(2^{-2}, \epsilon 2^{-2}) \to (2^{-3}, \epsilon 2^{-3})$	2.399
$(2^{-3}, \epsilon 2^{-3}) \to (2^{-4}, \epsilon 2^{-4})$	2.012
$(2^{-4}, \epsilon 2^{-4}) \to (2^{-5}, \epsilon 2^{-5})$	1.886
$(2^{-5}, \epsilon 2^{-5}) \to (2^{-6}, \epsilon 2^{-6})$	1.997

TABLE 4.1

On the left hand side, we see a listing of relative L^2-errors between the HMM-approximation u_H and the homogenized solution u_0. On the right hand side, we see the corresponding experimental orders of convergence, where we denote $e_H := \|u_0 - u_H\|_{L^2(\Omega)}$.

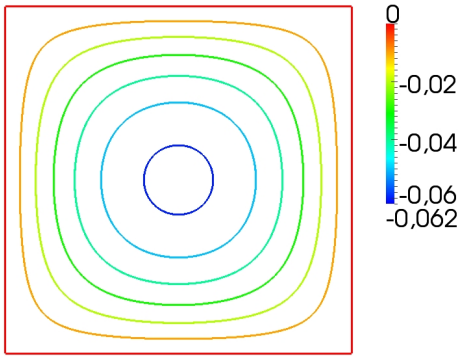

FIG. 4.1. *The figure shows a comparison between the isolines of the homogenized solution and the HMM approximation for $H = 2^{-6}$. We have a color shading from red (indicating the maximum value 0) to blue (indicating the minimum value -0.0625).*

The numerical experiments demonstrate that we obtain very accurate approximations of the homogenized solution u_0 by using the Heterogeneous Multiscale Method combined with the damped Newton algorithm proposed in Section 3. Each approximation u_H is obtained after 3 to 4 Newton steps for the discrete macro-problem. In the left list of Table 4.1 we depict the relative L^2-errors between HMM-approximation and homogenized solution. We observe that we rapidly reach a very high accuracy. The associated experimental orders of convergence are shown in the right listing of Table 4.1. Here, the experimental order of convergence (EOC) for two errors $e_H := \|u_0 - u_H\|_{L^2(\Omega)}$ and $e_{\frac{H}{2}}$ (i.e. for $(H,h) \to (\frac{H}{2}, \frac{h}{2})$) is defined by the ratio

$$\frac{\log\left(\frac{\|e_H\|_{L^2(\Omega)}}{\|e_{\frac{H}{2}}\|_{L^2(\Omega)}}\right)}{\log(2)}.$$

We see that the multiscale method exhibits the expected second order convergence for the L^2-error. In Figure 4.1 we see a comparison of the isolines between the homogenized solution u_0 and the HMM approximation u_H for $H = 2^{-6}$. We observe

that there is no perceptible difference between these isolines. This shows that the algorithm is indeed accurate and effective.

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