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ON THE USE OF NON-LINEAR TVD FILTERS IN FINITE-VOLUME SIMULATIONS*

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Abstract. This paper aims to demonstrate the effect of digital filtering on solution of advection dominated phenomena using finite-volume methods. The non-oscillatory post-processing procedure is based on the simple filter presented in [4]. Some simple one-dimensional tests are presented to show the efficiency of this stabilization technique. Further references are given to the authors papers using this technique as well as to other literature discussing this or other filtering techniques applicable for solution of some other CFD problems.

Key words. data filtering, finite-volume, stabilization, smoothing

AMS subject classifications. 65M08, 65M06, 93E14

1. Introduction. One of the most challenging problems in CFD is the solution of convection dominated problems. In these cases the physical viscosity (or diffusivity) is very small. The numerical solution obtained using certain schemes suffers from non-physical oscillations. This is why many specialized, non-oscillatory discretizations have been developed and successfully used (ENO/WENO, TVD, etc.,see e.g. *Tadmor* [10]). The drawback of the use of such non-oscillatory discretizations is that they are often computationally expensive and hard to implement into already existing codes .

One of the alternative ways of obtaining oscillation-free solution, consists in the post-processing of the resulting data, rather than in modification of the discretization algorithm itself. This idea comes from the area of signal (e.g. image or acoustic signal) processing. It has been found that the algorithms used for the signal denoising can successfully be applied to remove the numerical point-to-point oscillations that often appear in the numerical solution of physical problems.

This approach was described e.g. in *Lafon & Osher* [5]. Some more sophisticated, characteristic based filters were proposed *Yee, Sandham, & Djomehri* [11] and used for turbulent flows simulation in *Lo, Blaisdelly, & Lyrintzis* [6]. A more recent work *Ortleb, Meister, & Sonar* [7] shows the application of a spectral and digital Total Variation filter for discontinuous Galerkin method on unstructured grids.

Although there are various approaches adopted in the wide range of available filters, the general requirements on such noise-filtering algorithm could be summarized into three points:

i) <u>*Conservativity*</u> - The application of the filtering algorithm should not modify the "energy/mass content" of the signal. This results into some kind of an area preserving requirement in the case of single-variable signal post-processing. The violation of this principle may e.g. lead to a wrong shock propagation speed in the numerical solution of the Riemann problem.

ii) <u>*Computational efficiency*</u> - The filter should only be applied in the regions where the oscillations appear. Moreover the filtering procedure should be

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designed to require as few computational operations as possible.

iii) *Quality control* - The user should be given a chance to measure the quality (non-oscillatority) of the solution in order to decide when to stop the post-processing. This could e.g. be done by controlling the Total Variation (TV) of the solution. In this way the filters could be designed to enforce the TVD property of the whole numerical scheme.

2. Digital filter. The filter presented here follows the paper by *Engquist et al.* [4], where a family of simple discrete filters has been proposed and studied. The general idea of this filtering algorithm can be briefly summarized as follows:

1) *Localize* - The oscillations appear as the local minima/maxima of the solution. Thus the filtering will only be applied to the points of local extrema. Non-extremal points will remain unchanged.

2) *Measure* - The amplitude of an oscillation could be estimated e.g. as a fraction of backward/forward difference of the filtered quantity.

3) *Remove* - As the value in the actual point is reduced/increased to remove the oscillatory maximum/minimum, the neighboring value(s) should also be modified to retain the overall conservativity of the algorithm.

Now let's take a closer look to one of the simplest versions of such algorithm. Let's have a set of grid values ϕ_i, for $i = 0, \dots, N$, which approximate the scalar function values $\phi(x_i)$ at regularly spaced grid points x_i. For each of the grid points in the range $i = 1, \dots, N-1$ do the following:

1) To localize the oscillation, compute the backward and forward differences of ϕ, i.e. $(\delta_-\phi)_i = \phi_i - \phi_{i-1}$ and $(\delta_+\phi)_i = \phi_{i+1} - \phi_i$. If $(\delta_-\phi)_i \cdot (\delta_+\phi)_i < 0$, then there is a local extrema in x_i, that needs to be adjusted.

2) Estimate the amplitude of the oscillation at point x_i as

$$osc = \min\Big\{(\delta\phi)_{min},\ 0.5(\delta\phi)_{max}\Big\} \quad \text{, where}$$

$$(\delta\phi)_{min} = \min\{|(\delta_-\phi)_i|,\ |(\delta_+\phi)_i|\} \text{ and } (\delta\phi)_{max} = \max\{|(\delta_-\phi)_i|,\ |(\delta_+\phi)_i|\}$$

3) Adjust the local extrema by subtracting/adding the value $\delta\phi$ to the ϕ_i, i.e. set $\phi_i = \phi_i + \delta\phi$, where $\delta\phi = \mathrm{sign}((\delta_+\phi)_i) \cdot osc$. To preserve the conservativity of the method, redistribute the subtracted/added value osc, to the neighboring point(s). E.g. adjust the value ϕ_{i-1} or ϕ_{i+1}, depending on which of them is further from ϕ_i. I.e. if $|(\delta_-\phi)_i| > |(\delta_+\phi)_i|$ set $\phi_{i-1} = \phi_{i-1} - \delta\phi$, or $\phi_{i+1} = \phi_{i+1} - \delta\phi$, if $|(\delta_-\phi)_i| < |(\delta_+\phi)_i|$.

This filter, originally proposed in [4], was further extended and studied in detail in *Shyy, Chen, Mittal, & Udaykumar* [9]. The above algorithm could be fine-tuned by choosing how much and how often the filter is applied. It means by the appropriate choice of relaxation parameter ω, by the which the correction value $\delta\phi$ can be multiplied, and by the number n_f of the filter passes. The filtering should be repeated until the stopping criterion (TV bound, number of filter passes, ...) is satisfied.

TVD variant of the filter. The above algorithm is just a basic variant of the discrete filtering procedure. Although it is very efficient in removing (or at least controlling) the point-to-point numerical oscillations, it can not guarantee the TVD property of the solution. An improved algorithm has been already presented in the original paper [4], leading (provided some supplementary conditions are satisfied) to a TVD method. The most important part is to keep filtering (correcting) the solution at the point i, till the following conditions are satisfied.

$$\max(\phi_{i-1}^n, \phi_i^n, \phi_{i+1}^n) \geq \phi_i^{n+1} \text{at discrete maxima} \tag{2.1}$$

$$\min(\phi_{i-1}^n, \phi_i^n, \phi_{i+1}^n) \leq \phi_i^{n+1} \text{at discrete minima} \tag{2.2}$$

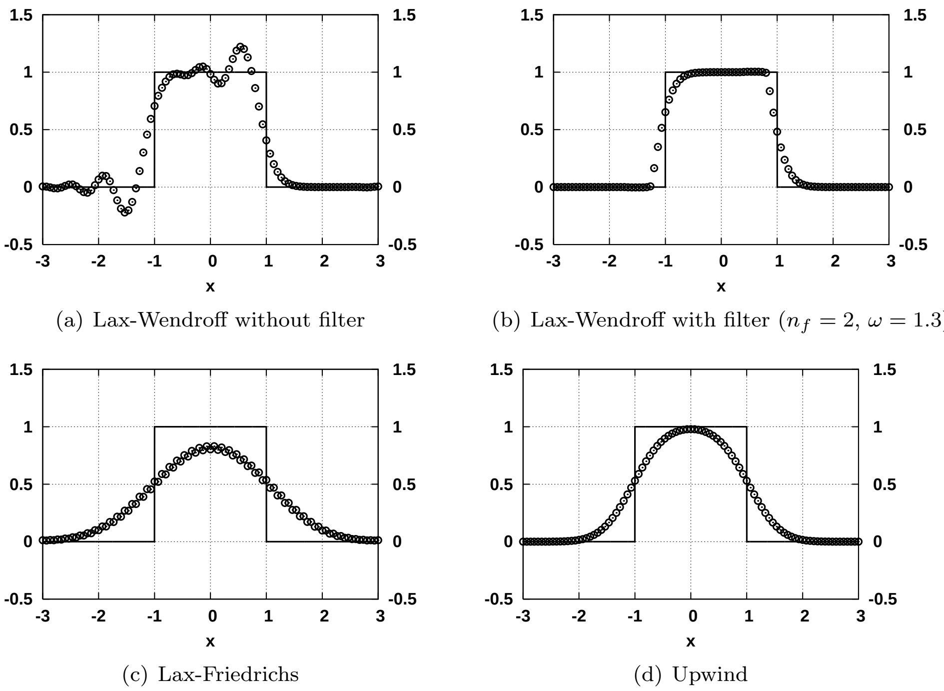
This condition is required to avoid the growth of local extrema in time. Using this assumption, and further requiring that no consecutive extrema appear in the solution, the TVD property of the algorithm was proved in [4].

3. Simple tests. In order to see the effect of the filter on numerical solution, a simple one-dimensional advection equation has been solved

$$\phi_t + \phi_x = 0 \tag{3.1}$$

with initial data distribution $\phi(x, t = 0) = \phi_0(x)$ and periodic boundary conditions. The exact solution is easy. The initial data are just shifted in time along the x coordinate, and due to periodicity of boundary condition reappear at the initial position after some time T_P. Various smooth and discontinuous data were used to show the behavior of the filter.

The effect of the filter could be seen from the following figures. The Lax-Wendroff scheme is taken here as a prototype of oscillatory discretization. Unless stated otherwise, the simulation stopping time was $T = 10 \times T_P$. In figure 3.1 the filtered Lax-Wendroff scheme is compared to its non-filtered version as well as to some low-order, much more dissipative schemes.



(a) Lax-Wendroff without filter

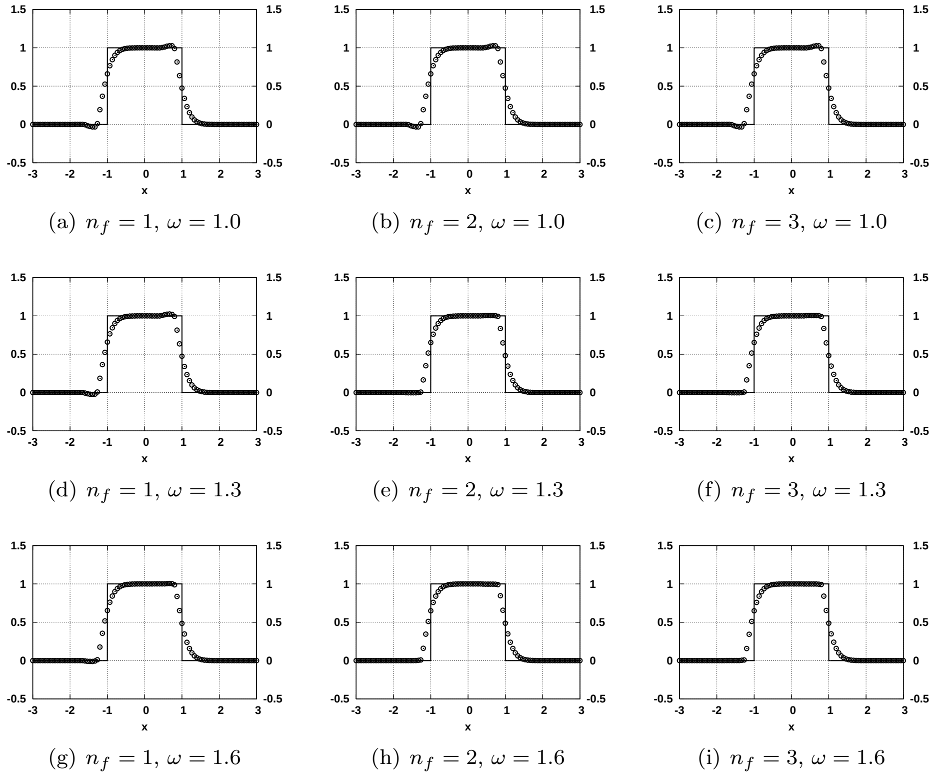
(b) Lax-Wendroff with filter ($n_f = 2$, $\omega = 1.3$)

(c) Lax-Friedrichs

(d) Upwind

FIG. 3.1. *Comparison of the filtered Lax-Wendroff scheme with some other classical numerical schemes CFL=0.5. (○ numerical solution, —exact solution).*

It is obvious that the filtering has largely improved the quality of numerical solution, which remained oscillation-free. When compared to the lower order methods, the filtering evidently retained more resolution. The influence of filter tuning parameters, i.e. the relaxation parameter ω and the number of filter passes n_f, is demonstrated in the figure 3.2. The results suggest that repeated application of the filter has similar effects as using higher relaxation parameter ω. It should however be kept in mind, that each pass of the filter has its computational cost. Moreover it can be shown (see e.g. [9]) that frequent application of the filter may affect also larger wavelengths of the solution (signal).



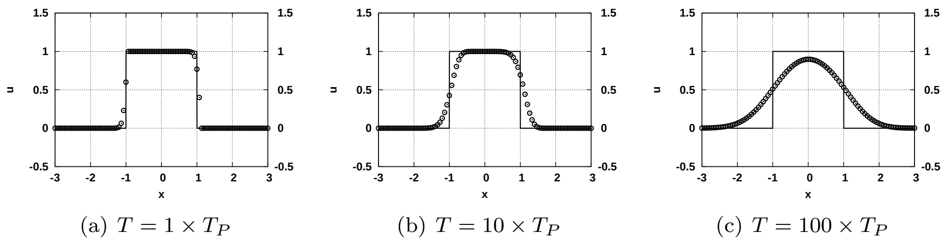
(a) $n_f = 1$, $\omega = 1.0$ (b) $n_f = 2$, $\omega = 1.0$ (c) $n_f = 3$, $\omega = 1.0$

(d) $n_f = 1$, $\omega = 1.3$ (e) $n_f = 2$, $\omega = 1.3$ (f) $n_f = 3$, $\omega = 1.3$

(g) $n_f = 1$, $\omega = 1.6$ (h) $n_f = 2$, $\omega = 1.6$ (i) $n_f = 3$, $\omega = 1.6$

FIG. 3.2. *Results of the filtered Lax-Wendroff scheme for different tuning parameters ω and n_f. ($\circ$ numerical solution, $|$ exact solution).*

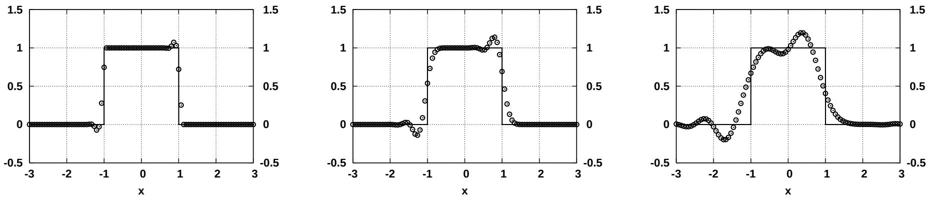
The advantage of filtering becomes more evident when longer time-evolution of the solution is observed. The typical low-order scheme as upwind, will totally smear the original piecewise data as it is shown in the figure 3.3.



(a) $T = 1 \times T_P$ (b) $T = 10 \times T_P$ (c) $T = 100 \times T_P$

FIG. 3.3. *Comparison of the Upwind solution for different solution times. ($\circ$ numerical solution, $|$ exact solution).*

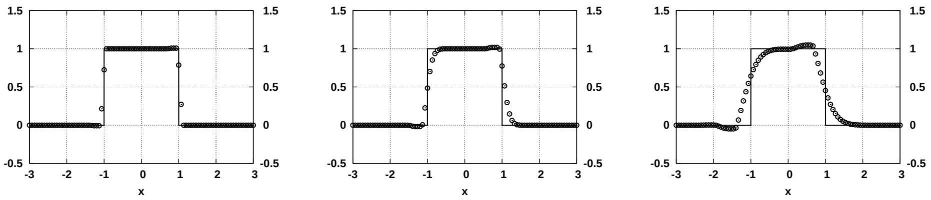
The Lax-Wendroff scheme will on the other hand introduce a growing in time numerical oscillations that will spoil the whole solution (see figure 3.4).



(a) $T = 1 \times T_P$ (b) $T = 10 \times T_P$ (c) $T = 100 \times T_P$

FIG. 3.4. *Comparison of the solution obtained by non-filtered* Lax-Wendroff scheme *at different times T with CFL=0.99. (○ numerical solution, — exact solution).*

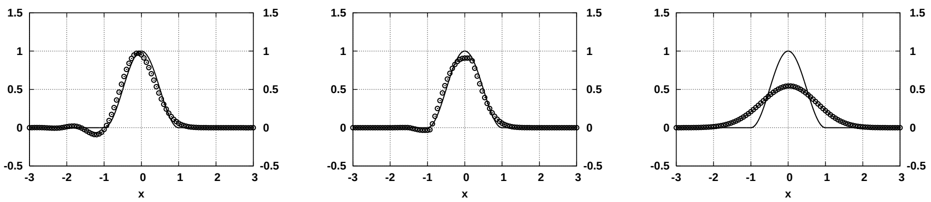
Much better results can be achieved using a filter to postprocess the solution after each time-step. The filtered version of Lax-Wendroff scheme produces results shown in the figure 3.5.



(a) $T = 1 \times T_P$ (b) $T = 10 \times T_P$ (c) $T = 100 \times T_P$

FIG. 3.5. *Comparison of the solution obtained by* filtered Lax-Wendroff scheme *($n_f = 2$, $\omega = 1.3$) at different times T with CFL=0.99. (○ numerical solution, — exact solution).*

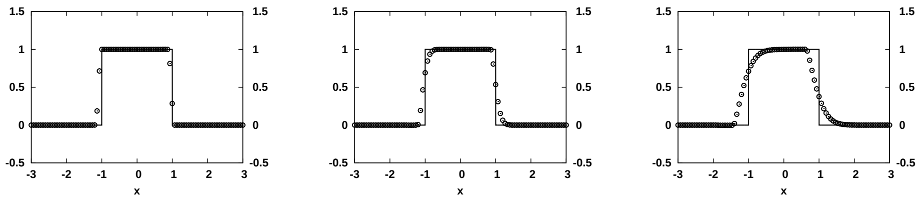
The solution is not oscillation free, but the time growth of oscillations is well controlled by the filter. Stronger filtering might be imposed, but at the price of more computational work or loss of sharpness of the discontinuity. The advection of smooth data is only marginally affected by the filter, as it only acts at the points of local extrema. This is in contrast to upwind solution (or some simple artificial viscosty techniques) where the strong numerical diffusivity is applied globally. This is shown in the figure 3.6



(a) Lax-Wendroff (b) Lax-Wendroff with filter (c) Upwind

FIG. 3.6. *Smooth data advection. Comparison of the solution obtained by non-filtered and filtered Lax-Wendroff scheme ($n_f = 2$, $\omega = 1.3$) with the Upwind scheme at time $T = 100 \times T_P$ with CFL=0.99. (○ numerical solution, — exact solution).*

It is good to note that all the results shown in the above figures 3.1–3.6 were achieved with fixed number of filter passes. A little bit better (in the sense of smoothness) results can be achieved using the TVD version of the filter, where the number of filter passes is decided by the algorithm, based on the quality (i.e. Total Variation) of the solution. The price to pay is the more computationally expensive algorithm and more filter passes required. The solution using Total Variation controlled filter is shown in the figure 3.7.



(a) $T = 1 \times T_P$ (b) $T = 10 \times T_P$ (c) $T = 100 \times T_P$

FIG. 3.7. *Comparison of the solution obtained by* Lax-Wendroff scheme with TVD filter *($\omega = 1.0$) at different times T with CFL=0.99. ($\circ$ numerical solution, — exact solution).*

The effect of the above discussed family of filters can be clearly seen from qualitative comparison of results shown in the figures 3.4 (Lax-Wendroff scheme without filter), 3.5 (Lax-Wendroff scheme with filter), 3.7 (Lax-Wendroff scheme with TVD filter). These qualitative observations can be supported by comparing a norm of the error of numerical solution for all of the above mentioned cases. The discrete ℓ_1 error norm can be defined as

$$\|err\|_1 = \sum_i |\phi_i - \phi^*(x_i)| \tag{3.2}$$

where ϕ_i is the numerical solution and $\phi^*(x_i)$ is the exact solution of the Riemann problem at the node x_i. The evolution in time of the error for the three chosen cases shown in figures 3.4, 3.5 and 3.7 is shown in the following graph in figure 3.8

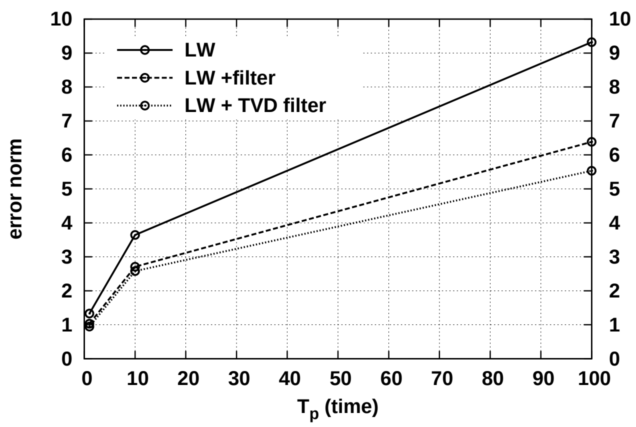


FIG. 3.8. *Comparison of the solution errors at different times T.*

The reduction of the solution error due to the use of filters is obvious. The main purpose of filtering is however to remove non-physical oscillations. This kind of effect is best demonstrated by comparing the Total Variation (TV) of solutions. The Total Variation of numerical solution is defined as:

$$\mathsf{TV} = \sum_i |\phi_i - \phi_{i-1}| \tag{3.3}$$

The evolution in time of the Total Variation for the three cases from figures 3.4, 3.5 and 3.7 is summarized in figure 3.9.

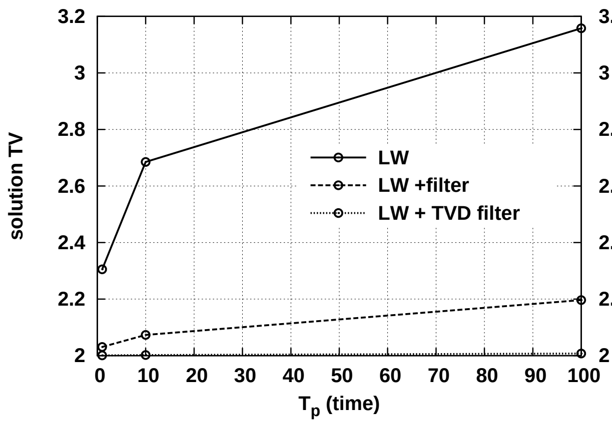


FIG. 3.9. *Comparison of the solution Total Variation at different times T.*

It is evident that the Lax-Wendroff scheme without any filter performs quite badly as the Total Variation of solution grows significantly in time due to numerical oscillations. This effect can largely be reduced by using a simple filter with fixed number of passes ($n_f = 2$, $\omega = 1.3$). There are some over/under-shoots, but their growth in time is well controlled by the filter. Stronger damping is introduced by the TVD variant of the filter that can guarantee the preservation of TV in time without risking an overdamping of solution (like upwind or Lax-Friedrichs schemes do) which would lead to decrease of the solution Total Variation. The price to pay for this efficient TVD filtering is the loos of control over the number of passes of the filter, which can (especially in multidimensional problems) lead to significant increase of computational time.

4. Practical application. The above described filter was e.g. used in the 3D simulation of blood coagulation in flowing blood. Detailed description of the problem can be found in earlier works by *Bodnár & Sequeira* [2] and *Sequeira, Santos, & Bodnár* [8]. Simply speaking, when the blood vessel wall is locally damaged, a cascade of biochemical reactions is initiated, which results in formation of a clot that acts as an obstacle in blood flow. The flow however supplies the basic constituents needed for the blood clot to be built. In this way the interaction of flow (of blood) and structure (clot) is established. The model used in the aforementioned study assumes that the clot can simply be modeled by locally increasing the fluid (blood) viscosity to simulate the highly viscous behavior of clot. The clot (resp. blood) viscosity is assumed to depend on the local concentration of *fibrin*, which is the final product of the cascade of biochemical reactions describing the blood coagulation.

The biochemistry model consists of a set of 23 coupled advection-diffusion-reaction equations. This system has to be solved numerically. One of the major problems in numerically solving these equations is that the physical diffusion coefficients are extremely small (but non-negligible). In order to do not spoil the solution by excessive numerical diffusion the numerical discretization has to be almost non-diffusive. This is the reason why the discontinuous boundary data, simulating the vessel wall injury, generate important oscillations in the numerical solution of concentrations. The over- and especially undershoots are very dangerous, leading to negative concentrations and consequently (due to concentration dependent viscosity) to the blow-up of the whole solution. This is why it is extremely important to keep the numerical oscillations under control.

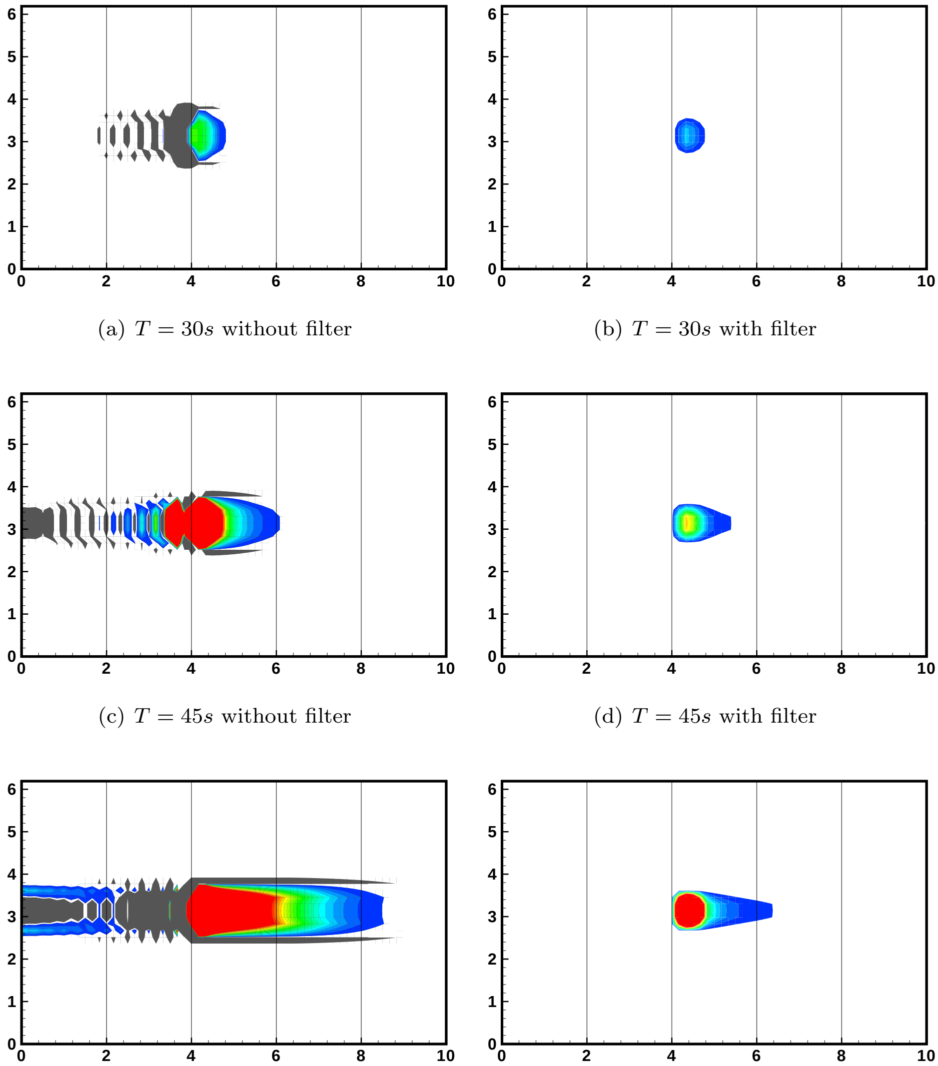
The case solved here represents the blood flow in a simple cylindrical tube (a segment of blood vessel). The local damage of blood vessel wall is simulated in small circular region on the wall where the boundary data (for some concentrations) are suddenly changed. This leads to start of the spatio-temporal evolution of all the chemical constituents that consequently affect the flow. The complete details for this case can be found in [2]. A simple central scheme is used for both, flow and biochemistry model solution. The results obtained without filter and with the use of filter are compared. The surface concentration of fibrin on the vessel wall is shown in figure 4.1. The coordinates are non-dimensionalized using the vessel radius R.

It is evident that the solution obtained without the use of filter is highly oscillatory with non-physical undershoots leading to negative concentrations (marked in gray color in the figure 4.1). The blood coagulation model is highly non-linear and therefore the solution oscillations can accelerate or decelerate the chemical reactions and cause unrealistic clot growth predictions. This behavior was not acceptable and it later led to solution blow-up and crash of simulation.

Applying the simple filter ($n_f = 3$, $\omega = 1.0$) in each grid direction has reduced the numerical oscillations (while retaining the conservativity of the method) and removed the under- and overshoots. The numerical simulation remained stable which has allowed to follow the blood clot formation over a long period of time. The filtering technique was essential in this case as it has stabilized the numerical solution without introducing excessive numerical diffusion.

5. Conclusions & Remarks. The discrete filters described in this paper have turned out to be very powerful and versatile tool in controlling numerical oscillations in advection dominated CFD problems. Besides of the above shown simple tests, they have been successfully applied to various multidimensional problems including biomedical applications *Bodnár & Sequeira* [2] and *Sequeira et al.* [8], or free-surface simulations presented in *Yost & Rao* [12] or *Bodnár & Příhoda* [1].

One of the positive aspects of this filtering technique is that it does not affects the solution as much as some classical artificial viscosity terms, including those imbedded in some modern non-linear schemes. The only locally applied filtering introduces very low numerical diffusion, which is important in cases where it may exceed the physical diffusivity and spoil the numerical solution. This behavior was essential in solving the blood coagulation problem in [2] and [8], where the diffusion coefficients were extremely small. Similar is the problem of solving free-surface flows using VOF-like methods where the gas-liquid interface needs to be kept as sharp as possible.



(a) $T = 30s$ without filter

(b) $T = 30s$ with filter

(c) $T = 45s$ without filter

(d) $T = 45s$ with filter

(e) $T = 60s$ without filter

(f) $T = 60s$ with filter

FIG. 4.1. *Contours of the fibrin concentration on the blood vessel wall.*

One of the drawbacks of this specific filter is that it is exclusively designed for structured grids. It's extension to multidimensional unstructured grids might be difficult (or at least non-trivial) and computationally expensive.

It is interesting to note that although the above filtering algorithm has been described in a fully discrete way, it is also possible to reformulate it in a continuous way to show its relation to some techniques known in signal and image processing. This has been done in *Bürgel & Sonar* [3].

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