Proceedings of ALGORITMY 2012
pp. 230–238

STRUCTURE-PRESERVING FINITE DIFFERENCE SCHEME FOR VORTEX FILAMENT MOTION *

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Abstract. Finite difference scheme for the model equation of vortex filament in three dimensional fluid is proposed. We show that the scheme inherits length-preserving and energy structures from the original model. We also show that solvability of the scheme since the scheme is implicit and nonlinear and give an error estimate of the iteration. Finally, numerical results are shown.

Key words. vortex filament, space curves, finite difference scheme, length-preserving, energy structure.

AMS subject classifications. 65N06, 35K55, 76M23, 35Q35

1. Introduction. The Localized Induction Equation(LIE) is a model of the movement of a vortex filament in three dimensional fluid [2, 1]. In this model, vortex filament is described by a space curve. The position of the vortex filament is denoted by $x(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t)) \in \mathbb{R}^3$, where s and t mean the arc length and time, respectively. Then, (LIE) model is described by

$$x_t = x_s \times x_{ss}.$$

Here, $\times$ denotes the exterior product. By introducing a new variable $u(s,t) := x_s(s,t)$, we have the following equation:

$$u_t = u \times u_{ss}, \tag{1.1}$$

where $u = u(s,t) = (u_1(s,t), u_2(s,t), u_3(s,t)) \in \mathbb{R}^3$. By taking an inner product of (1.1) with u, it is easily shown that $u(s,t) \equiv u(s,0)$ for any s and $t > 0$, that is, the solution of the above equation keeps its length at each point. Thus, in this paper we impose the initial condition

$$u(s,0) = u_0(s), \quad |u_0| = 1$$

for $s \in \Omega = [0,L]$ and periodic boundary condition for simplicity.

Next, we consider the energy-structure. Let $E(u(t)) := ||u_s(t)||^2_{L^2(\Omega)}$ be the energy. Then, we have the energy conservation property:

$$E(u(t)) = E(u_0) \tag{1.2}$$

for any $t > 0$. In [5, 6], the first author proposed a structure-preserving finite difference scheme for the Landau-Lifshitz equation and show that the scheme satisfies (i) length-preserving property and (ii) energy dissipation or conservation property and also show

*This work was supported by Grant-in-Aid for Challenging Exploratory Research No.: 24654026.
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the convergence result. However, an iteration procedure is needed for each time step since the proposed scheme is implicit and nonlinear. Then, by a finite termination of the iteration, there appears errors on the length at each point and the energy. (Note that some estimates of termination errors of the iteration are also shown in [6].) However, the length-preserving property is directly connected to a stability of the scheme and also length of the solution curve to (LIE) model. Thus, in this paper we propose a finite difference scheme and also an iteration which keeps the length of the solution in spite of a finite termination. Moreover, we give an error estimate on the energy in the iteration procedure.

The paper is organized as follows: In the next section, we propose a finite difference scheme which inherits the length-preserving property and energy-structure and also length-preserving iteration for each time step. In Section 3, we analyze the iteration and show some error estimates. In Section 4, test numerical calculation are presented for exact solution. In the last section we mention conclusion remarks.

2. Proposed scheme and iteration.

2.1. Structure-preserving scheme. We set a spatial mesh size $\Delta s > 0$ as $L = N\Delta s$ for some integer N and also set a time step $\Delta t > 0$ arbitrary. We discretize the problem as follows : (S)

$$\frac{U_n^{j+1} - U_n^j}{\Delta t} = U_n^{(j,j+1)} \times \Delta_d U_n^{(j,j+1)}, \qquad 0,1,\cdots,N-1, \quad j = 0,1,2,\cdots, \tag{2.1}$$

$$U_n^0 = u_0(n\Delta x), \qquad n = 0,1,2,\cdots N-1, \tag{2.2}$$

$$U_N^j = U_0^j, \quad U_{-1}^j = U_{N-1}^j, \qquad j = 0,1,2,\cdots. \tag{2.3}$$

Here, $U_n^j = (U_{1,n}^j, U_{2,n}^j, U_{3,n}^j)$ denote a finite difference solution at $s_n = n\Delta s, t_j = j\Delta t$. The notation $U_n^{(j,j+1)}$ and the discrete Laplace operator Δ_d are defined by

$$U_n^{(j,j+1)} := \frac{U_n^{j+1} + U_n^j}{2}$$

and

$$\Delta_d U_n := \frac{U_{n+1} - 2U_n + U_{n-1}}{\Delta s^2} = D^+ D^- U_n$$

where

$$D^+ U_n := \frac{U_{n+1} - U_n}{\Delta s}, \qquad D^- U_n := \frac{U_n - U_{n-1}}{\Delta s}.$$

Here, we introduce the following norms:

$$||U||_2 = \left(\sum_{n=0}^{N-1} |U_n|^2 \Delta s\right)^{1/2}, ||U||_\infty = \max_{0\le n\le N-1} |U_n|.$$

The following shows that (S) inherits the length-preserving property and energy structure from the original problem.

Proposition 2.1. *For any mesh size, we have*

$$|U_n^j| = 1$$

for any j and n and

$$E_d(U^j) = E_d(U^0)$$

for any j. Here, $E_d(U) := ||D^+U||_2^2$.

Proof. By taking an inner product of (2.1) with $U_n^{(j,j+1)}$, it is easily shown that $|U_n^{j+1}| = |U_n^j|$ for any n and j. Thus, we have the first assertion.

Next, we show the energy equality. By taking an inner product of (2.1) with $\Delta_d U_n^{(j,j+1)}$, we have

$$\frac{U_n^{j+1} - U_n^j}{\Delta t} \Delta_d U_n^{(j,j+1)} = 0.$$

Summing up through all n, we have

$$||D^+U^{j+1}||_2^2 = ||D^+U^j||_2^2.$$

Here, we use the summation by parts:

$$\sum_{n=0}^{N-1} (D^- a_n) b_n = -\sum_{n=0}^{N-1} a_n (D^+ b_n)$$

where a_n and b_n are N-periodic. Hence, we have the second assertion. □

REMARK 2.2. *We can easily construct a structure-preserving finite difference scheme for the Dirichlet and the Neumann boundary conditions.*

2.2. Proposed iteration method for (S). Since the scheme (S) is implicit and nonlinear, we need an iterative procedure to obtain $\{U^{j+1}\}$ from $\{U^j\}$. We here introduce a length preserving iteration. For each time step j, we construct a sequence V_n^m by the following iteration:

$$\frac{V_n^{m+1} - U_n^j}{\Delta t} = \frac{V_n^{m+1} + U_n^j}{2} \times \Delta_d \frac{V_n^m + U_n^j}{2}, \tag{2.4}$$
$$n = 0, 1, 2, \cdots, N-1, \quad m = 0, 1, 2, \cdots,$$

$$V_n^0 = U_n^j, \qquad n = 0, 1, 2, \cdots, N-1, \tag{2.5}$$

$$V_{-1}^m = V_{N-1}^m, \quad V_N^m = V_0^m, \qquad m = 0, 1, 2, \cdots. \tag{2.6}$$

By taking an inner product of (2.4) with $V_n^{m+1} + U_n^j$, we have $|V_n^{m+1}| = |V_n^m|$ and also $|V_n^m| = 1$ by (2.5) for any m, n. Thus, the above iteration has a length-preserving property.

We can rewrite the above iteration into a matrix form:

$$A_{m,n} V_n^{m+1} = B_{m,n} U_n^j. \tag{2.7}$$

Here, $A_{m,n}$ and $B_{m,n}$ are the following matrices:

$$A_{m,n} = \begin{pmatrix} 1 & -c_{m,n}^{(3)} & c_{m,n}^{(2)} \\ c_{m,n}^{(3)} & 1 & -c_{m,n}^{(1)} \\ -c_{m,n}^{(2)} & c_{m,n}^{(1)} & 1 \end{pmatrix} \quad \text{and} \quad B_{m,n} = \begin{pmatrix} 1 & c_{m,n}^{(3)} & -c_{m,n}^{(2)} \\ -c_{m,n}^{(3)} & 1 & c_{m,n}^{(1)} \\ c_{m,n}^{(2)} & -c_{m,n}^{(1)} & 1 \end{pmatrix}$$

where $c_{m,n}^{(k)}$ $(k = 1, 2, 3)$ are the k-th component of the vector $\Delta t \Delta_d (V_n^m + U_n^j)/4$. Now, we see that there exists an inverse of $A_{m,n}$ since $\det A_{m,n} = 1 + (c_{m,n}^{(1)})^2 + (c_{m,n}^{(2)})^2 + (c_{m,n}^{(3)})^2 \geq 1$. Therefore, we can solve (2.7) as

$$V_n^{m+1} = {A_{m,n}}^{-1} B_{m,n} U_n^j. \tag{2.8}$$

That is, this procedure is explicit and thus we can easily calculate V^{m+1} from V^m.

The convergence of $\{V_n^m\}_{m=0}^\infty$ is shown in the next subsection. We here suppose that there exists the limit. Then, the limit satisfies the scheme (S) as U_n^{j+1}. For numerical calculation, we set a proper threshold $k_j > 0$ for each j and terminate the iteration when the inequality

$$||V^{m+1} - V^m||_\infty \leq k_j \tag{2.9}$$

holds. Let m_j be the number m at the termination. Then, we set

$$U_n^{j+1} := V_n^{m_j+1} \tag{2.10}$$

for all n. Obviously, this U_n^{j+1} satisfies $|U_n^{j+1}| = 1$ for all n.

2.3. Solvability of (S). We here show the convergence of $\{V_n^m\}_{m=0}^\infty$.

THEOREM 2.3. *If*

$$\eta := \frac{\Delta t}{(\Delta s)^2} < \frac{1}{2}, \tag{2.11}$$

then,

$$||V^{m+2} - V^{m+1}||_\infty \leq \rho ||V^{m+1} - V^m||_\infty, \tag{2.12}$$

where $\rho = 2\eta/(1-2\eta)$. *Moreover, if* $\eta < 1/4$, *then* $\{V_n^m\}_{m=0}^\infty$ *has a limit as* m *tends to infinity for each* n.

Proof. Let $W_n^m = (V_n^m + U_n^j)/2$. Obviously, $|W_n^m| \leq 1$ since $|V_n^m| = |U_n^j| = 1$. By taking a difference between (2.4) at m and $(m+1)$ steps, we obtain

$$\begin{aligned}
\frac{V_n^{m+2} - V_n^{m+1}}{\Delta t} &= W_n^{m+2} \times \Delta_d W_n^{m+1} - W_n^{m+1} \times \Delta_d W_n^m \\
&= W_n^{m+2} \times \Delta_d W_n^{m+1} - W_n^{m+1} \times \Delta_d W_n^{m+1} \\
&\quad + W_n^{m+1} \times \Delta_d W_n^{m+1} - W_n^{m+1} \times \Delta_d W_n^m \\
&= \frac{V_n^{m+2} - V_n^{m+1}}{2} \times \Delta_d W_n^{m+1} + W_n^{m+1} \times \Delta_d \frac{V_n^{m+1} - V_n^m}{2}.
\end{aligned}$$

Thus, we have

$$\frac{||V^{m+2} - V^{m+1}||_\infty}{\Delta t} \leq \frac{2}{(\Delta s)^2} ||V^{m+2} - V^{m+1}||_\infty + \frac{2}{(\Delta s)^2} ||V^{m+1} - V^m||_\infty$$

and thus

$$(1 - 2\eta) ||V^{m+2} - V^{m+1}||_\infty \leq 2\eta ||V^{m+1} - V^m||_\infty.$$

Since $1 - 2\eta > 0$, we have $\rho > 0$ and

$$||V^{m+2} - V^{m+1}||_\infty \leq \rho ||V^{m+1} - V^m||_\infty.$$

Thus, we have the first assertion.

If $\eta < 1/4$, then $0 < \rho < 1$. For any $\ell > m > 0$, we have

$$||V^\ell - V^m||_\infty \leq \frac{\rho^m}{1-\rho}||V^1 - V^0||_\infty.$$

Hence, we have the second assertion. □

3. Analysis of iteration. We already show the proposed iteration has a length-preserving property in the previous section. We here estimate a truncation error of the energy.

THEOREM 3.1. *Assume the assumption in the previous theorem, that is,*

$$\frac{\Delta t}{\Delta x^2} < \frac{1}{4}.$$

Then, the solution $U_n^{j+1} := V_n^{m_j+1}$ *satisfies the following estimate:*

$$\left| ||D^+U^{j+1}||_2^2 - ||D^+U^j||_2^2 \right| \leq 4Lk_j. \tag{3.1}$$

Proof. We first note that the iteration at $m = m_j$ is the following:

$$\frac{V_n^{m_j+1} - U_n^j}{\Delta t} = \frac{V_n^{m_j+1} + U_n^j}{2} \times \Delta_d \frac{V_n^{m_j} + U_n^j}{2}. \tag{3.2}$$

By taking an inner product with $\Delta_d(V_n^{m_j} + U_n^j)$, we get

$$(V_n^{m_j+1} - U_n^j)\Delta_d(V_n^{m_j+1} + U_n^j) + (V_n^{m_j+1} - U_n^j)\Delta_d(-\hat{V}_n) = 0,$$

where $\hat{V}_n = V_n^{m_j+1} - V_n^{m_j}$. By taking the summation and using the summation by parts, we have

$$\begin{aligned} -||D^+V^{m_j+1}||_2^2 + ||D^+U^j||_2^2 &= \sum_{n=0}^{N-1} \Delta x (V_n^{m_j+1} - U_n^j)\Delta_d \hat{V}_n \\ &= \sum_{n=0}^{N-1} \Delta x \Delta t \frac{V_n^{m_j} + U_n^j}{2} \times \Delta_d \frac{V_n^{m_j} + U_n^j}{2} \Delta_d \hat{V}_n. \end{aligned}$$

Therefore,

$$\begin{aligned} \left| -||D^+V^{m_j+1}||_2^2 + ||D^+U^j||_2^2 \right| &\leq \frac{4\Delta t}{\Delta s^2} \left| \sum_{n=0}^{N-1} \Delta s \Delta_d \hat{V}_n \right| \\ &\leq 4L||\hat{V}||_\infty \\ &\leq 4Lk_j. \end{aligned}$$

Thus, we have the assertion. □

$\max_{0\le s_n\le 1, 0\le t_j\le 1} \left\| \|U_n^j\| - 1 \right\|$	$3.11708125 \times 10^{-16}$
$\max_{0\le t_j\le 1} \left\| \frac{E_d(U^n) - E_d(U^0)}{E_d(U^0)} \right\|$	$1.0758463 \times 10^{-10}$

TABLE 4.1
Maximum error of length and maximum relative error of the energy

4. Numerical result. We check the efficiency of the proposed scheme and iteration using an exact solution [7].

Let $\alpha, k \in \mathbb{R}$ and set

$$\begin{aligned} u_1(s,t) &= \sin\alpha\cos(ks - (|k|^2\cos\alpha)t), \\ u_2(s,t) &= \sin\alpha\sin(ks - (|k|^2\cos\alpha)t), \\ u_3(s,t) &= \cos\alpha. \end{aligned}$$

Then, the function $u(s,t) = (u_1(s,t), u_2(s,t), u_3(s,t))$ is a exact solution.

We use the parameters: $k = 4\pi, \alpha = 1$ and also use the numerical parameters: $L = 1$, $\Delta s = 0.02$ $(N = 50)$, $\Delta t = 0.00008$ and $k_j = k = 1.0 \times 10^{-15}$, respectively.

Figure 4.1 shows a view of the first component of the numerical solution in (a) and the exact solution in (b) and also shows the behavior of the energy of numerical solution in (c). Figure 4.2 shows a time evolution of the vortex filament curves $x = x(s,t)$. We see that coiled curve moves upward and the solution curve keeps its shape during a time evolution. Table 4.1 shows the maximum error of the length and maximum relative error of the energy in the time interval $[0,1]$. We see that the numerical solution approximates the exact one and inherits length-preserving and energy conservative properties sufficiently.

5. Conclusion remarks. We construct the finite difference scheme which inherits length-preserving property and energy-structure. Moreover, we also propose length-preserving iteration. From these properties, we have a stability of the scheme by the boundedness of the length of the solution and the energy. We also show the solvability of the scheme and error estimate of energy in the iterative procedure with the termination. We finally show numerical results and check the efficiency of the proposed scheme. We here remark that we can construct the structure-preserving scheme for other boundary conditions, for example, the Dirichlet and the Neumann boundary conditions. Moreover, we can show the convergence of the finite difference solution to the solution of the original problem. However, by the limit of page, we skip this point. We show this in forthcoming paper and the presentation.

REFERENCES

[1] R. J. Arms and F. R. Hama, *Localized Induction Concept on a Curved Vortex and Motion of an Elliptic Vortex Ring*, Phys. Fluids **8** (1965), 553–559.

[2] L. S. Da Rios, *Sul moto d'un liquido indefinito con un filetto vorticoso di forma qualunque*, Rend. Circ. Mat. Palermo, **22** (1906), no. 1, 117–135.

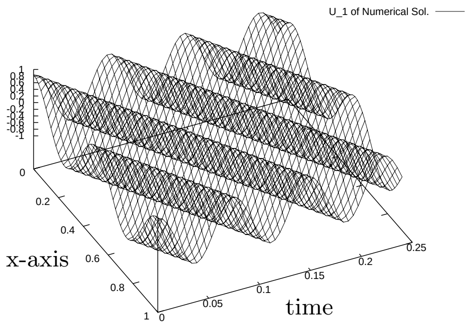
[3] Weinan E, Xiao-ping Wang, *Numerical methods for the Landau-Lifshitz equation*, SIAM J. NUMER. ANAL. **Vol. 38** (2001), 1647–1665.

[4] D. Furihata, *Finite Difference Schemes for* $\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^{\alpha} \frac{\delta G}{\delta u}$ *That Inherit Energy Conservation or Dissipation Property*, J. Comput. Phys. **156** (1999), 181–205.

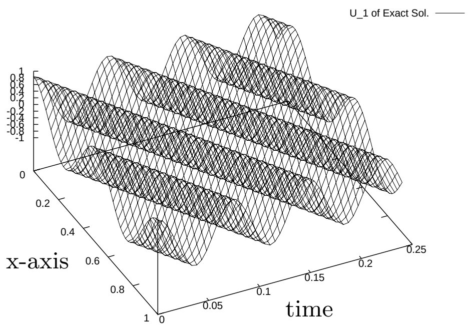
[5] A. Fuwa, T. Ishiwata and M. Tsutsumi, *Finite difference schemes for Landau-Lifshitz equation*, Proceedings of Czech–Japanese Seminar in Applied Mathematics 2006, COE Lecture Note Vol. 6, Kyushu University, 2007, 107–113.

[6] A. Fuwa, T. Ishiwata and M. Tsutsumi, *Finite difference scheme for the Landau-Lifshitz equation*, Japan J. Ind. Appl. Math. **29** (2012), 83–110.

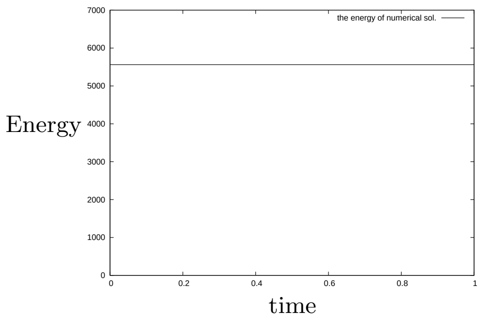
[7] M. Lakshmanan, Th. W. Ruijgrok, C. J. Thompson, *On the dynamics of a continuum spin system*, Physica **84A** (1976), 579–590.



(a) U_1 of the numerical solution

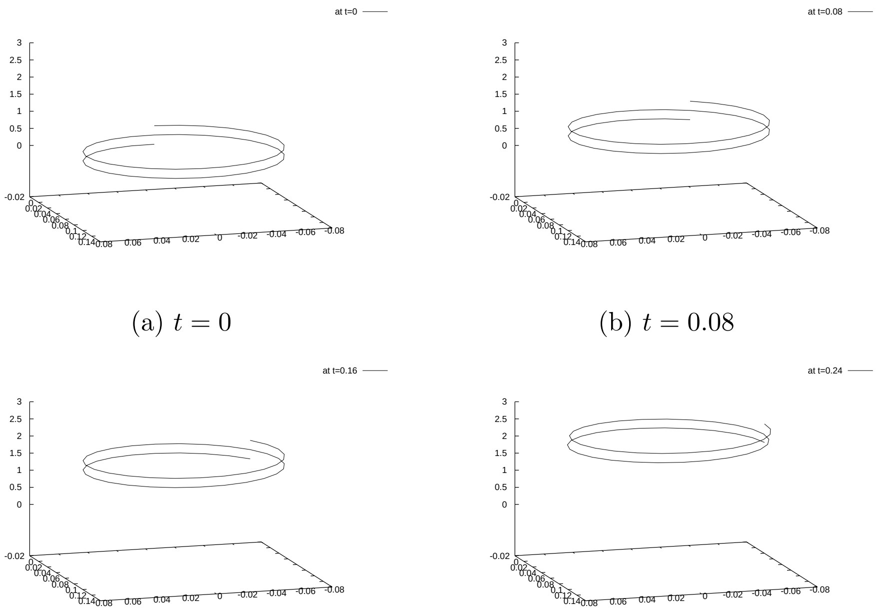


(b) u_1 of the exact solution



(c) The energy of numerical solution

FIG. 4.1. *Time evolution of solutions*



(a) $t = 0$ (b) $t = 0.08$

(c) $t = 0.16$ (d) $t = 0.24$

FIG. 4.2. *Time evolution of vortex filament* $x = x(s, t)$