Proceedings of ALGORITMY 2012
pp. 342–352

SOME SECOND ORDER TIME ACCURATE FOR A FINITE VOLUME METHOD FOR THE WAVE EQUATION USING A SPATIAL MULTIDIMENSIONAL GENERIC MESH*

ABDALLAH BRADJI†

Abstract. The present work is an extension of the previous one [1] which dealt with error analysis of a finite volume scheme of first order (both in time and space) for second order hyperbolic equations on general nonconforming multidimensional spatial meshes introduced recently in [4]. The aim of this contribution is to get some second–order time accurate schemes for a finite volume method for second order hyperbolic equations using the same class of spatial generic meshes stated above. We present a family of implicit time schemes to approximate the wave equation. The time discretization is performed using a one–parameter Newmark method. We prove that, when the discrete flux is calculated using a stabilized discrete gradient, the convergence order is $k^2 + h_{\mathcal{D}}$, where $h_{\mathcal{D}}$ (resp. k) is the mesh size of the spatial (resp. time) discretization. This estimate is valid for discrete norms $\mathbb{L}^\infty(0, T; H_0^1(\Omega))$ and $\mathcal{W}^{1,\infty}(0, T; L^2(\Omega))$! under the regularity assumption $u \in \mathcal{C}^4([0, T]; \mathcal{C}^2(\overline{\Omega}))$ for the exact solution u. These error estimates are useful because they allow to obtain approximations to the exact solution and its first derivatives of order $k^2 + h_{\mathcal{D}}$.

Key words. Second order hyperbolic equations; Wave equation; Non–conforming grid; Second order time accurate; Stabilized discrete gradient; Fully discretization scheme; Multidimensional spatial domain

AMS subject classifications. 65M08, 65M12, 65M15, 35L10

1. Motivation and aim of this paper.

We consider the wave equation, as a model for second order hyperbolic equations:

$$u_{tt}(x, t) - \Delta u(x, t) = f(x, t),\ (x, t) \in \Omega \times (0, T), \tag{1.1}$$

where Ω is an open polygonal bounded subset in $\mathbb{R}^d$, $T > 0$, and f is a given function. An initial condition is given by: for given functions u^0 and u^1 defined on Ω

$$u(x, 0) = u^0(x) \text{ and } u_t(x, 0) = u^1(x)\ x \in \Omega, \tag{1.2}$$

Homogeneous Dirichlet boundary conditions are given by

$$u(x, t) = 0,\ (x, t) \in \partial\Omega \times (0, T). \tag{1.3}$$

Recently, in the previous work [1], it is provided an implicit finite volume scheme approximating the wave problem (1.1)–(1.3). The feature of the finite volume scheme presented in [1] is that the spatial meshes considered are ones used in [4] to approximate stationary equations. The general class of nonconforming multidimensional meshes, see Definition 2.1 given below, introduced recently in [4] has the following advantages:

- The scheme can be applied on any type of grid: conforming or non conforming, 2D and 3D, or more, made with control volumes which are only assumed to be polyhedral (the boundary of each control volume is a finite union of subsets of hyperplanes).

*This work was supported by Algerian Ministry of Higher Education and Scientific Research
†Department of Mathematics, University of Annaba–Algeria (bradji@cmi.univ-mrs.fr; abdallah-bradji@univ-annaba.org).

- When the family of the discrete fluxes are satisfying some suitable conditions, the matrices of the generated linear systems are sparse, symmetric, positive and definite.
- A discrete gradient for the exact solution is formulated and converges to the gradient of the exact solution.

It is proved that the implicit scheme presented in [1] is of order one in both space and time. The aim of the present contribution is to *improve* the order with respect to time using the same mesh described in Definition 2.1. For this reason, we shall use a Newmark's method as a discretization in time. Newmark methods are used as a discretization in time for the wave equation for instance when the spatial discretization is performed using finite difference method in [8], the variational methods in [9], spectral methods in [10], or finite element methods in [7].

2. Definition of the scheme. The discretization of Ω is performed using the mesh $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ described in [4, Definition 2.1] which we recall here for the sake of completeness.

DEFINITION 2.1 (Definition of the spatial mesh, cf. [4]). *Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. A discretization of Ω, denoted by $\mathcal{D}$, is defined as the triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:*

1. *$\mathcal{M}$ is a finite family of non empty connected open disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{K\in\mathcal{M}}\overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K}\setminus K$ be the boundary of K; let $\mathrm{m}\,(K) > 0$ denote the measure of K and h_K denote the diameter of K.*
2. *$\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, σ is a non empty open subset of a hyperplane of $\mathbb{R}^d$, whose $(d-1)$–dimensional measure is strictly positive. We also assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma\in\mathcal{E}_K}\overline{\sigma}$. For any $\sigma \in \mathcal{E}$, we denote by $\mathcal{M}_\sigma = \{K;\ \sigma \in \mathcal{E}_K\}$. We then assume that, for any $\sigma \in \mathcal{E}$, either $\mathcal{M}_\sigma$ has exactly one element and then $\sigma \subset \partial\Omega$ (the set of these interfaces, called boundary interfaces, denoted by $\mathcal{E}_{\mathrm{ext}}$) or $\mathcal{M}_\sigma$ has exactly two elements (the set of these interfaces, called interior interfaces, denoted by $\mathcal{E}_{\mathrm{int}}$). For all $\sigma \in \mathcal{E}$, we denote by x_σ the barycentre of σ. For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by! $\boldsymbol{n}_{K,\sigma}$ the unit vector normal to σ outward to K.*
3. *$\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$, such that for all $K \in \mathcal{M}$, $x_K \in K$ and K is assumed to be x_K–star-shaped, which means that for all $x \in K$, the property $[x_K, x] \subset K$ holds. Denoting by $d_{K,\sigma}$ the Euclidean distance between x_K and the hyperplane including σ, one assumes that $d_{K,\sigma} > 0$. We then denote by $\mathcal{D}_{K,\sigma}$ the cone with vertex x_K and basis σ.*

The time discretization is performed with a constant time step $k = \frac{T}{N+1}$, where $N \in \mathbb{N}^\star$, and we shall denote $t_n = nk$, for $n \in [\![0, N+1]\!]$. Throughout this paper, the letter C stands for a positive constant independent of the parameters of the space and time discretizations and its values may be different in different appearance.
We define the space $\mathcal{X}_\mathcal{D}$ as the set of all $\big((v_K)_{K\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}\big)$, and $\mathcal{X}_{\mathcal{D},0} \subset \mathcal{X}_\mathcal{D}$ is the set of all $v \in \mathcal{X}_\mathcal{D}$ such that $v_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\mathrm{ext}}$. Let $H_\mathcal{M}(\Omega) \subset \mathbb{L}^2(\Omega)$ be the space of piecewise constant functions on the control volumes of the mesh $\mathcal{M}$. For all $v \in \mathcal{X}_\mathcal{D}$, we denote by $\Pi_\mathcal{M}\, v \in H_\mathcal{M}(\Omega)$ the function defined by $\Pi_\mathcal{M}\, v(x) = v_K$, for a.e. $x \in K$, for all $K \in \mathcal{M}$.

For all $\varphi \in \mathcal{C}(\Omega)$, we define $\mathcal{P}_{\mathcal{D}}\varphi = \big((\varphi(x_K))_{K\in\mathcal{M}}\,,\,(\varphi(x_\sigma))_{\sigma\in\mathcal{E}}\big) \in \mathcal{X}_{\mathcal{D}}$. We denote by $\mathcal{P}_{\mathcal{M}}\varphi \in H_{\mathcal{M}}(\Omega)$ the function defined by $\mathcal{P}_{\mathcal{M}}\,\varphi(x) = \varphi(x_K)$, for a.e. $x \in K$, for all $K \in \mathcal{M}$.

In order to analyze the convergence, we need to consider the size of the discretization $\mathcal{D}$ defined by $h_{\mathcal{D}} = \sup\{\mathrm{diam(K)}, K \in \mathcal{M}\}$ and the regularity of the mesh given by $\theta_{\mathcal{D}} = \max\left(\max_{\sigma\in\mathcal{E}_{\mathrm{int}},K,L\in\mathcal{M}} \dfrac{d_{K,\sigma}}{d_{L,\sigma}}, \max_{K\in\mathcal{M},\sigma\in\mathcal{E}_K} \dfrac{h_K}{d_{K,\sigma}}\right)$. The scheme we want to consider in this note (A detailed work [3] which includes general framework is in progress.) is based on the use of the discrete gradient given in [4]. For $u \in \mathcal{X}_{\mathcal{D}}$, we define, for all $K \in \mathcal{M}$

$$\nabla_{\mathcal{D}}\, u(x) = \nabla_{K,\sigma}\, u, \ \text{ a. e. } x \in \mathcal{D}_{K,\sigma}, \tag{2.1}$$

where $\mathcal{D}_{K,\sigma}$ is the cone with vertex x_K and basis σ and

$$\nabla_{K,\sigma} u = \nabla_K\, u + \left(\frac{\sqrt{d}}{d_{K,\sigma}}\,(u_\sigma - u_K - \nabla_K\, u\cdot(x_\sigma - x_K))\right)\mathbf{n}_{K,\sigma}, \tag{2.2}$$

where $\nabla_K\, u = \dfrac{1}{\mathrm{m}(K)}\displaystyle\sum_{\sigma\in\mathcal{E}_K}\mathrm{m}(\sigma)\,(\,u_\sigma - u_K)\,\mathbf{n}_{K,\sigma}$ and d is the space dimension.

The family of finite volume schemes approximating (1.1)–(1.3) we want to study in this work is based on the use of a Newmark's method as discretization in time, see [7, 9, 10]. For a parameter $\gamma \in]\frac{1}{2}, 1]$, we define the finite volume approximate solution as $(\,u^n_{\mathcal{D}})_{n=0}^{N+1} \in \mathcal{X}_{\mathcal{D},0}^{N+2}$ with $u^n_{\mathcal{D}} = \big((\,u^n_K)_{K\in\mathcal{M}}\,,(\,u^n_\sigma)_{\sigma\in\mathcal{E}}\big)$, for all $n \in [\![0, N+1]\!]$ and

1. discretization of the initial conditions (1.2):

$$\langle\, u^0_{\mathcal{D}}, v\rangle_F = -\left(\,\Delta\, u^0, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)}\,,\ \forall\, v \in \mathcal{X}_{\mathcal{D},0}, \tag{2.3}$$

and

$$\langle\,\frac{u^1_{\mathcal{D}} - u^0_{\mathcal{D}}}{k}, v\rangle_F = -\left(\,\Delta\,\bar{u}^1, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)}\,,\ \forall\, v \in \mathcal{X}_{\mathcal{D},0}, \tag{2.4}$$

2. discretization of equation (1.1): for any $n \in [\![\,1, N]\!]$, $v \in \mathcal{X}_{\mathcal{D},0}$

$$\begin{aligned}&\left(\,\partial^2\,\Pi_{\mathcal{M}}\, u^{n+1}_{\mathcal{D}}, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)} + \frac{1}{2}\langle\,\gamma\, u^{n+1}_{\mathcal{D}} + 2(1-\gamma)u^n_{\mathcal{D}} + \gamma\, u^{n-1}_{\mathcal{D}}, v\rangle_F \\ &= \frac{1}{2}\,(\,\gamma\, f(t_{n+1}) + 2(1-\gamma)f(t_n) + \gamma\, f(t_{n-1}), \Pi_{\mathcal{M}}\, v)_{\mathbb{L}^2(\Omega)}\,,\end{aligned} \tag{2.5}$$

where

$$\langle\, u, v\rangle_F = \int_\Omega \nabla_{\mathcal{D}}\, u(x)\cdot\nabla_{\mathcal{D}}\, v(x)dx,\ \forall\, u, v \in \mathcal{X}_{\mathcal{D}}, \tag{2.6}$$

$$\partial^2\, v^{n+1} = \frac{v^{n+1} - 2v^n + v^{n-1}}{k^2},\ \forall\, n \in [\![\,1, N]\!], \tag{2.7}$$

$$\bar{u}^1 = u^1 + \frac{k}{2}(\Delta\, u^0 + f(0)), \tag{2.8}$$

and $(\,\cdot,\cdot)_{\mathbb{L}^2(\Omega)}$ denotes the $\mathbb{L}^2$ inner product.

As stated before, for the sake of simplicity we will only focus on the case $\gamma \in]\frac{1}{2}, 1]$. The case $\gamma \in [0, \frac{1}{2}]$ will be detailed in [3]. It is useful to mention that it is possible to obtain a convergence order four (instead of two for general values of $\gamma \in [0, 1]$, see Theorem 3.1 below) in time (with optimal order in space) when $\gamma = \frac{1}{6}$ when modifying slightly the expression (2.8) of $\bar{u}^1$.

3. Main results. The main result of the present contribution is the following theorem.

THEOREM 3.1 (Error estimates for the finite volume scheme (2.3)–(2.5) when $\frac{1}{2} < \gamma \le 1$). *Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. Assume that the solution (weak) of (1.1)–(1.3) satisfies $u \in \mathcal{C}^4([0, T]; \mathcal{C}^2(\overline{\Omega}))$. Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^\star$, and denote by $t_n = nk$, for $n \in [\![0, N+1]\!]$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 2.1. Assume that $\theta_{\mathcal{D}}$ satisfies $\theta \ge \theta_{\mathcal{D}}$.*
Then there exists a unique solution $(u^n_{\mathcal{D}})_{n=0}^{N+1} \in \mathcal{X}^{N+2}_{\mathcal{D},0}$ for problem (2.3)–(2.5).
For any $n \in [\![0, N+1]\!]$ and for any $\gamma \in]\frac{1}{2}, 1]$, we define the error $e^n_{\mathcal{M}} \in H_{\mathcal{M}}(\Omega)$ in the level n by:

$$e^n_{\mathcal{M}} = \mathcal{P}_{\mathcal{M}}\, u(t_n) - \Pi_{\mathcal{M}}\, u^n_{\mathcal{D}}. \tag{3.1}$$

Then, for all $\frac{1}{2} < \gamma \le 1$, the following error estimates hold

- *Discrete $\mathbb{L}^\infty(0, T; H^1_0(\Omega))$–estimate: for all $n \in [\![0, N+1]\!]$*

$$\| e^n_{\mathcal{M}} \|_{1,2,\mathcal{M}} \le C(k^2 + h_{\mathcal{D}}) \| u \|_{\mathcal{C}^4([0,T];\, \mathcal{C}^2(\overline{\Omega}))}, \tag{3.2}$$

- *Discrete $\mathcal{W}^{1,\infty}(0, T; \mathbb{L}^2(\Omega))$–estimate: for all $n \in [\![1, N+1]\!]$*

$$\| \partial^1 e^n_{\mathcal{M}} \|_{\mathbb{L}^2(\Omega)} \le C(k^2 + h_{\mathcal{D}}) \| u \|_{\mathcal{C}^4([0,T];\, \mathcal{C}^2(\overline{\Omega}))}, \tag{3.3}$$

  where $\partial^1 v^n = \dfrac{1}{k}\left(v^n - v^{n-1}\right)$.

- *Error estimate in the gradient approximation: for all $n \in [\![0, N+1]\!]$*

$$\| \nabla_{\mathcal{D}}\, u^n_{\mathcal{D}} - \nabla\, u(t_n) \|_{\mathbb{L}^2(\Omega)} \le C(k^2 + h_{\mathcal{D}}) \| u \|_{\mathcal{C}^4([0,T];\, \mathcal{C}^2(\overline{\Omega}))}. \tag{3.4}$$

The following lemma will help us to prove Theorem 3.1.

LEMMA 3.2 (An *a priori estimate* for the family of finite volume schemes (2.3)–(2.5) when $1 \ge \gamma > \frac{1}{2}$). *Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^\star$, and denote by $t_n = nk$, for $n \in [\![0, N+1]\!]$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of Definition 2.1. Assume that $\theta_{\mathcal{D}}$ satisfies $\theta \ge \theta_{\mathcal{D}}$. Assume in addition that there exists $(\eta^n_{\mathcal{D}})_{n=0}^{N+1} \in \mathcal{X}^{N+2}_{\mathcal{D},0}$ such that for any $n \in [\![1, N]\!]$, for all $v \in \mathcal{X}_{\mathcal{D},0}$*

$$\begin{aligned}\left(\Pi_{\mathcal{M}}\, \partial^2 \eta^{n+1}_{\mathcal{D}}, \Pi_{\mathcal{M}}\, v \right)_{\mathbb{L}^2(\Omega)} &+ \frac{1}{2} \langle \gamma\, \eta^{n+1}_{\mathcal{D}} + 2(1-\gamma)\eta^n_{\mathcal{D}} + \gamma\, \eta^{n-1}_{\mathcal{D}}, v \rangle_F \\ &= (\mathcal{S}^n, \Pi_{\mathcal{M}}\, v)_{\mathbb{L}^2(\Omega)}\,,\end{aligned} \tag{3.5}$$

where $\mathcal{S}^n \in \mathbb{L}^2(\Omega)$, for all $n \in [\![1, N]\!]$ and γ is a parameter satisfying $\gamma \in]\frac{1}{2}, 1]$. Then the following estimates hold, for all $j \in [\![1, N]\!]$

$$\begin{aligned}&\| \Pi_{\mathcal{M}}\, \partial^1 \eta^{j+1}_{\mathcal{D}} \|^2_{\mathbb{L}^2(\Omega)} + (2\gamma - 1) C\, | \eta^{j+1}_{\mathcal{D}} |^2_{\mathcal{X}} \\ &\le C \left(\| \Pi_{\mathcal{M}}\, \partial^1 \eta^1_{\mathcal{D}} \|^2_{\mathbb{L}^2(\Omega)} + | \eta^1_{\mathcal{D}} |^2_{\mathcal{X}} + | \eta^0_{\mathcal{D}} |^2_{\mathcal{X}} + (\mathcal{S})^2 \right),\end{aligned} \tag{3.6}$$

where

$$\mathcal{S} = \max_{n=1}^{N} \| \mathcal{S}^n \|_{\mathbb{L}^2(\Omega)}. \tag{3.7}$$

Proof. The following simple equality will be useful

$$\eta_K^{n+1} - \eta_K^{n-1} = k(\partial^1 \eta_K^{n+1} + \partial^1 \eta_K^n). \tag{3.8}$$

Taking $v = \eta_{\mathcal{D}}^{n+1} - \eta_{\mathcal{D}}^{n-1}$ in (3.5) and using (3.8) to get

$$\begin{aligned} \|\partial^1 \Pi_{\mathcal{M}}\, \eta_{\mathcal{D}}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} - \|\partial^1 \Pi_{\mathcal{M}}\, \eta_{\mathcal{D}}^{n}\|^2_{\mathbb{L}^2(\Omega)} &+ (1-\gamma)\left(\langle \eta_{\mathcal{D}}^n, \eta_{\mathcal{D}}^{n+1}\rangle_F - \langle \eta_{\mathcal{D}}^n, \eta_{\mathcal{D}}^{n-1}\rangle_F\right)\\ &+ \frac{\gamma}{2}\left(\langle \eta_{\mathcal{D}}^{n+1}, \eta_{\mathcal{D}}^{n+1}\rangle_F - \langle \eta_{\mathcal{D}}^{n-1}, \eta_{\mathcal{D}}^{n-1}\rangle_F\right)\\ &= \left(\mathcal{S}^n, \Pi_{\mathcal{M}}\left(\eta^{n+1} - \eta^{n-1}\right)\right)_{\mathbb{L}^2(\Omega)}. \end{aligned} \tag{3.9}$$

On the other hand

$$\begin{aligned} &(1-\gamma)\left(\langle \eta_{\mathcal{D}}^n, \eta_{\mathcal{D}}^{n+1}\rangle_F - \langle \eta_{\mathcal{D}}^n, \eta_{\mathcal{D}}^{n-1}\rangle_F\right) + \frac{\gamma}{2}\left(\langle \eta_{\mathcal{D}}^{n+1}, \eta_{\mathcal{D}}^{n+1}\rangle_F - \langle \eta_{\mathcal{D}}^{n-1}, \eta_{\mathcal{D}}^{n-1}\rangle_F\right)\\ &= \langle \eta_{\mathcal{D}}^{n+1}, \eta_{\mathcal{D}}^n\rangle_F - \langle \eta_{\mathcal{D}}^n, \eta_{\mathcal{D}}^{n-1}\rangle_F + \frac{\gamma}{2}\langle \eta_{\mathcal{D}}^{n+1} - \eta_{\mathcal{D}}^n, \eta_{\mathcal{D}}^{n+1} - \eta_{\mathcal{D}}^n\rangle_F\\ &- \frac{\gamma}{2}\langle \eta_{\mathcal{D}}^n - \eta_{\mathcal{D}}^{n-1}, \eta_{\mathcal{D}}^n - \eta_{\mathcal{D}}^{n-1}\rangle_F. \end{aligned} \tag{3.10}$$

Thanks to (3.10), (3.9) can be written as

$$E_{\mathcal{D}}^{n+1} - E_{\mathcal{D}}^n = \left(\mathcal{S}^n, \Pi_{\mathcal{M}}\left(\eta^{n+1} - \eta^{n-1}\right)\right)_{\mathbb{L}^2(\Omega)}, \tag{3.11}$$

where

$$\begin{aligned} E_{\mathcal{D}}^{n+1} &= \|\partial^1 \Pi_{\mathcal{M}}\, \eta_{\mathcal{D}}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} + \langle \eta_{\mathcal{D}}^{n+1}, \eta_{\mathcal{D}}^n\rangle_F\\ &+ \frac{\gamma}{2}\langle \eta_{\mathcal{D}}^{n+1} - \eta_{\mathcal{D}}^n, \eta_{\mathcal{D}}^{n+1} - \eta_{\mathcal{D}}^n\rangle_F. \end{aligned} \tag{3.12}$$

Summing (3.11) over $n \in [\![1, j]\!]$, where $j \in [\![1, N]\!]$, we get

$$E_{\mathcal{D}}^{j+1} = \sum_{n=1}^{j} \left(\mathcal{S}^n, \Pi_{\mathcal{M}}\left(\eta^{n+1} - \eta^{n-1}\right)\right)_{\mathbb{L}^2(\Omega)} + E_{\mathcal{D}}^1, \tag{3.13}$$

We have, using (3.12)

$$\begin{aligned} E_{\mathcal{D}}^{j+1} &= \|\partial^1 \Pi_{\mathcal{M}}\, \eta_{\mathcal{D}}^{j+1}\|^2_{\mathbb{L}^2(\Omega)} + \frac{\gamma}{2}\left(\langle \eta_{\mathcal{D}}^{j+1}, \eta_{\mathcal{D}}^{j+1}\rangle_F + \langle \eta_{\mathcal{D}}^{j}, \eta_{\mathcal{D}}^{j}\rangle_F\right)\\ &+ (1-\gamma)\langle \eta_{\mathcal{D}}^{j+1}, \eta_{\mathcal{D}}^{j}\rangle_F. \end{aligned} \tag{3.14}$$

On the other hand, we can justify easily that

$$\langle \eta_{\mathcal{D}}^{j+1}, \eta_{\mathcal{D}}^{j}\rangle_F \geq -\frac{1}{2}\left(\langle \eta_{\mathcal{D}}^{j+1}, \eta_{\mathcal{D}}^{j+1}\rangle_F + \langle \eta_{\mathcal{D}}^{j}, \eta_{\mathcal{D}}^{j}\rangle_F\right). \tag{3.15}$$

This with (3.14) yields that (recall that $1 \geq \gamma$ which means that $1-\gamma \geq 0$)

$$E_{\mathcal{D}}^{j+1} \geq \|\partial^1 \Pi_{\mathcal{M}}\, \eta_{\mathcal{D}}^{j+1}\|^2_{\mathbb{L}^2(\Omega)} + \frac{2\gamma-1}{2}\left(\langle \eta_{\mathcal{D}}^{j+1}, \eta_{\mathcal{D}}^{j+1}\rangle_F + \langle \eta_{\mathcal{D}}^{j}, \eta_{\mathcal{D}}^{j}\rangle_F\right). \tag{3.16}$$

This with (3.13) leads to

$$\|\partial^1 \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{j+1}\|_{\mathbb{L}^2(\Omega)}^2 + \frac{2\gamma-1}{2}\left(\langle \eta_{\mathcal{D}}^{j+1}, \eta_{\mathcal{D}}^{j+1}\rangle_F + \langle \eta_{\mathcal{D}}^{j}, \eta_{\mathcal{D}}^{j}\rangle_F\right) \le \sum_{n=1}^{j}\left(\mathcal{S}^n, \Pi_{\mathcal{M}}\left(\eta^{n+1}-\eta^{n-1}\right)\right)_{\mathbb{L}^2(\Omega)} + E_{\mathcal{D}}^1. \tag{3.17}$$

Gathering (3.17) with (3.8), the fact that $\langle \eta_{\mathcal{D}}^{j}, \eta_{\mathcal{D}}^{j}\rangle_F \ge 0$ (which stems from [4, Lemma 4.2, Page 1026]), $\gamma > \frac{1}{2}$, the triangle inequality and the Cauchy Schwarz inequality leads to (recall that $\mathcal{S}$ is defined in (3.7))

$$\|\partial^1 \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{j+1}\|_{\mathbb{L}^2(\Omega)}^2 + \frac{2\gamma-1}{2}\langle \eta_{\mathcal{D}}^{j+1}, \eta_{\mathcal{D}}^{j+1}\rangle_F \le 2k\mathcal{S}\sum_{n=1}^{j+1}\|\partial^1 \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{n}\|_{\mathbb{L}^2(\Omega)} + E_{\mathcal{D}}^1. \tag{3.18}$$

This with the inequality $ab \le \epsilon a^2 + b^2/\epsilon$, for all $\epsilon > 0$, and [4, Lemma 4.2, Page 1026] implies, for all $j \in [\![1, N]\!]$ (recall that $k(N+1) = T$ and $k/T = 1/(N+1) \le 1/2$)

$$\|\Pi_{\mathcal{M}} \partial^1 \eta_{\mathcal{D}}^{j+1}\|_{\mathbb{L}^2(\Omega)}^2 + (2\gamma-1)C\,|\eta_{\mathcal{D}}^{j+1}|_{\mathcal{X}}^2 \le \frac{2k}{T}\sum_{n=2}^{j}\left(\|\Pi_{\mathcal{M}} \partial^1 \eta_{\mathcal{D}}^{n}\|_{\mathbb{L}^2(\Omega)}^2 + (2\gamma-1)C\,|\eta_{\mathcal{D}}^{n}|_{\mathcal{X}}^2\right) + 2E_{\mathcal{D}}^1 + 8T^2\left(\mathcal{S}\right)^2 + \|\Pi_{\mathcal{M}} \partial^1 \eta_{\mathcal{D}}^{1}\|_{\mathbb{L}^2(\Omega)}^2. \tag{3.19}$$

On the other hand, using the fact that $\gamma \le 1$, $\langle \eta_{\mathcal{D}}^1 - \eta_{\mathcal{D}}^0, \eta_{\mathcal{D}}^1 - \eta_{\mathcal{D}}^0\rangle_F \ge 0$ (which stems from [4, Lemma 4.2, Page 1026]), (3.12) implies

$$E_{\mathcal{D}}^1 \le \|\partial^1 \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{1}\|_{\mathbb{L}^2(\Omega)}^2 + \frac{M}{2}\left(|\eta_{\mathcal{D}}^1|_{\mathcal{X}}^2 + |\eta_{\mathcal{D}}^0|_{\mathcal{X}}^2\right). \tag{3.20}$$

This with (3.19), the discrete version of the Gronwall's lemma and the fact that $(N+1)k = T$ implies the required estimate (3.6) of Lemma 3.2. □

Sketch of the proof of Theorem 3.1: The uniqueness of $\left(u_{\mathcal{D}}^n\right)_{n\in[\![0,N+1]\!]}$ satisfying (2.3)–(2.5) can be deduced from the [4, Lemma 4.2]. As usual, we can use this uniqueness to prove the existence. To prove (3.2)–(3.4), we compare the solution $\left(u_{\mathcal{D}}^n\right)_{n\in[\![0,N+1]\!]}$ satisfying (2.3)–(2.5) with the solution (it exists and it is unique thanks to [4, Lemma 4.2]): for any $n \in [\![0, N+1]\!]$, find $\bar{u}_{\mathcal{D}}^n \in \mathcal{X}_{\mathcal{D},0}$ such that, see (2.6)

$$\langle \bar{u}_{\mathcal{D}}^n, v\rangle_F = -\langle \bar{u}_{\mathcal{D}}^n, v\rangle_F = -\left(\Delta u(t_n), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \ \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{3.21}$$

Taking $n = 0$ in (3.21), using the fact that $u(0) = u^0$, and comparing this with (2.3), we get the following property which will be used below

$$\bar{u}_{\mathcal{D}}^0 = u_{\mathcal{D}}^0. \tag{3.22}$$

One remarks that the solution of (3.21) is the same one of [2, (12)], one can use error estimates [2, (13), (15), and (16)] as error estimates for the solution of (3.21).
Writing (3.21) in the steps $n+1$ and $n-1$ yields, for all $n \in [\![0, N]\!]$ and for all

$v \in \mathcal{X}_{\mathcal{D},0}$

$$\frac{1}{2}\langle \gamma \bar{u}_{\mathcal{D}}^{n+1} + 2(1-\gamma)\bar{u}_{\mathcal{D}}^{n} + \gamma \bar{u}_{\mathcal{D}}^{n-1}, v\rangle_F$$
$$= -\frac{1}{2}\left(\gamma\, \Delta(t_{n+1}) + 2(1-\gamma)\Delta(t_n) + \gamma\, \Delta(t_{n-1}), \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)}. \tag{3.23}$$

Subtracting the previous equation from (2.5) to get

$$\left(\Pi_{\mathcal{M}}\, \partial^2\, \eta_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)} + \frac{1}{2}\langle \gamma \eta_{\mathcal{D}}^{n+1} + 2(1-\gamma)\eta_{\mathcal{D}}^{n} + \gamma \eta_{\mathcal{D}}^{n-1}, v\rangle_F$$
$$= \left(\mathcal{S}_{\mathcal{D}}^{n,\gamma}, \Pi_{\mathcal{M}}\, v\right)_{\mathbb{L}^2(\Omega)}, \tag{3.24}$$

where $\eta_{\mathcal{D}}^{n} = u_{\mathcal{D}}^{n} - \bar{u}_{\mathcal{D}}^{n}$, for all $n \in [\![0, N+1]\!]$ and

$$\mathcal{S}_{\mathcal{D}}^{n,\gamma} = \frac{1}{2}\left(\gamma\, f(t_{n+1}) + 2(1-\gamma)\, f(t_n) + \gamma\, f(t_{n-1})\right)$$
$$+ \frac{1}{2}\left(\gamma\, \Delta\, u(t_{n+1}) + 2(1-\gamma)\Delta\, u(t_n) + \gamma \Delta\, u(t_{n-1})\right)$$
$$- \Pi_{\mathcal{M}} \partial^2 \bar{u}_{\mathcal{D}}^{n+1}. \tag{3.25}$$

Equation (3.24) with Lemma 3.2 and (3.22) (which yields $\eta_{\mathcal{D}}^{0} = 0$) implies that, for all $n \in [\![1, N]\!]$

$$\|\Pi_{\mathcal{M}}\, \partial^1\, \eta_{\mathcal{D}}^{n+1}\|_{\mathbb{L}^2(\Omega)}^2 + (2\gamma - 1)\, C|\, \eta_{\mathcal{D}}^{n+1}|_{\mathcal{X}}^2$$
$$\leq C\left(\|\Pi_{\mathcal{M}}\, \partial^1\, \eta_{\mathcal{D}}^{1}\|_{\mathbb{L}^2(\Omega)}^2 + |\eta_{\mathcal{D}}^{1}|_{\mathcal{X}}^2 + (\mathcal{S})^2\right), \tag{3.26}$$

with $\mathcal{S}$ is defined by (3.7) by replacing $\mathcal{S}^n$ with $\mathcal{S}_{\mathcal{D}}^{n,\gamma}$.
To estimate the terms on the right hand side of the previous inequality, we consider

$$\xi_{\mathcal{D}}^{n} = \bar{u}_{\mathcal{D}}^{n} - \mathcal{P}_{\mathcal{D}}\, u(\cdot, t_n),\ \forall\, n \in [\![0, N+1]\!]. \tag{3.27}$$

It is useful to remark that (recall that $\eta_{\mathcal{D}}^{n} = u_{\mathcal{D}}^{n} - \bar{u}_{\mathcal{D}}^{n}$)

$$u_{\mathcal{D}}^{n} - \mathcal{P}_{\mathcal{D}}\, u(t_n) = \eta_{\mathcal{D}}^{n} + \xi_{\mathcal{D}}^{n}. \tag{3.28}$$

1. *Estimate of* $\|\Pi_{\mathcal{M}}\, \partial^1\, \eta_{\mathcal{D}}^{1}\|_{\mathbb{L}^2(\Omega)}$: using (3.28), we get

$$\|\Pi_{\mathcal{M}}\, \partial^1\, \eta_{\mathcal{D}}^{1}\|_{\mathbb{L}^2(\Omega)} \leq \sum_{i=1}^{4} \mathbb{T}_i, \tag{3.29}$$

where

$$\mathbb{T}_1 = \|\Pi_{\mathcal{M}} \partial^1 \xi_{\mathcal{D}}^{1}\|_{\mathbb{L}^2(\Omega)},\ \ \mathbb{T}_2 = \|\Pi_{\mathcal{M}}\, \partial^1\, u_{\mathcal{D}}^{1} - \bar{u}^1\|_{\mathbb{L}^2(\Omega)},$$

$$\mathbb{T}_3 = \|\, \bar{u}^1 - \partial^1\, u(t_1)\|_{\mathbb{L}^2(\Omega)},\ \mathbb{T}_4 = \|\, \partial^1\, u(t_1) - \mathcal{P}_{\mathcal{M}} \partial^1\, u(t_1)\|_{\mathbb{L}^2(\Omega)}. \tag{3.30}$$

Estimate [2, (15)], when $j = 1$, with (3.27) leads to

$$\mathbb{T}_1 \leq\, C\, h_{\mathcal{D}} \|\, u\|_{\mathcal{C}^1([0,T];\, \mathcal{C}^2(\overline{\Omega}))}. \tag{3.31}$$

Equation (2.4) can be written as

$$\langle \partial^1 u_{\mathcal{D}}^1, v\rangle_F = -\left(\Delta \bar{u}^1, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \ \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{3.32}$$

This with [4, (4.25)] and the triangle inequality implies that

$$\mathbb{T}_2 \le C\, h_{\mathcal{D}} \| u\|_{\mathcal{C}^1([0,T];\ \mathcal{C}^2(\overline{\Omega}))}. \tag{3.33}$$

Using the Taylor expansions to get

$$\mathbb{T}_4 \le C\, h_{\mathcal{D}} \| u\|_{\mathcal{C}^1([0,T];\ \mathcal{C}^1(\overline{\Omega}))}. \tag{3.34}$$

A convenient Taylor expansion implies that

$$\mathbb{T}_3 \le Ck^2 \|u\|_{\mathcal{C}^3([0,T];\ \mathcal{C}(\overline{\Omega}))}. \tag{3.35}$$

Thanks to (3.29), (3.31), and (3.33)–(3.35) we have

$$\|\Pi_{\mathcal{M}}\, \partial^1\, \eta_{\mathcal{D}}^1\|_{\mathbb{L}^2(\Omega)} \le C\,(k^2 + h_{\mathcal{D}})\| u\|_{\mathcal{C}^3([0,T];\ \mathcal{C}^2(\overline{\Omega}))}. \tag{3.36}$$

2. *Estimate of* $|\eta_{\mathcal{D}}^1|_{\mathcal{X}}$: let us first remark that thanks to (2.3) and (2.4), we have

$$\langle u_{\mathcal{D}}^1, v\rangle_F = -\left(\Delta\,(u^0 + k\bar{u}^1), \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^2(\Omega)}, \ \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{3.37}$$

In order to bound $|\eta_{\mathcal{D}}^1|_{\mathcal{X}} = |\, u_{\mathcal{D}}^1 - \bar{u}_{\mathcal{D}}^1|_{\mathcal{X}}$, we use the triangle inequality to get

$$\begin{aligned} |\eta_{\mathcal{D}}^1|_{\mathcal{X}} \le\ & |\, u_{\mathcal{D}}^1 - \mathcal{P}_{\mathcal{D}}\,(\omega)|_{\mathcal{X}} + |\,\mathcal{P}_{\mathcal{D}}\,(\omega) - \mathcal{P}_{\mathcal{D}}\, u(t_1)|_{\mathcal{X}} \\ & + |\,\mathcal{P}_{\mathcal{D}}\, u(t_1) - \bar{u}_{\mathcal{D}}^1|_{\mathcal{X}}, \end{aligned} \tag{3.38}$$

where, using (1.1) and (1.2)

$$\omega = u^0 + k\bar{u}^1 = u^0 + ku_t(0) + \frac{k^2}{2} u_{tt}(0). \tag{3.39}$$

This with the proof of [4, (4.29)] and suitable Taylor expansions, we get

$$|\eta_{\mathcal{D}}^1|_{\mathcal{X}} \le C\,(k^2 + h_{\mathcal{D}})\| u\|_{\mathcal{C}^3([0,T];\ \mathcal{C}^2(\overline{\Omega}))}. \tag{3.40}$$

3. *Estimate of* $\mathcal{S}$: substituting f by $u_{tt} - \Delta\, u$, see (1.1), in the expansion of $\mathcal{S}_{\mathcal{D}}^{n,\gamma}$, we get

$$\begin{aligned} \mathcal{S}_{\mathcal{D}}^{n,\gamma} = & \frac{1}{2}\left(\gamma\, u_{tt}(t_{n+1}) + 2(1-\gamma)u_{tt}(t_n) + \gamma\, u_{tt}(t_{n-1})\right) \\ & - \Pi_{\mathcal{M}} \partial^2 \bar{u}_{\mathcal{D}}^{n+1}. \end{aligned} \tag{3.41}$$

Thanks to the Taylor expansion, [2, (15)] (when $j = 2$), we have

$$\mathcal{S} \le C\,(k^2 + h_{\mathcal{D}})\| u\|_{\mathcal{C}^4([0,T];\ \mathcal{C}^2(\overline{\Omega}))}. \tag{3.42}$$

Gathering now (3.26), (3.36), (3.40), and (3.42) yields, for all $n \in [\![2, N+1]\!]$

$$\|\Pi_{\mathcal{M}}\, \partial^1\, \eta_{\mathcal{D}}^n\|_{\mathbb{L}^2(\Omega)} \le C\,(k^2 + h_{\mathcal{D}})\| u\|_{\mathcal{C}^4([0,T];\ \mathcal{C}^2(\overline{\Omega}))}, \tag{3.43}$$

and

$$|\,\eta_{\mathcal{D}}^n|_{\mathcal{X}} \le C\,(k^2 + h_{\mathcal{D}})\| u\|_{\mathcal{C}^4([0,T];\ \mathcal{C}^2(\overline{\Omega}))}. \tag{3.44}$$

We now combine (3.43)–(3.44) with [2, (13), (15), and (16)] to prove the required estimates (3.2)–(3.4).

- *Proof of estimate* (3.2): estimate (3.44) with [4, (4.6)] implies, for all $n \in [\![2, N+1]\!]$

$$\|\Pi_{\mathcal{M}}\ \eta^n_{\mathcal{D}}\|_{1,2,\mathcal{M}} \le C\,(k^2 + h_{\mathcal{D}})\|\,u\|_{\mathcal{C}^4([0,T];\ \mathcal{C}^2(\overline{\Omega}))}. \tag{3.45}$$

This with (3.28), the fact that $\Pi_{\mathcal{M}}\ \xi^n_{\mathcal{D}} = \Pi_{\mathcal{M}}\bar{u}^n_{\mathcal{D}} - \mathcal{P}_{\mathcal{M}}\,u(t_n)$, estimate [2, (13)] , and the triangle inequality implies estimate (3.2) for all $n \in [\![2, N+1]\!]$. The case when $n = 1$ in (3.2) can be proved by gathering (3.40), [4, (4.6)], and the case $n = 1$ of [2, (13)]. Property (3.22) with the case $n = 0$ of [2, (13)] yields the case $n = 0$ of (3.2).

- *Proof of estimate* (3.3): the case when $n \in [\![2, N+1]\!]$ of (3.3) can be proved by gathering (3.43), the case when $j = 1$ in [2, (15)], and the triangle inequality. The case $n = 1$ of (3.3) can be proved by gathering (3.36), the case when $n = 1$ and $j = 1$ in [2, (15)], and the triangle inequality.
- *Proof of estimate* (3.4): gathering (3.40) and (3.44), and [4, Lemma 4.2] leads to, for all $n \in [\![1, N+1]\!]$

$$\|\,\nabla_{\mathcal{D}}\eta^n_{\mathcal{D}}\|_{\mathbb{L}^2(\Omega)} \le C\,(k^2 + h_{\mathcal{D}})\|\,u\|_{\mathcal{C}^4([0,T];\ \mathcal{C}^2(\overline{\Omega}))}. \tag{3.46}$$

Combining (3.46), [2, (16)], and the triangle inequality yields (3.4) for all $n \in [\![1, N+1]\!]$. The case $n = 0$ of (3.4) can be deduced directly from the case $n = 0$ of [2, (16)] by using (3.22).

The proof Theorem 3.1 is completed. □

The following corollaries are useful applications of Theorem 3.1.

Corollary 3.3 (Approximation of order two in time for the exact solution u and its first spatial derivatives). *Consider the case $\gamma \in]\frac{1}{2}, 1]$ in the finite volume schemes (2.3)–(2.5). Under the same assumptions of Theorem 3.1, let $(\,u^n_{\mathcal{D}})^{N+1}_{n=0} \in \mathcal{X}^{N+2}_{\mathcal{D},0}$ be the unique solution of problem (2.3)–(2.5). Then:*

1. *$\Pi_{\mathcal{M}}u^n_{\mathcal{D}}$ approximates $u(t_n)$ by order $(k^2 + h_{\mathcal{D}})$, in $\mathbb{L}^2(\Omega)$–norm, uniformly in n.*
2. *The i–th component of the discrete gradient $\nabla_{\mathcal{D}}u^n_{\mathcal{D}}$, defined by (2.1)–(2.2) by replacing u with $u^n_{\mathcal{D}}$, approximates the i–th component of the gradient $\nabla u(t_n)$ by order $(k^2 + h_{\mathcal{D}})$ uniformly in n, in $\mathbb{L}^2(\Omega)$–norm.*

Corollary 3.4 (Approximation of order two in time for the time derivative u_t). *Consider the case $\gamma \in]\frac{1}{2}, 1]$ in the finite volume schemes (2.3)–(2.5). Under the same assumptions of Theorem 3.1, let $(\,u^n_{\mathcal{D}})^{N+1}_{n=0} \in \mathcal{X}^{N+2}_{\mathcal{D},0}$ be the unique solution of problem (2.3)–(2.5). We consider the element $(\,\Xi^n_{\mathcal{D}})^{N+1}_{n=0} \in \mathcal{X}^{N+2}_{\mathcal{D}}$ given by: $\Xi^0_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}}(u^1)$,*

$$\Xi^n_{\mathcal{D}} = \frac{u^{n+1}_{\mathcal{D}} - u^{n-1}_{\mathcal{D}}}{2k},\ \forall\, n \in [\![1, N]\!],\ \text{and}\ \Xi^{N+1}_{\mathcal{D}} = \frac{3u^{N+1}_{\mathcal{D}} - 4u^N_{\mathcal{D}} + u^{N-1}_{\mathcal{D}}}{2k}.$$

The following error estimate holds

$$\|\,\mathcal{P}_{\mathcal{M}}\,u_t(t_n) - \Pi_{\mathcal{M}}\,\Xi^n_{\mathcal{D}}\|_{\mathbb{L}^2(\Omega)} \le C\,(k^2 + h_{\mathcal{D}})\|\,u\|_{\mathcal{C}^4([0,T];\ \mathcal{C}^2(\overline{\Omega}))}. \tag{3.47}$$

3.1. A numerical example supporting Theorem 3.1. The present subsection is devoted to provide a numerical test to justify theoretical results provided in Theorem 3.1 in two dimensions and $\gamma = 1$ in (2.5). We consider $\Omega = (0,1)^2$ meshed

with the rectangular meshes described as in [6, Pages 756–758] (which is a particular case of the mesh introduced recently in [4]), with uniform meshes with mesh size h. For the sake of simplicity, we will consider the discrete gradient described in [5, (211)–(212), Page 333] (which is also a particular case of the discrete gradient introduced recently in [4]). The exact solution is given by $u(x,y,t) = \sin(\pi x)\sin(\pi y)\cos(\sqrt{2}\pi t)$, where $(x,y,t) \in (0,1)^2 \times (0,1)$. By this way $u^0(x,y) = \sin(\pi x)\sin(\pi y)$, $(x,y) \in (0,1)^2$, $u^1 \equiv 0$, and $f \equiv 0$ in (1.1)–(1.3).
By some computations, we find that the discrete gradient is given by, for $K_{ij} =](i-1)h, ih[\times](j-1)h, jh[$

$$\nabla_{\mathcal{D}} u_{\mathcal{D}}|_{K_{ij}} = \left(\frac{u_{i+1,j} - u_{i-1,j}}{2h}, \frac{u_{i,j+1} - u_{i,j-1}}{2h} \right), \ K_{ij} \cap \partial\Omega = \emptyset, \tag{3.48}$$

and with slightly modification when $K_{ij} \cap \partial\Omega \neq \emptyset$.
The following results are obtained using a Scilab programme with $k = h$:

$1/h$	$\dfrac{\lvert \text{Error} \rvert_{W^{1,\infty}(L^2)}}{k^2+h}$	$\dfrac{\lvert \text{Error} \rvert_{L^\infty(H^1)}}{k^2+h}$	$\dfrac{\lvert \text{Gradient of Error} \rvert_{L^\infty(L^2)}}{k^2+h}$
55	4.7915629	1.0453143	1.0348686
60	4.8074899	0.9630278	0.9524088
65	4.8204142	0.8926979	0.8820714
70	4.8310864	0.8319055	0.8213756

From the previous tests, we remark that the error has the same behavior as $k^2 + h = h^2 + h$ which supports our theoretical results quoted in Theorem 3.1.

4. Conclusion.

The present work is an extension of the previous work [1] which dealt with error analysis of a finite volume scheme for second order hyperbolic equations on general nonconforming multidimensional spatial meshes introduced recently in [4]. We considered the wave equation (as a model for second order hyperbolic equations). We presented a one–parameter family of finite volume schemes in which the spatial discretization is performed using the generic mesh introduced in [4] and the discretization in time is performed using a second order Newmark's method.
The considered family of the finite volume schemes can be applied on any type of spatial grid: conforming or non conforming, 2D and 3D, or more, made with control volumes which are only assumed to be polyhedral. The matrices generated by these scheme are sparse, symmetric, positive and definite. We proved that the convergence order of the stated family of finite volume schemes is *optimal* in space and is two in time. The analysis of the convergence order is performed in several discrete norms which allow us to derive approximations for not only the exact solution but also for its first derivatives whose the convergence order is *optimal* in space and is two in time. For the sake of simplicity, we only studied the case when the parameter γ involved in (2.5) is satisfying $1 \geq \gamma > \frac{1}{2}$. The case when γ is satisfying $\frac{1}{2} \geq \gamma \geq 0$ will be detailed together with a general framework in [3].

REFERENCES

[1] BRADJI, A. , *A theoretical analysis of a new finite volume scheme for second order hyperbolic equations on general nonconforming multidimensional spatial meshes*, Numer. Methods Partial Differ. Equations, to appear.
[2] BRADJI, A., FUHRMANN, J., *Error estimates of the discretization of linear parabolic equations on general nonconforming spatial grids*, C. R. Math. Acad. Sci. Paris, 348/19-20 (2010), pp. 1119–1122.

[3] Bradji, A. , *Convergence analysis of some high–order time accurate schemes for a finite volume method for second order hyperbolic equations on general nonconforming multidimensional spatial meshes*, in Progress.

[4] Eymard, R., Gallouët, T. and Herbin, R., *Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes SUSHI: a scheme using stabilization and hybrid interfaces*, IMA J. Numer. Anal., 30/4 (2010), pp. 1009–1043.

[5] Eymard, R., Gallouët, T. and Herbin, R., *A cell–centred finite–volume approximation for anisotropic diffusion operators on unstructured meshes in any space dimension*, IMA Journal of Numerical Analysis, 26/2 (2006), pp. 326–353.

[6] Eymard, R., Gallouët, T. and Herbin, R., *Finite Volume Methods*, Handbook of Numerical Analysis. P. G. Ciarlet and J. L. Lions (eds.), VII/(2000), pp. 723–1020.

[7] Karaa, S., *Finite element θ–schemes for the acoustic wave equation*, Adv. Appl. Math. Mech., 3/2 (2011), pp. 181–203.

[8] Kim, S. and Lim, H., *High–order schemes for acoustic waveform simulation*, Appl. Numer. Math., 57/4 (2007), pp. 402–414.

[9] Raviart, P. A. and Thomas, J. M., *Introduction à l'Analyse Numérique des Equations aux Dérivées Partielles*, Masson, Paris, 1983.

[10] Zampieri, E. and Pavarino, L. F., *Approximation of acoustic waves by explicit Newmark's schemes and spectral element methods*, J. Comput. Appl. Math., 185/2 (2006), 308–325.