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FINITE ELEMENT ERROR ESTIMATES FOR NONLINEAR CONVECTIVE PROBLEMS *

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Abstract. This paper is concerned with the analysis of the finite element method applied to nonstationary nonlinear convective problems. Using special estimates of the convective terms, we prove apriori error estimates for a semidiscrete and implicit scheme. For the semidiscrete scheme we need to apply so-called continuous mathematical induction and a nonlinear Gronwall lemma. For the implicit scheme, we prove that there does not exist a Gronwall-type lemma capable of proving the desired estimates using standard arguments. To overcome this obstacle, we use a suitable continuation of the discrete implicit solution and again use continuous mathematical induction to prove the error estimates. The technique presented can be extended to locally Lipschitz-continuous convective nonlinearities.

Key words. Nonlinear convection equation, finite element method, apriori error estimates, continuous mathematical induction, continuation

AMS subject classifications. 65M15, 65M60, 65M12

1. Introduction. During the long history of the finite element method (FEM), much work has been devoted to its theoretical analysis, especially for elliptic or parabolic problems, where the diffusion terms have a "nice" structure. Concerning convective (and convection-dominated) problems, the situation is not so simple. Most results deal with linear problems and/or modifications of the FEM, e.g. by including stabilization terms ([7], [10]), weighted basis functions ([9], [10]), upwinding ([1], [10]) and layer-adapted meshes ([5], [10]).

In this paper, we analyze the standard finite element method applied to nonlinear convective problems. We use a technique based on estimates of the convective terms derived originally in [12] for explicit discretizations of the discontinuous Galerkin (DG) method. For the explicit scheme, the proof relies heavily on mathematical induction and in order to apply the technique to the method of lines, we apply so-called continuous mathematical induction. For an implicit scheme, we must proceed more carefully, the discrete problem does not contain enough information to perform the key induction step. For this purpose, we construct a suitable continuation of the implicit solution and again apply continuous mathematical induction. This paper presents an overview of [8], where also the explicit case is treated in detail and the technique is extended to locally Lipschitz-continuous convective nonlinearities as well.

2. Continuous problem. Let $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, be a bounded open (polyhedral) domain. We treat the following nonlinear convective problem. Find $u : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\text{a)} \quad \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = g \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$\text{b)} \quad u\big|_{\Gamma_D \times (0,T)} = 0, \tag{2.2}$$

$$\text{d)} \quad u(x,0) = u^0(x), \quad x \in \Omega. \tag{2.3}$$

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Here $g : \Omega \times (0,T) \to \mathbb{R}$ and $u^0 : \Omega \to \mathbb{R}$ are given functions and $\Gamma_D \subset \partial\Omega$ has positive measure. In our analysis, we need to assume that $\Gamma_N := \partial\Omega \backslash \Gamma_D$ is an outflow boundary for either u or u_h, i.e. we assume e.g. $\Gamma_N^{(t)} \subseteq \{x \in \partial\Omega; \mathbf{f}'(u(x,t)).\mathbf{n} \geq 0\}$.

We assume that the convective fluxes $\mathbf{f} = (f_1, \cdots, f_d) \in (C_b^2(\mathbb{R}))^d = (C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R}))^d$, hence $\mathbf{f}$ and $\mathbf{f}' = (f_1', \cdots, f_d')$ are *globally Lipschitz continuous.* For improved estimates via Lemma 4.2, we shall assume $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$. In [8] the presented results the presented error analysis is extended, assuming only *local* Lipschitz continuity and boundedness, i.e. $\mathbf{f} \in (C^2(\mathbb{R}))^d$ or $\mathbf{f} \in (C^3(\mathbb{R}))^d$.

By $(\cdot,\cdot)$ we denote the standard $L^2(\Omega)-$scalar product and by $\|\cdot\|$ the $L^2(\Omega)$-norm. By $\|\cdot\|_\infty$, we denote the $L^\infty(\Omega)$-norm. For simplicity of notation, we shall drop the argument Ω in Sobolev norms, e.g. $\|\cdot\|_{H^{p+1}}$ denotes the $H^{p+1}(\Omega)-$norm. We shall also denote the Bochner norms over the whole interval $[0,T]$ in concise form, e.g. $\|u\|_{L^\infty(H^{p+1})}$ denotes the $L^\infty(0,T;H^{p+1}(\Omega))$-norm.

3. Discretization. Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$, i.e. a partition into a finite number of closed simplexes with mutually disjoint interiors. We assume standard conforming properties: two neighboring elements from $\mathcal{T}_h$ share an entire face, edge or vertex. We set $h = \max_{K\in\mathcal{T}_h} \mathrm{diam}(K)$.

We consider a system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$, of triangulations of the domain Ω which are shape regular and satisfy the inverse assumption, cf. [3]. Let $p \geq 1$ be an integer. The approximate solution will be sought in the space of globally continuous piecewise polynomial functions $S_h = \{v \in C(\overline{\Omega}); v|_{\Gamma_D} = 0, v|_K \in P^p(K) \forall K \in \mathcal{T}_h\}$, where $P^p(K)$ denotes the space of polynomials on K of degree $\leq p$.

We discretize the continuous problem in a standard way. Multiply (2.1) by a test function $\varphi_h \in S_h$, integrate over Ω and apply Green's theorem.

DEFINITION 3.1. *We say that $u_h \in C^1([0,T];S_h)$ is the space-semidiscretized finite element solution of problem (2.1) - (2.3), if $u_h(0) = u_h^0 \approx u^0$ and*

$$\frac{d}{dt}\big(u_h(t),\varphi_h\big) + b\big(u_h(t),\varphi_h\big) = l\big(\varphi_h\big)(t), \quad \forall \varphi_h \in S_h,\ t \in (0,T). \tag{3.1}$$

Here, we have introduced an approximation $u_h^0 \in S_h$ of the initial condition u^0 and the *convective form* defined for $v, \varphi \in H^1(\Omega)$:

$$b(v,\varphi) = -\int_\Omega \mathbf{f}(v)\cdot \nabla\varphi \,\mathrm{d}x + \int_{\Gamma_N} \mathbf{f}(v)\cdot \mathbf{n}\,\varphi\,\mathrm{d}S,$$

where $\mathbf{n}$ is the unit outer normal to Ω. Finally, we introduce the *right-hand side form*

$$l(\varphi)(t) = \int_\Omega g(t)\varphi\,\mathrm{d}x.$$

We note that a sufficiently regular exact solution u of problem (2.1) satisfies

$$\frac{d}{dt}\big(u(t),\varphi_h\big) + b\big(u(t),\varphi_h\big) = l\big(\varphi_h\big)(t), \quad \forall \varphi_h \in S_h,\ \forall t \in (0,T), \tag{3.2}$$

which implies the *Galerkin orthogonality property* of the error.

4. Key estimates of the convective terms. As usual in apriori error analysis, we assume that the weak solution u is sufficiently regular, namely

$$u, u_t \in L^2\big(0,T;H^{p+1}(\Omega)\big), \quad u \in L^\infty(0,T;W^{1,\infty}(\Omega)), \tag{4.1}$$

where $u_t := \frac{\partial u}{\partial t}$. For $v \in L^2(\Omega)$ we denote by $\Pi_h v$ the $L^2(\Omega)$-projection of v on S_h:

$$\Pi_h v \in S_h, \quad (\Pi_h v - v, \varphi_h) = 0, \qquad \forall\, \varphi_h \in S_h.$$

Let $\eta_h(t) = u(t) - \Pi_h u(t) \in H^{p+1}(\Omega)$ and $\xi_h(t) = \Pi_h u(t) - u_h(t) \in S_h$ for $t \in (0,T)$. Then we can write the error e_h as $e_h(t) := u(t) - u_h(t) = \eta_h(t) + \xi_h(t)$. Standard approximation results give us estimates for $\eta_h(t)$ in terms of power of h, e.g. $||\eta|| \leq Ch^{p+1}|u|_{H^{p+1}}$, cf. [3]. By C we denote a generic constant independent of h, which may have different values in different parts of the text.

LEMMA 4.1. *There exists a constant $C \geq 0$ independent of h, t, such that*

$$b\big(u_h(t), \xi(t)\big) - b\big(u(t), \xi(t)\big) \leq C\Big(1 + \frac{\|e_h(t)\|_\infty}{h}\Big)\big(h^{2p+1}|u(t)|^2_{H^{p+1}} + \|\xi(t)\|^2\big). \quad (4.2)$$

Proof. The proof follows the arguments of [12], where similar estimates are derived for periodic boundary conditions or compactly supported solutions in 1D. The proof for mixed Dirichlet-Neumann boundary conditions is contained in [8]. Here, we only note that the estimate is quite straightforward and is based on performing second order Taylor expansions of $\mathbf{f}$ in the interior and boundary terms. □

We can improve estimate (4.2) if we suppose $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$ and $\Gamma_N = \emptyset$, obtaining a factor of $h^{-1}\|e_h\|^2_\infty$ instead of $h^{-1}\|e_h\|_\infty$ and h^{2p+2} instead of h^{2p+1}, cf. [8]:

LEMMA 4.2. *Let $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$ and $\Gamma_N = \emptyset$. There exists a constant $C \geq 0$ independent of h, t, such that*

$$b\big(u_h(t), \xi(t)\big) - b\big(u(t), \xi(t)\big) \leq C\Big(1 + \frac{\|e_h(t)\|^2_\infty}{h}\Big)\big(h^{2p+2}|u(t)|^2_{H^{p+1}} + \|\xi(t)\|^2\big).$$

5. Error analysis for the method of lines. We proceed in a standard way. Due to Galerkin orthogonality, we subtract (3.2) and (3.1) and set $\varphi_h := \xi_h(t) \in S_h$. Since $\big(\frac{\partial \xi_h}{\partial t}, \xi_h\big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_h\|^2$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_h(t)\|^2 = b\big(u_h(t), \xi_h(t)\big) - b\big(u(t), \xi_h(t)\big) - \Big(\frac{\partial \eta_h(t)}{\partial t}, \xi_h(t)\Big).$$

For the last right-hand side term, we use the Cauchy and Young's inequalities and standard estimates for η. For the convective terms, we use Lemma 4.1. Integration from 0 to $t \in [0,T]$ yields

$$\|\xi_h(t)\|^2 \leq C\int_0^t \Big(1 + \frac{\|e_h(\vartheta)\|_\infty}{h}\Big)\Big(h^{2p+1}|u(\vartheta)|^2_{H^{p+1}} + h^{2p+2}|u_t(\vartheta)|^2_{H^{p+1}} + \|\xi_h(\vartheta)\|^2\Big)\mathrm{d}\vartheta, \quad (5.1)$$

where $C \geq 0$ is independent of h, t. For simplicity, we have assumed that $\xi_h(0) = 0$, i.e. $u_h^0 = \Pi_h u^0$. Otherwise we must assume e.g. $\|\xi_h(0)\|^2 \leq Ch^{2p+1}|u^0|^2_{H^{p+1}}$ and include this term in the estimate.

We notice that if we knew *apriori* that $\|e_h\|_\infty = O(h)$ then the unpleasant term $\|e_h\|_\infty h^{-1}$ in (5.1) would be $O(1)$. Thus we could simply apply the standard Gronwall lemma to obtain the desired error estimates. We state this formally:

LEMMA 5.1. *Let $t \in [0,T]$ and $p \geq d/2$. If $\|e_h(\vartheta)\| \leq h^{1+d/2}$ for all $\vartheta \in [0,t]$, then there exists a constant C_T independent of h, t such that*

$$\max_{\vartheta \in [0,t]} \|e_h(\vartheta)\|^2 \leq C_T^2 h^{2p+1}. \quad (5.2)$$

Proof. The assumptions imply, by the inverse inequality and estimates of η, that

$$\|e_h(\vartheta)\|_\infty \le \|\eta_h(\vartheta)\|_\infty + \|\xi_h(\vartheta)\|_\infty \le Ch|u(t)|_{W^{1,\infty}} + C_I h^{-d/2}\|\xi_h(\vartheta)\| \tag{5.3}$$
$$\le Ch + C_I h^{-d/2}\|e_h(\vartheta)\| + C_I h^{-d/2}\|\eta_h(\vartheta)\| \le Ch + Ch^{p+1-d/2}|u(\vartheta)|_{H^{p+1}(\Omega)} \le Ch,$$

where the constant C is independent of h, ϑ, t. Using this estimate in (5.1) gives us

$$\|\xi_h(t)\|^2 \le \tilde{C}h^{2p+1} + C\int_0^t \|\xi_h(\vartheta)\|^2 \,\mathrm{d}\vartheta, \tag{5.4}$$

where the constants $\widetilde{C}, C$ are independent of h, t. Gronwall's inequality applied to (5.4) states that there exists a constant $\widetilde{C}_T$, independent of h, t, such that

$$\max_{\vartheta\in[0,t]} \|\xi_h(\vartheta)\|^2 + \frac{1}{2}\int_0^t |\xi_h(\vartheta)|^2_{\Gamma_N} \,\mathrm{d}\vartheta \le \widetilde{C}_T h^{2p+1},$$

which allong with similar estimates for η gives us (5.2). □

Now it remains to get rid of the *apriori* assumption $\|e_h\|_\infty = O(h)$. In [12] this is done for an explicit scheme using mathematical induction. Starting from $\|e_h^0\| = O(h^{p+1/2})$, the following induction step is proved:

$$\|e_h^n\| = O(h^{p+1/2}) \quad\Longrightarrow\quad \|e_h^{n+1}\|_\infty = O(h) \quad\Longrightarrow\quad \|e_h^{n+1}\| = O(h^{p+1/2}). \tag{5.5}$$

For the method of lines we continuous time and hence cannot use mathematical induction straightforwardly. However, we can divide $[0, T]$ into a finite number of sufficiently small intervals $[t_n, t_{n+1}]$ on which "*e_h does not change too much*" and use induction with respect to n. This is essentially a *continuous mathematical induction* argument, a concept introduced in [2].

LEMMA 5.2 (Continuous mathematical induction). *Let $\varphi(t)$ be a propositional function depending on $t \in [0, T]$ such that*

(*i*) $\varphi(0)$ *is true,*

(*ii*) $\exists\delta_0 > 0:\ \varphi(t)$ *implies* $\varphi(t+\delta),\ \forall t \in [0, T]\ \forall\delta \in [0, \delta_0] : t + \delta \in [0, T]$.

Then $\varphi(t)$ holds for all $t \in [0, T]$.

The independence of δ_0 on t can be replaced e.g. by some closure property, cf. [4] for an overview.

Remark 1 Due to the regularity assumptions, the functions $u(\cdot), u_h(\cdot)$ are continuous mappings from $[0, T]$ to $L^2(\Omega)$. Since $[0, T]$ is a compact set, $e_h(\cdot)$ is a *uniformly continuous* function from $[0, T]$ to $L^2(\Omega)$. By definition,

$$\forall\epsilon > 0\ \exists\delta > 0:\ s, \bar{s} \in [0, T], |s - \bar{s}| \le \delta \implies \|e_h(s) - e_h(\bar{s})\| \le \epsilon.$$

THEOREM 5.3 (Semidiscrete error estimate). *Let $p > (1 + d)/2$. Let $h_1 > 0$ be such that $C_T h_1^{p+1/2} = \frac{1}{2}h_1^{1+d/2}$, where C_T is the constant from Lemma 5.1. Then for all $h \in (0, h_1]$ we have the estimate*

$$\max_{\vartheta\in[0,T]} \|e_h(\vartheta)\|^2 \le C_T^2 h^{2p+1}. \tag{5.6}$$

Proof. Since $p > (1+d)/2$, h_1 is uniquely determined and $C_T h^{p+1/2} \leq \frac{1}{2}h^{1+d/2}$ for all $h \in (0, h_1]$. We define the propositional function φ by

$$\varphi(t) \equiv \Big\{ \max_{\vartheta\in[0,t]} \|e_h(\vartheta)\|^2 \leq C_T^2 h^{2p+1}\Big\}.$$

We shall use continuous mathematical induction (Lemma 5.2) to show that φ holds on $[0,T]$, hence $\varphi(T)$ holds, which is equivalent to (5.6).
(i) $\varphi(0)$ holds, since this is the error of the initial condition.
(ii) *Induction step*: We fix an arbitrary $h \in (0, h_1]$. By Remark 1, there exists $\delta_0 > 0$, such that if $t \in [0,T), \delta \in [0, \delta_0]$, then $\|e_h(t+\delta) - e_h(t)\| \leq \frac{1}{2}h^{1+d/2}$. Now let $t \in [0,T)$ and assume $\varphi(t)$ holds. Then $\varphi(t)$ implies $\|e_h(t)\| \leq C_T h^{p+1/2} \leq \frac{1}{2}h^{1+d/2}$. Let $\delta \in [0, \delta_0]$, then by uniform continuity

$$\|e_h(t+\delta)\| \leq \|e_h(t)\| + \|e_h(t+\delta) - e_h(t)\| \leq \tfrac{1}{2}h^{1+d/2} + \tfrac{1}{2}h^{1+d/2} = h^{1+d/2}.$$

This and $\varphi(t)$ implies that $\|e_h(s)\| \leq h^{1+d/2}$ for $s \in [0,t] \cup [t, t+\delta] = [0, t+\delta]$. By Lemma 5.1, φ holds on $[0, t+\delta]$. As a special case, we obtain the "induction step" $\varphi(t) \Longrightarrow \varphi(t+\delta)$ for all $\delta \in [0, \delta_0]$. □

If we assume $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$ we can use the improved estimate of Lemma 4.2 which gives the more favorable factor $h^{-1}\|e_h(\vartheta)\|_\infty^2$ instead of $h^{-1}\|e_h(\vartheta)\|_\infty$ in the estimate of the convective terms. Hence, we only need the weaker apriori assumption $\|e_h(\vartheta)\|_\infty = O(h^{1/2})$ to cancel this unpleasant term. This leads to the weaker assumption $\|e_h(\vartheta)\| \leq h^{(1+d)/2}$ for all $\vartheta \in [0,t]$ in Lemma 5.1. Therefore, in Theorem 5.3 we only need to assume $p > (d-1)/2$.

THEOREM 5.4 (Improved semidiscrete error estimate). *Let $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$ and $\Gamma_N = \emptyset$. Let $p > (d-1)/2$ and $h_1 > 0$ be such that $C_T h_1^{p+1} = \frac{1}{2}h_1^{(1+d)/2}$. Then for all $h \in (0, h_1]$ we have the estimate*

$$\max_{\vartheta\in[0,T]} \|e_h(\vartheta)\|^2 \leq C_T^2 h^{2p+2}.$$

Remark 2 For the method of lines we can also use a *nonlinear Gronwall-type lemma* to prove Theorem 5.3 directly, without using the continuous mathematical induction argument, cf. [8]. As we shall see in Lemma 6.2, this is not possible in the case of an implicit scheme, since an analogous discrete Gronwall lemma cannot exist.

6. Error estimates for an implicit scheme. In this section, we shall introduce and analyze the DG scheme with a standard first order implicit time discretization. Here we cannot use the approach of [12] for the explicit scheme, since we were unable to prove the first implication in the induction step (5.5). On the other hand, in Lemma 6.2 we prove that for the implicit scheme we cannot use a discrete Gronwall-type lemma as mentioned in Remark 2. In Section 5 the key ingredient was the continuity of e_h with respect to time, which guarantees that the error cannot suddenly blow up and can be therefore controlled in an inductive manner. However, for the implicit scheme we have a discrete temporal structure, hence no suitable concept of continuity. To overcome this obstacle, we introduce an appropriate continuation of the discrete solution and error with respect to time. This is constructed using an auxiliary problem, essentially a modification of the discrete implicit scheme. This allows us to derive error estimates for the continuated solution and consequently for the original implicit scheme.

We consider a partition $0 = t_0 < t_1 < \cdots < t_{N+1} = T$ of the time interval $[0,T]$ and set $\tau_n = t_{n+1} - t_n$ for $n = 0, \cdots, N$. The exact solution $u(t_n)$ will be approximated by $u_h^n \in S_h$.

Definition 6.1. *We say that* $\{u_h^n\}_{n=0}^{N+1} \subset S_h$ *is an implicit finite element solution of problem (2.1) - (2.3), if* $u_h^0 = \Pi_h u^0 \in S_h$ *and for* $n = 0, \cdots, N$

$$\Big(\frac{u_h^{n+1} - u_h^n}{\tau_n}, \varphi_h\Big) + b\big(u_h^{n+1}, \varphi_h\big) = l\big(\varphi_h\big)(t_{n+1}), \quad \forall \varphi_h \in S_h. \tag{6.1}$$

Similarly as in Section 4, we define $\eta_h^n = u(t_n) - \Pi_h u(t_n) \in L^2(\Omega)$ and $\xi_h^n = \Pi_h u(t_n) - u_h^n \in S_h$. We write the error e_h^n as $e_h^n := u(t_n) - u_h^n = \eta_h^n + \xi_h^n$.

To obtain error estimates for the implicit scheme, we would now subtract (3.2) and (6.1), test with $v_h := \xi_h^{n+1}$ and apply the derived estimates of b and the evolutionary terms. At this point, we would apply some (nonlinear) discrete Gronwall lemma to obtain the desired error estimates. However, it is simple to see that no such Gronwall-type lemma can exist and we need to proceed more carefully.

Lemma 6.2. *There does not exist a Gronwall-type lemma which could prove the desired error estimate (6.9)* only *from the error equation of the implicit scheme tested by* ξ_h^{n+1} *and the derived estimates of individual terms contained therein.*

Proof. We write the error equation on the first time level $n = 0$:

$$\big(e_h^1 - e_h^0, \varphi_h\big) + \tau_0\big(b(u(t_1), \varphi_h) - b(u_h^1, \varphi_h)\big) = \big(u(t_1) - u(t_0) - \tau_0 u_t(t_1), \varphi_h\big).$$

We set $\varphi_h := \xi_h^1$ and apply estimate (4.2) for b and standard approximation results to the evolutionary terms, [6]:

$$\begin{aligned} &\|\xi_h^1\|^2 + \|\xi_h^1 - \xi_h^0\|^2 \\ &\quad\le \|\xi_h^0\|^2 + C\tau_0\Big(1 + \frac{\|e_h^1\|_\infty}{h}\Big)\big(h^{2p+1} + \|\xi_h^1\|^2\big) + C\tau_0\big(h^{2p+2} + \tau_0^2 + \|\xi_h^1\|^2\big). \end{aligned} \tag{6.2}$$

Let us denote $X := \|e_h^1\|$. We shall prove that for all h, the right-hand side of (6.2) "*grows faster with respect to* X" than the left-hand side as $X \to \infty$. Therefore (6.2) is satisfied not only for small $X = O(h^{p+1/2} + \tau)$, but also for all X sufficiently large. Hence, (6.2) does not imply the error estimate (6.9) even for $n = 0$, in other words there does not exist a Gronwall-type (or any other) lemma which could derive (6.9) from (6.2). We proceed as follows: let X be sufficiently large. Then

$$\begin{aligned} \|\xi_h^1\| &\le \|e_h^1\| + \|\eta_h^1\| \le X + Ch^{p+1} \le 2X, \\ \|\xi_h^1\| &\ge \|e_h^1\| - \|\eta_h^1\| \ge X - Ch^{p+1} \ge X/2, \\ \|\xi_h^1 - \xi_h^0\| &\le \|\xi_h^1\| + \|\xi_h^0\| \le 2X + Ch^{p+1} \le 3X, \\ \|e_h^1\|_\infty &\ge X|\Omega|^{-1/2}, \end{aligned}$$

where the last inequality follows directly from Hölders inequality. Applying these estimates, we can estimate the left-hand side of (6.2) as

$$LHS = \|\xi_h^1\|^2 + \|\xi_h^1 - \xi_h^0\|^2 + \tau_0|\xi_h^1|_{\Gamma_N}^2 \le 4X^2 + 9X^2.$$

On the other hand, we get for the right-hand side of (6.2)

$$RHS \ge C\tau_0\Big(1 + \frac{\|e_h^1\|_\infty}{h}\Big)\|\xi_h^1\|^2 \ge C\tau_0\Big(1 + \frac{X}{|\Omega|^{1/2}h}\Big)X^2/4.$$

We want to determine, for what X is (6.2) satisfied, i.e. when is $LHS \leq RHS$. This happens e.g. if

$$LHS \leq 4X^2 + 9X^2 \leq C\tau_0\Big(1 + \frac{X}{|\Omega|^{1/2}h}\Big)X^2/4 \leq RHS,$$

i.e. when X satisfies

$$C\tau_0\Big(1 + \frac{X}{|\Omega|^{1/2}h}\Big)X^2/4 - 4X^2 - 9X^2 \geq 0. \tag{6.3}$$

But the leading term X^3 in (6.3) has a positive coefficient $C\tau_0|\Omega|^{-1/2}h^{-1}/4$. Hence inequality (6.3) – and therefore inequality (6.2) – holds for all X sufficiently large. □

6.1. Auxiliary problem and continuation of the discrete solution. Problem (6.1) represents a nonlinear equation on each time level t^{n+1} for the unknown function u_h^{n+1}. First, we prove that u_h^{n+1} exists uniquely and depends continuously on τ_n. For this purpose we define an "abstract" formulation of problem (6.1):

Definition 6.3. *(Auxiliary problem) Let $t \in [0,T], \tau \in [0,T]$ and $U_h \in S_h$. We seek $u_\tau \in S_h$ such that*

$$\big(u_\tau - U_h, \varphi_h\big) + \tau b\big(u_\tau, \varphi_h\big) = \tau l\big(\varphi_h\big)(t), \quad \forall \varphi_h \in S_h. \tag{6.4}$$

If we take $\tau := \tau_n, U_h := u_h^n, t := t_{n+1}$ and define $u_h^{n+1} := u_\tau$, the auxiliary problem (6.4) reduces to equation (6.1), which defines the approximate solution u_h^{n+1}. On the other hand, if we take $\tau := 0$ the solution of (6.4) is $u_\tau = u_h^n$. In between these two cases u_τ changes continuously with τ:

Lemma 6.4. *Let $t \in [0,T], h \in (0,h_0), U_h \in S_h$ and $\tau \in \big[0, C_\tau h\big)$, where C_τ is a given constant. Then u_τ, the solution of (6.4), exists, is uniquely determined, $\|u_\tau\|$ is uniformly bounded with respect to τ and $\|u_\tau\|$ depends continuously on $\tau \in \big[0, C_\tau h\big)$.*

Proof. Problem (6.4) is a nonlinear equation for u_τ on the finite-dimensional space S_h. Existence and uniqueness follow from the nonlinear Lax-Milgram theorem, cf. [11], one only needs Lipschitz-continuity and monotonicity of the nonlinear forms. Continuity with respect to τ is obtained by subtracting (6.4) for τ and $\bar{\tau}$, apply the monotonicity estimates and let $\bar{\tau} \to \tau$, For details of the proof, see [8]. □

As stated above, by taking $U_h := u_h^n$ in (6.4) we obtain $u_\tau = u_h^{n+1}$ for $\tau := \tau_n$ and $u_\tau = u_h^n$ for $\tau := 0$. For general $\tau \in [0, \tau_n]$, u_τ depends continuously on the parameter τ. This allows us to construct a function $\tilde{u}_h \in C([0,T]; S_h)$ which "interpolates" the values $\{u_h^n\}_{n=0}^N$ and which is constructed using essentially the implicit problem itself.

Definition 6.5 (Continuated discrete solution). *Let $\tilde{u}_h : [0,T] \to S_h$ be such that for $s \in [t_n, t_{n+1}]$ we define $\tilde{u}_h(s) := u_\tau$, the solution of the auxiliary problem (6.4) with $\tau := s - t_n$, $t{:=}t_{n+1}$ and $U_h := u_h^n$. Furthermore, we define $\tilde{e}_h := u - \tilde{u}_h$ and $\tilde{\xi}_h := \Pi_h u - \tilde{u}_h$.*

Under the assumptions of Lemma 6.4, $\tilde{u}_h$ is uniquely determined, $\tilde{u}_h \in C([0,T]; S_h)$ and $\tilde{e}_h \in C([0,T]; L^2(\Omega))$, due to the regularity assumptions (4.1). Also, $\tilde{u}_h(t_n) = u_h^n$, $\tilde{e}_h(t_n) = e_h^n$ and $\tilde{\xi}_h(t_n) = \xi_h^n$, for $n = 0, \cdots, N$. Therefore, estimates of $\tilde{e}_h(\cdot)$ imply estimates of e_h^n. Finally, we note that $\tilde{e}_h = \eta_h + \tilde{\xi}_h$.

6.2. Estimates based on continuous mathematical induction.. Since $\tilde{u}_h$ is constructed using the auxiliary problem (6.4), which is essentially the original implicit scheme (6.1) with special data, we can derive error estimates for $\tilde{u}_h$ in a standard manner. We start by proving a discrete analogy of Lemma 5.1. For simplicity we assume a uniform partition of $[0,T]$, i.e. $\tau_n := \tau$ for all $n = 0, \cdots, N$.

LEMMA 6.6. *Let $p \geq d/2$. Let $s \in (t_n, t_{n+1}]$ for some $n \in \{0, \cdots, N\}$. If $\|\tilde{e}_h(s)\| \leq h^{1+d/2}$ and $\|\tilde{e}_h(t_k)\| \leq h^{1+d/2}$, for $k = 0, \cdots, n$, then we have the estimate*

$$\max_{t\in\{t_0,\cdots,t_n,s\}} \|\tilde{e}_h(t)\|^2 \leq C_T^2\big(h^{2p+1} + \tau^2\big), \tag{6.5}$$

where the constant C_T is independent of s, n, h, τ.

Proof. Since $\tilde{e}_h = u - \tilde{u}_h$ and $\tilde{u}_h$ is defined by the Auxiliary problem (6.4) with $\tau = s - t_n$, $U_h = u_h^n$, in order to obtain an equation for $\tilde{e}_h$, we subtract (6.4) from (3.2). Furthermore, in (3.2) we introduce the time difference instead of the time derivative u_t. Thus $\tilde{e}_h(s)$ satisfies

$$\begin{aligned}\big(\tilde{e}_h(s) - \tilde{e}_h(t_n), \varphi_h\big) + (s - t_n)\big(b(u(s), \varphi_h) - b(\tilde{u}_h(s), \varphi_h)\big)\\ = \big(u(s) - u(t_n) - (s - t_n)u_t(s), \varphi_h\big).\end{aligned} \tag{6.6}$$

We set $\varphi_h := \tilde{\xi}_h(s)$ and use the fact that $2(a - b, a) = \|a\|^2 - \|b\|^2 + \|a - b\|^2$. Furthermore, we estimate the convective terms using Lemma 4.1 and standard estimates for the right-hand side evolutionary terms, cf. [6]. Thus we obtain the inequality

$$\begin{aligned}\|\tilde{\xi}_h(s)\|^2 - \|\tilde{\xi}_h(t_n)\|^2 + \|\tilde{\xi}_h(s) - \tilde{\xi}_h(t_n)\|^2 \leq C\tau\Big(1 + \frac{\|\tilde{e}_h(s)\|_\infty}{h}\Big)\Big(h^{2p+1}|u|^2_{L^\infty(H^{p+1})}\\ + h^{2p+2}\|u_t\|^2_{L^\infty(H^{p+1})} + \tau^2\|u_{tt}\|^2_{L^\infty(L^2)} + \|\tilde{\xi}_h(s)\|^2\Big).\end{aligned} \tag{6.7}$$

Similarly as in (5.3), we may show that the assumptions imply $\|\tilde{e}_h(s)\|_\infty \leq Ch$. Thus (6.7) reduces to

$$\|\tilde{\xi}_h(s)\|^2 \leq \|\tilde{\xi}_h(t_n)\|^2 + C\tau\big(h^{2p+1} + \tau^2 + \|\tilde{\xi}_h(s)\|^2\big). \tag{6.8}$$

Similarly as $\tilde{e}_h(s)$ satisfies (6.6), $\tilde{e}_h(t_k)$ satisfies the following equation for all $k = 0, \cdots, n-1$:

$$\begin{aligned}\big(\tilde{e}_h(t_{k+1}) - \tilde{e}_h(t_k), \varphi_h\big) + \tau\big(b(u(t_{k+1}), \varphi_h) - b(\tilde{u}_h(t_{k+1}), \varphi_h)\big)\\ = \big(u(t_{k+1}) - u(t_k) - \tau u_t(t_{k+1}), \varphi_h\big).\end{aligned}$$

We set $\varphi_h := \tilde{\xi}_h(t_{k+1})$ and proceed as in estimates (6.7), (6.8) to obtain

$$\|\tilde{\xi}_h(t_{k+1})\|^2 \leq \|\tilde{\xi}_h(t_k)\|^2 + C\tau\big(h^{2p+1} + \tau^2 + \|\tilde{\xi}_h(t_{k+1})\|^2\big).$$

Summing from 0 to k, we obtain

$$\|\tilde{\xi}_h(t_{k+1})\|^2 \leq CT\big(h^{2p+1} + \tau^2\big) + C\tau\sum_{l=1}^{k+1}\|\tilde{\xi}_h(t_l)\|^2.$$

Assuming without loss of generality $C\tau \leq \frac{1}{2}$, we can absorb $C\tau\|\tilde{\xi}_h(t_{k+1})\|^2$, the last term in the right-hand side sum, by left-hand side, obtaining

$$\frac{1}{2}\|\tilde{\xi}_h(t_{k+1})\|^2 \leq CT\big(h^{2p+1} + \tau^2\big) + C\tau\sum_{l=1}^{k}\|\tilde{\xi}_h(t_l)\|^2.$$

Applying the discrete Gronwall lemma, we get for $k = 0, \cdots, n-1$

$$\max_{l=0,\cdots,k+1} \|\tilde{\xi}_h(t_l)\|^2 \leq 2e^{CT} CT\big(h^{2p+1} + \tau^2\big).$$

From (6.8) we obtain an estimate for $\|\tilde{\xi}_h(s)\|^2$. Estimates for η_h give us (6.5). □

Remark 3 The functions $u(\cdot), \tilde{u}_h(\cdot)$ are continuous mappings from $[0,T]$ to $L^2(\Omega)$. Therefore, $\tilde{e}_h(\cdot)$ is a *uniformly continuous* function from $[0,T]$ to $L^2(\Omega)$. By definition,

$$\forall \epsilon > 0\ \exists \delta > 0 :\ s, \bar{s} \in [0,t], |s - \bar{s}| \leq \delta \implies \|\tilde{e}_h(s) - \tilde{e}_h(\bar{s})\| \leq \epsilon.$$

THEOREM 6.7 (Implicit error estimate). *Let $p > (1+d)/2$. Let $h_1 > 0$ be such that $C_T h_1^{p+1/2} = \frac{1}{4} h_1^{1+d/2}$. Then for all $h \in (0, h_1), \tau < \min\{\frac{1}{4}h^{1+d/2}, C_\tau h\}$ we have the estimate (C_T is the constant form Lemma 6.6)*

$$\max_{n\in\{0,\cdots,N+1\}} \|e_h^n\|^2 \leq C_T^2\big(h^{2p+1} + \tau^2\big). \tag{6.9}$$

Proof. We have $p > (1+d)/2$, therefore h_1 exists and $C_T(h^{p+1/2} + \tau) \leq \frac{1}{2}h^{1+d/2}$ for all $h \in (0, h_1], \tau < \min\{\frac{1}{4}h^{1+d/2}, C_\tau h\}$. We define the statement φ by

$$\varphi(t) \equiv \big\{ \max_{\vartheta\in[0,t]} \|\tilde{e}_h(\vartheta)\| \leq C_T\big(h^{p+1/2} + \tau\big)\big\}.$$

As in the proof of Theorem 5.3, we shall use continuous mathematical induction (Lemma 5.2) to show that φ holds on $[0,T]$, hence $\max_{\vartheta\in[0,T]} \|\tilde{e}_h(\vartheta)\| \leq h^{1+d/2}$, thus (6.9) follows from Lemma 6.6.

(i) $\varphi(0)$ holds, since this is the error of the initial condition.

(ii) *Induction step*: We fix an arbitrary $h \in (0, h_1]$. By Remark 3, there exists $\delta_0 > 0$, such that if $t \in [0,T), \delta \in [0,\delta_0]$, then $\|\tilde{e}_h(t+\delta) - \tilde{e}_h(t)\| \leq \frac{1}{2}h^{1+d/2}$. Now let $t \in [0,T)$ and assume $\varphi(t)$ holds. Then $\varphi(t)$ implies $\|\tilde{e}_h(t)\| \leq C_T(h^{p+1/2} + \tau) \leq \frac{1}{2}h^{1+d/2}$. Let $\delta \in [0, \delta_0]$, then by uniform continuity

$$\|\tilde{e}_h(t+\delta)\| \leq \|\tilde{e}_h(t)\| + \|\tilde{e}_h(t+\delta) - \tilde{e}_h(t)\| \leq \tfrac{1}{2}h^{1+d/2} + \tfrac{1}{2}h^{1+d/2} = h^{1+d/2}.$$

This and $\varphi(t)$ implies that $\|\tilde{e}_h(s)\| \leq h^{1+d/2}$ for $s \in [0,t] \cup [t, t+\delta] = [0, t+\delta]$. By Lemma 6.6, φ holds on $[0, t+\delta]$. As a special case, we obtain the "induction step" $\varphi(t) \Longrightarrow \varphi(t+\delta)$ for all $\delta \in [0, \delta_0]$. □

As in Theorem 5.4, we can use the improved estimate of Lemma 4.2 using the preceding arguments. This leads to the following theorem.

THEOREM 6.8 (Improved implicit error estimate). *Let $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$ and $\Gamma_N = \emptyset$. Let $p > (d-1)/2$. Let $h_1 > 0$ be such that $C_T h_1^{p+1} = \frac{1}{4}h_1^{(1+d)/2}$. Then for all $h \in (0, h_1), \tau < \min\{\frac{1}{4}h^{(1+d)/2}, C_\tau h\}$ we have the estimate*

$$\max_{n\in\{0,\cdots,N+1\}} \|e_h^n\|^2 \leq C_T^2\big(h^{2p+2} + \tau^2\big).$$

We conclude with several remarks.

- The *CFL* condition required in Theorems 6.7 and 6.8 effectively imposes $\tau = O(h^{1+d/2})$ and $\tau = O(h^{(1+d)/2})$, respectively. This is rather restrictive from the perspective of an implicit scheme. This condition arises due to the key step in our analysis, where we require the final error to be smaller than the apriori assumption, i.e. $C_T(h^{p+1/2} + \tau) \leq \frac{1}{2}h^{1+d/2}$ for Theorem 6.7.

- We note that we have treated the simplest first order temporal discretization. Were we to analyze a scheme of k-th order in time, we would require that $C_T(h^{p+1/2}+\tau^k) \leq \frac{1}{2}h^{1+d/2}$, which leads to the less restrictive CFL condition $\tau = O(h^{(1+d/2)/k})$ or $\tau = O(h^{(1+d)/(2k)})$, using Lemma 4.2.
- Essentially, we are still limited by the CFL condition $\tau = O(h)$ of Lemma 6.4 to guarantee existence and continuity of the continuated discrete solution.

7. Conclusion. We have presented an apriori error analysis of the finite element method of order p for a nonstationary nonlinear convective problem on $\Omega \subset \mathbb{R}^d$. Building on results from [12], which dealt with an explicit time discretization of the discontinuous Galerkin scheme, we proved apriori $L^\infty(L^2)$ error estimates for the method of lines and the backward Euler method. The technique used is quite similar to standard finite element error estimates for parabolic problems.

- For the method of lines, using the apriori assumption $\|e_h(t)\|_\infty = O(h), t \in [0,T]$, we proved that if $p > (1+d)/2$, then $\|e_h\|_{L^\infty(L^2)} \leq C_T h^{p+1/2}$. Using continuous mathematical induction we have eliminated the apriori assumption. The same result can also be proved by a nonlinear Gronwall-type lemma.
- For an implicit scheme we proved there does not exist a discrete Gronwall-type lemma capable of proving the desired error estimate only from the error equation and estimates of its individual terms.
- Using an appropriate auxiliary problem derived from the implicit scheme, we introduced a suitable continuation $\tilde{u}_h$ with respect to time of the discrete solution u_h^n. Using continuous mathematical induction we proved error estimates for $\tilde{u}_h$, which imply estimates for u_h^n. For the first order implicit scheme we have that if $p > (1+d)/2$, then $\sup_{n=0,\cdots,N} \|e_h^n\| \leq C_T(h^{p+1/2}+\tau)$. This was proved under the rather restrictive CFL-like condition $\tau = O(h^{1+d/2})$.
- For $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$ with the absence of Neumann boundary conditions, we obtained all the derived estimates under the less restrictive assumption $p > (d-1)/2$. Under these conditions the error estimates improve to $O(h^{p+1})$ and $O(h^{p+1}+\tau)$, respectively. For the implicit scheme we also obtained an improved CFL condition $\tau = O(h^{(1+d)/2})$.
- In [8], the results are extended to the *locally Lipschitz* case $\mathbf{f} \in (C^2(\mathbb{R}))^d$ and $\mathbf{f} \in (C^3(\mathbb{R}))^d$ using (continuous) mathematical induction directly, without the need to modify the original continuous problem.

There are several open problems connected with the presented analysis:

- Eliminating the order conditions $p > (1+d)/2$ and $p > (d-1)/2$.
- Eliminating the rather unnatural CFL conditions $\tau = O(h^{1+d/2})$ and $\tau = O(h^{(1+d)/2})$ in the implicit scheme.
- Obtaining the improved convergence rate $O(h^{p+1})$ also for $\mathbf{f} \in (C^2(\mathbb{R}))^d$.

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