Proceedings of ALGORITMY 2012
pp. 404–415

NUMERICAL STUDIES OF VARIATIONAL-TYPE TIME-DISCRETIZATION TECHNIQUES FOR TRANSIENT OSEEN PROBLEM

NAVEED AHMED* AND GUNAR MATTHIES†

Abstract. In this paper, we combine continuous Galerkin-Petrov (cGP) and discontinuous Galerkin (dG) time stepping schemes with local projection method applied to inf-sup stable discretization of the transient Oseen problem. Using variational-type time-discretization methods of polynomial degree k, we show that the cGP(k) and dG(k) methods are accurate of order $k+1$, in the whole time interval. Moreover, in the discrete time points, the cGP(k)-method is super-convergent of order $2k$ and the dG(k)-method is of order $2k+1$. Furthermore, the dependence of the results on the choice of the stabilization parameters are discussed.

Key words. Oseen equations, stabilized finite elements, continuous Galerkin-Petrov, discontinuous Galerkin

AMS subject classifications. 65M12, 65M15, 65M60

1. Introduction. In this paper, we consider the numerical solution of the transient Oseen equations in the case of dominant convection. It is well known that standard Galerkin finite element methods are unsuitable for the solution of convection-dominated problem. On the other hand, if the mesh size in space becomes small then often the system obtained by space discretization becomes more and more stiff. To this end, one needs a stable time and space discretization. The classical stabilization method are the streamline-upwind Petrov-Galerkin (SUPG) [1] and pressure-stabilization Petrov-Galerkin (PSPG) [2] methods. Concerning steady incompressible flow problems, this class of residual based stabilization techniques is still very popular. Besides the robustness of the method, a fundamental drawback is the addition of several terms for ensuring the strong consistency of the method. Using inf-sup stable pairs of finite element spaces for approximating velocity and pressure, we can skip the PSPG term to obtain a so-called reduced stabilized scheme [4, 3]. Numerical studies of time-dependent incompressible flow problems can be found in [5] where finite difference schemes in time are combined with SUPG in space. The extension to the transient Stokes problem of different stabilization methods including Galerkin/least squares (GLS) method in small time step limit are studied in [6, 7, 8]. In [7], it has been shown for the small time step limit that even the first order backward difference methods perturbs the stability of the numerical scheme. This behavior is caused by the finite difference operator appearing in the stabilization terms of the SUPG to guarantee consistency and produces a non-symmetric term which is difficult to handle.

Furthermore, the strong coupling between velocity and pressure in the stabilization terms makes the analysis difficult. In order to relax the strong consistency in the SUPG or PSPG type stabilization, there are stabilization techniques such as

*Universität Kassel, Fachbereich 10 Mathematik und Naturwissenschaften, Institut für Mathematik, Heinrich-Plett-Straße 40, 34132 Kassel, Germany (nahmed@mathematik.uni-kassel.de).

†Universität Kassel, Fachbereich 10 Mathematik und Naturwissenschaften, Institut für Mathematik, Heinrich-Plett-Straße 40, 34132 Kassel, Germany. (matthies@mathematik.uni-kassel.de)

continuous interior penalty CIP [9, 10] or local projection stabilization methods as two-level [11, 12] and one-level enrichment approach [13], we will use the one-level LPS method [13]. When applied to time-dependent problems, time derivative needs not to be included into the stabilization term in contrast to the SUPG method. Also, the stabilization terms of CIP and the two-level variant of LPS introduce additional coupling between the degrees of freedom that do not belong to the same cell. Hence, the sparsity of the element matrices decreases and one needs appropriate data structures. Although, the LPS method is weakly consistent only, the consistency error can be bounded such that the optimal order of convergence is maintained.

In order to handle the difficulty of stiff systems, we will consider two classes of variational type time discretizations to solve the time-dependent problems. The first one is the discontinuous Galerkin (dG) time stepping scheme in which both trial and test functions are discontinuous in time. In the second scheme, the trial functions are continuous in time whereas the test functions are discontinuous in time. This method can be viewed as a Galerkin-Petrov method. The continuous Galerkin-Petrov (cGP) method has been studied in [14] for heat equation. A numerical comparison of cGP and dG methods applied to the heat equation is given in [15] and to transient Stokes equations in [16]. Recently, in [17, 18], the cGP method has been investigated for linear and nonlinear ordinary differential equations. The cGP methods are A-stable whereas it is well-known that the dG methods are even strongly A-stable (or L-stable according to [19]). The space-time dG-method for nonstationary convection-diffusion-reaction problems has been analyzed in [20]. The local projection stabilization methods for incompressible flow problems has been studied in [11, 12, 13, 21].

2. Model problem. Find $u : \Omega \times (0,T) \to R^2$ and $p : \Omega \times (0,T) \to R$ such that

$$\begin{aligned} u' - \nu\Delta u + b \cdot \nabla u + \sigma u + \nabla p &= f \qquad \text{in} \quad \Omega \times (0,T) \\ \nabla \cdot u &= 0 \qquad \text{in} \quad \Omega \times (0,T) \\ u &= 0 \qquad \text{on} \quad \partial\Omega \\ u(\cdot,0) &= u_0 \qquad \text{in} \quad \Omega. \end{aligned} \tag{2.1}$$

Here $\Omega \subset R^d$ $(d = 2,3)$ is a bounded domain with polyhedral boundary $\partial\Omega$, $u(t,x)$ is the fluid velocity, $p(t,x)$ is the fluid pressure, $f(t,x)$ is vector function representing the external force, and u_0 is the initial velocity. For the sake of simplicity, the simplest homogeneous Dirichlet boundary conditions will be considered.

Let us introduce some standard notation. The space of square integrable functions in a domain Ω is denoted by $L^2(\Omega)$ and the space of functions whose distributional derivatives of order up to $m \geq 0$ belong to $L^2(\Omega)$ by $H^m(\Omega)$. We denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ and by $\|\cdot\|_0$ the associated L^2-norm. The norm in $H^m(\Omega)$ is defined as

$$\|v\|_m = \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|_0^2 \right)^{1/2} \quad \text{with the seminorm} \quad |v|_m = \left(\sum_{|\alpha| = m} \|D^\alpha v\|_0^2 \right)^{1/2}.$$

We also use the Bochner spaces. Let X be a Banach space with norm $\|\cdot\|_X$ and let

m be an integer. Then we define:

$$C\big([0,T];X\big) := \Big\{v : [0,T] \to X : v \text{ continuous}\Big\},$$
$$L^2\big(0,T;X\big) := \Big\{v : (0,T) \to X : \int_0^T \|v\|_X^2\,dt < \infty\Big\},$$
$$H^m\big(0,T,X\big) := \Big\{v \in L^2\big([0,T];X\big) : v^{(j)} \in L^2\big([0,T];X\big), \quad 1 \le j \le m\Big\},$$

where the derivatives $v^{(j)}$ are considered in the sense of distribution on $(0,T)$. In the following we use the short notation $Y(X) := Y(0,T;X)$. The norm in the above defined spaces are given as follows

$$\|v\|_{C(X)} = \sup_{t\in[0,T]} \|v(t)\|_X, \quad \|v\|_{L^2(X)}^2 = \int_0^T \|v\|_X^2\,dt, \quad \|v\|_{H^m(X)}^2 = \int_0^T \sum_{j=0}^m \|v^{(j)}\|_X^2\,dt.$$

Let $V := [H_0^1(\Omega)]^d$ and $Q := L_0^2(\Omega)$. A variational form of (2.1) reads as follows:

Find $(u,p) : [0,T] \to V \times Q$ such that

$$\begin{aligned} (u',v) + a(u,v) + b(p,v) - b(q,u) &= (f,v) \qquad \text{a.e. in } (0,T) \quad \forall (v,q) \in V \times Q \\ u(0) &= u_0 \qquad \text{a.e. in } \Omega \end{aligned} \tag{2.2}$$

where the bilinear forms are defined as follows

$$a(u,v) = \nu(\nabla u, \nabla v) + (b\cdot\nabla u, v) + \sigma(u,v), \qquad b(p,v) = -(p, \nabla\cdot v).$$

In what follows, we shall denote by f', f'', and $f^{(k)}$ the first, second, and kth order time derivative of f, respectively.

3. Space discretization. We are given a family $\mathcal{T}_h$ of shape-regular decompositions of Ω into d-simplices, quadrilaterals or hexhedra. The diameter of a cell K is denoted by h_K. Let $V_h \subset V$ be a finite element space of continuous, piecewise polynomial function over $\mathcal{T}_h$. The pressure is described using a finite element space $Q_h \subset Q$ of continuous or discontinuous functions with respect to $\mathcal{T}_h$. We will consider inf-sup stable pair (V_h, Q_h) throughout this paper.

Assumption A1. The pair (V_h, Q_h) fulfills the discrete inf-sup condition, i.e., there exists a positive constant β_0 such that

$$\inf_{q_h\in Q_h} \sup_{v_h\in V_h} \frac{(v_h, \mathrm{div} v_h)}{|v_h|_1 \|q_h\|_0} \ge \beta_0 > 0$$

uniformly in h.

The semi-discrete formulation of (2.2) reads:

Find $(u_h, p_h) : [0,T] \to V_h \times Q_h$ such that for $u_h(0) = u_{h,0}$

$$(u_h', v_h) + a(u_h, p_h) + b(p_h, v_h) - b(q_h, u_h) = (f, v_h) \qquad \text{a.e. in } (0,T) \tag{3.1}$$

for all $(v_h, q_h) \in V_h \times Q_h$, where $u_h^0 \in V_h$ is a suitable approximation of u_0.

In general, problem (3.1) lacks stability for $\nu \ll 1$ due to dominating convection. To overcome this difficulty, we consider the stabilization by local projection and introduce some additional notations. Let $\mathcal{D}_h(K)$ be a finite dimensional space on the cell $K \in \mathcal{T}_h$ and $\pi_K : L^2(K) \to \mathcal{D}_h(K)$ the associated local L^2-projection into $\mathcal{D}_h(K)$. The global projection space is defined by

$$\mathcal{D}_h := \bigoplus_{K\in\mathcal{T}_h} \mathcal{D}_h(K).$$

Note that functions in these spaces are allowed to be discontinuous with respect to $\mathcal{T}_h$. The mapping $\pi_h : L^2(\Omega) \to L^2(\Omega)$ defined by $(\pi_h v)|_K := \pi_K(v|_K)$ for all $K \in \mathcal{T}_h$ is the L^2-projection into the projection space $\mathcal{D}_h$. The fluctuation operator κ_h is given by $\kappa_h := id - \pi_h$ where $id : L^2(\Omega) \to (\Omega)$ is the identity mapping.

The stabilizing term S_h is then defined as

$$S_h(u_h, v_h) := \sum_{K\in\mathcal{T}_h} \mu_K(\kappa_h(\nabla u_h), \kappa_h(\nabla v_h))_K.$$

Here, $(\cdot,\cdot)_K$ denotes the inner product in $L^2(K)$, and μ_K the user chosen non-negative constant. The stabilization term S_h gives additional control over the fluctuation of gradients. Note that one can replace the gradient ∇w_h by the derivative in the streamline direction $b \cdot \nabla w_h$ or (even better [22], [23]) by $b_K \cdot \nabla w_h$ where b_K is a piecewise approximation of b but one have to add the divergence term $(\nabla \cdot u_h, \nabla \cdot v_h)$ into S_h, see [13]. We define the stabilized bilinear form a_h

$$a_h(u, v) = a(u, v) + S_h(u, v)$$

and introduce the mesh dependent norm on the product space $V_h \times Q_h$

$$|||(v, q)||| := \Big(\nu|v|_1^2 + \sigma\|v\|_0^2 + \|q\|_0^2 + S_h(v, v)\Big)^{1/2}. \tag{3.2}$$

Now, the stabilized semi-discrete scheme reads:

Find $(u_h, p_h) : [0, T] \to V_h \times Q_h$ such that for $u_h(0) = u_{0,h}$

$$(u_h', v_h) + a_h(u_h, v_h) + b(p_h, v_h) - b(q_h, u_h) = (f, v_h) \qquad \text{a.e. in } (0, T) \tag{3.3}$$

for all $(v_h, q_h) \in V_h \times Q_h$.

Stability and convergence properties of the local projection stabilization method (3.3) are based on the following assumptions:

Assumption A2. There are interpolation operators $j_h : V \cap H^2(\Omega)^d \to V_h$ and $i_h : Q \cap H^2(\Omega) \to Q_h$ fulfilling the orthogonality property

$$(w - j_h w, q_h) = 0 \qquad \forall q_h \in \mathcal{D}_h,\ \forall w \in H^1(\Omega)$$

and the approximation property

$$\|w - j_h w\|_{0,K} + h_K|w - j_h w|_{1,K} \le C h_K^l \|w\|_{l,\omega(K)}$$

for all $w \in H^l\big(\omega(K)^d\big)$, $2 \le l \le r + 1$, $\forall K \in \mathcal{T}_h$ and

$$\|q - i_h q\|_{0,K} + h_K|q - i_h q|_{1,K} \le C h_K^l \|q\|_{l,\omega(K)}$$

for all $q \in H^l\big(\omega(K)\big)$, $2 \le l \le r$, $\forall K \in \mathcal{T}_h$, where $\omega(K)$ denotes a certain local neighborhood of K.

Assumption A3. Let the fluctuation operator satisfy the following approximation property

$$\|\kappa_h q\|_{0,K} \le C h_K^l |q|_{l,K}$$

$\forall K \in \mathcal{T}_h,\ \forall q \in H^l(K),\ 0 \le l \le r.$

For the stationary problem associated with (2.1) we have, see [13]

THEOREM 3.1. *Suppose that the spaces V_h, Q_h satisfy A1, A2. Let (u,p) be the solution of (2.2) and (U,P) the solution of (3.3). Let the user chosen parameter satisfy $\mu_K \sim 1$. Assume that $u \in V \cap H^{r+1}(\Omega)^d$ and $p \in H^r(\Omega)$. Then, there exists a positive constant C independent of ν, h and τ such that the error estimate*

$$|||(u-u_h, p-p_h)||| \le Ch^r\Big(\|u\|_{r+1} + \|p\|_r\Big) \tag{3.4}$$

holds true.

4. Time discretization. We discretize the problem (3.3) in time using the continuous Galerkin-Petrov (cGP) and discontinuous Galerkin (dG) methods. For this we decompose the time interval J into N sub-intervals $J_n := (t_{n-1}, t_n]$, where $n = 1, \dots, N$ $0 < t_1 < \cdots < t_{N-1} = T$, $\tau_n = t_n - t_{n-1}$ and $\tau = \max_{1\le n\le N} \tau_n$. In the following, the set of the time intervals $\mathcal{M}_\tau$ will be called the time-mesh. For a non-negative integer k, the fully discrete time-continuous and time-discontinuous velocity spaces are defined as follows

$$X_k^{\mathrm{c}} := \Big\{u \in C(J;V_h) : u|_{J_n} \in \mathcal{P}_k(J_n, V_h)\Big\}, \quad X_k^{\mathrm{dc}} := \Big\{u \in L^2(J;V_h) : u|_{J_n} \in \mathcal{P}_k(J_n, V_h)\Big\}.$$

Similarly, the fully discrete time-continuous and time-discontinuous Y_k^{dc} pressure spaces are defined by

$$Y_k^{\mathrm{c}} := \Big\{q \in C(J;Q_h) : q|_{J_n} \in \mathcal{P}_k(J_n, Q_h)\Big\}, \quad Y_k^{\mathrm{dc}} := \Big\{q \in L^2(J;Q_h) : q|_{J_n} \in \mathcal{P}_k(J_n, Q_h)\Big\}$$

where

$$\mathcal{P}_k(J_n, W_h) := \Big\{u : J_n \to W_h : u(t) = \sum_{i=0}^{k} U^i t^i,\ \forall t \in J_n, U^i \in W_h, \forall i\Big\}$$

The functions in spaces X_k^{dc} and Y_k^{dc} are allowed to be discontinuous at the nodes t_n. For the discrete functions, we define the left-sided and right-sided values u_n^- and u_n^+ and the jumps $[u]_n$ as

$$u_n^- := \lim_{t\to t_n-0} u(t), \qquad u_n^+ := \lim_{t\to t_n+0} u(t), \qquad [u]_n = u_n^+ - u_n^-.$$

4.1. Continuous Galerkin-Petrov method. In this method, we use the space X_k^{c} for velocity and Y_k^{c} for pressure as the fully discrete solution spaces and X_k^{dc} and Y_k^{dc} as the discrete test spaces. The fully discrete cGP(k) method is defined as follows:

Find $(U,P) \in X_k^{\mathrm{c}} \times Y_k^{\mathrm{c}}$ such that $U(0) = u_{h,0}$ and

$$\int_0^T \Big\{(U', v_{h,\tau}) + a_h(U, v_{h,\tau}) + b(P, v_{h,\tau}) - b(q_{h,\tau}, U)\Big\}dt = \int_0^T (f, v_{h,\tau})dt \tag{4.1}$$

for all $(v_{h,\tau}, q_{h,\tau}) \in X_k^{\mathrm{dc}} \times Y_k^{\mathrm{dc}}$. Since the discrete test spaces are discontinuous, problem (4.1) can be solved by a time-marching process where on each time interval J_n local problems in space have to be solved.

We introduce the mesh dependent norm associated with the cGP method as

$$\|(v,p)\|_{\mathrm{cGP}} = \left(\int_0^T \Big\{ \|v'\|_0^2 + |||(v,p)|||^2 \Big\} dt \right)^{1/2}$$

The following theorem gives the a priori estimates for the velocity and pressure, which are obtained by using the similar idea as in [18] for the fully discrete scheme (4.1) obtained by continuous Galerkin-Petrov time stepping scheme.

THEOREM 4.1. *Suppose that the spaces V_h, Q_h satisfy A1, A2 and $\mu_K \sim 1$. Let (u,p) be the solution of (2.2) and (U,P) the solution of (4.1). Assume that $u \in H^{k+1}(J; H_0^1(\Omega)^d) \cap H^1(J; H^{r+1}(\Omega)^d)$, $p \in H^{k+1}(J; H_0^1(\Omega)) \cap C(J; H^r(\Omega))$ and $u_0 \in H_0^1(\Omega)^d \cap H^{r+1}(\Omega)^d$. Then, there exists a positive constant C independent of ν, h and τ such that the error estimate*

$$\|(u-U, p-P)\|_{\mathrm{cGP}} \le C(\tau^k + h^r)$$

holds true.

4.2. Discontinuous Galerkin method. In this subsection we discretize the semi-discrete problem (3.3) with respect to time by the discontinuous Galerkin time stepping scheme. Here the discrete solution spaces for velocity and pressure are the same as the test spaces, namely X_k^{dc} and Y_k^{dc}. The fully discrete problem is defined as follows:

Find $(U,P) \in X_k^{\mathrm{dc}} \times Y_k^{\mathrm{dc}}$ such that

$$\sum_{n=1}^{N} \int_{J_n} \Big\{ (U', v_{h,\tau}) + a_h(U, v_{h,\tau}) - b(P, v_{h,\tau}) + b(q_{h,\tau}, U) \Big\} dt + \sum_{n=1}^{N-1} \big([U]_n, v_n^+ \big) + \big(U_0^+, v_0^+ \big) = (u_0, v_0^+) + \int_0^T \big(f, v_{h,\tau} \big) dt \tag{4.2}$$

for all $(v_{h,\tau}, q_{h,\tau}) \in X_k^{\mathrm{dc}} \times Y_k^{\mathrm{dc}}$.

Due to the discontinuity in time of the discrete test space, a time marching process can be used to solve (4.2). We consider the following mesh-dependent norm associated with the dG time discretization method

$$\big\|(v,p)\big\|_{\mathrm{dG}} := \left(\sum_{n=1}^{N} \int_{J_n} \big|\big|\big|(v,p)\big|\big|\big|^2 dt + \frac{1}{2}\big\|v_0^+\big\|_0^2 + \frac{1}{2} \sum_{n=1}^{N-1} \big\|[v]_n\big\|_0^2 + \frac{1}{2}\big\|v_N^-\big\|_0^2 \right)^{1/2}.$$

The following theorem provides the a priori error estimates for the fully discrete problem (4.2) obtained by discontinuous Galerkin time stepping scheme [26].

THEOREM 4.2. *Suppose that the spaces V_h, Q_h satisfy A1, A2 and $\mu_K \sim 1$. Let (u,p) be the solution of (2.2) and (U,P) the solution of (4.2). Assume that $u \in H^{k+1}(J; H_0^1(\Omega)^d) \cap H^1(J; H^{r+1}(\Omega)^d)$, $p \in H^{k+1}(J; H_0^1(\Omega)) \cap C(J; H^r(\Omega))$ and $u_0 \in H_0^1(\Omega)^d \cap H^{r+1}(\Omega)^d$. Then, there exists a positive constant C independent of ν, h and τ such that the error estimate*

$$\big\|(u-U, p-P)\big\|_{\mathrm{dG}} \le C\Big(\tau^{k+1/2} + h^r\Big)$$

holds true.

5. Numerical experiments for the Oseen equations. Numerical results for two-dimensional transient Oseen equations are presented in this section. The main objective here is to examine the accuracy of the two different time discretization schemes: continuous Galerkin-Petrov and discontinuous Galerkin time stepping methods. For space discretization, the one-level local projection stabilization method is used. All numerical calculations were performed with the finite element package MooNMD [24].

Mapped finite element spaces were used in the numerical computations where on the reference cell $\widehat{K}$ the enriched spaces are given by

$$Q_r^{\text{bubble}}(\widehat{K}) = Q_r(\widehat{K}) + \operatorname{span}\{\hat{b}x_i^{r-1}, \quad i = 1, 2\}.$$

Here $\hat{b}$ denote the biquadratic bubble on the reference square. The triple $(V_h, Q_h, \mathcal{D}_h) = (Q_3^{\text{bubble}}, P_3^{\text{disc}}, P_3^{\text{disc}})$ fulfills the assumptions A2 and A3. The stabilization parameters μ_K have been chosen as

$$\mu_K = \mu_0 \qquad \forall K \in \mathcal{T}_h$$

where μ_0 denotes a constant which will be given latter.

We consider the problem (2.1) in two-dimensional domain $\Omega = (0, 1)^2$ with non-homogeneous boundary conditions. The right-hand side f and boundary and initial conditions are chosen in order to ensure that the exact solution of (2.1) is given by

$$\begin{aligned}
u_1(t, x, y) &= \cos(\pi x) \sin(\pi y) \exp(-2\pi^2 t) \\
u_2(t, x, y) &= \sin(\pi x) \cos(\pi y) \exp(-2\pi^2 t) \\
p(t, x, y) &= -\frac{1}{4}\Big(\cos(2\pi x) + \cos(2\pi y)\Big) \exp(-4\pi^2 t)
\end{aligned}$$

This example is taken from [25]. We set $b = u$, $\sigma = 1$, $T = 1$ in (2.1). The simulations were performed with $\nu = 10^{-10}$, i.e., in the convection-dominated regime. Uniform quadrilateral grids were used with the coarsest grid (level 0) consisting of a single quadrilateral.

In order to illustrate the convergence rate in time, we have chosen the inf-sup stable finite element pair $(V_h, Q_h, \mathcal{D}_h) = (Q_3^{\text{bubble}}, P_3^{\text{disc}}, P_3^{\text{disc}})$ on a relatively fine mesh consisting of 1024 cells (level 6). The coefficient in the stabilization parameter is set to $\mu_0 = 0.1$.

We apply the time discretization schemes cGP(k) and dG(k) with an equidistant time step size $\tau = T/N$. In the following, we evaluate the results of our calculations by considering the following norms

$$\begin{aligned}
\|e_u\| &:= \left(\int_0^T \|e_u\|_{V_h}^2 dt\right)^{1/2}, \qquad \|e_p\| := \left(\int_0^T \|e_p\|_{Q_h}^2 dt\right)^{1/2}, \\
\|e_u\|_\infty &:= \max_{1\le n\le N} \|e_u(t_n)\|_{V_h}, \qquad \|e_p\|_\infty := \max_{1\le n\le N} \|e_p(t_n)\|_{Q_h}.
\end{aligned}$$

The behavior of the integral based norms of velocity $\|e_u\| := \|u(t) - U\|$ and pressure $\|e_p\| := \|p(t) - P\|$ for the time discretization schemes cGP(k) and dG(k), $k \in \{1, 2\}$, over the whole time interval $J = [0, 1]$ can be seen in Tables 5.1 and 5.2, respectively. The errors and convergence orders for the cGP-norm $\|(\cdot,\cdot)\|_{\text{cGP}}$ and dG-norm $\|(\cdot,\cdot)\|_{\text{dG}}$ are listed in Table 5.3. We see that the predicted orders in the integral based norms are confirmed, see Theorems 4.1 and 4.2.

TABLE 5.1
Errors and convergence orders in the integral based norms of the velocity and pressure.

τ	cGP(1) $\|e_u\|$	order	$\|e_p\|$	order	cGP(2) $\|e_u\|$	order	$\|e_p\|$	order
1/10	5.893-2		8.345-2		4.210-3		1.412-2	
1/20	1.571-2	1.91	2.395-2	1.80	5.936-4	2.83	2.385-3	2.57
1/40	3.993-3	1.98	6.178-3	1.96	7.683-5	2.95	3.465-4	2.78
1/80	1.002-3	1.99	1.551-3	1.99	9.691-6	2.99	4.667-5	2.89
1/160	2.508-4	2.00	3.881-4	2.00	1.214-6	3.00	6.056-6	2.95
1/320	6.272-5	2.00	9.701-5	2.00	1.518-7	3.00	7.715-7	2.97
1/640	1.568-5	2.00	2.425-5	2.00	1.899-8	3.00	9.777-8	2.98
theoret. order:		2		2		3		3

TABLE 5.2
Errors and convergence orders in the integral based norms of the velocity and pressure.

τ	dG(1) $\|e_u\|$	order	$\|e_p\|$	order	dG(2) $\|e_u\|$	order	$\|e_p\|$	order
1/10	2.110-2		5.416-2		3.243-3		1.000-2	
1/20	6.111-3	1.79	1.663-2	1.70	4.849-4	2.74	1.603-3	2.64
1/40	1.612-3	1.92	4.474-3	1.89	6.416-5	2.92	2.195-4	2.87
1/80	4.109-4	1.97	1.147-3	1.96	8.165-6	2.94	2.824-5	2.96
1/160	1.036-4	1.99	2.894-4	1.99	1.027-6	2.99	3.561-6	2.99
1/320	2.598-5	2.00	7.261-5	2.00	1.287-7	3.00	4.466-7	3.00
1/640	6.505-6	2.00	1.818-5	2.00				
theoret. order:		2		2		3		3

TABLE 5.3
Errors and convergence orders for cGP-norm (Theorem 4.1) and dG-norm (Theorem 4.2).

τ	cGP(1) $\|\|(e_u, e_p)\|\|$	order	cGP(2) $\|\|(e_u, e_p)\|\|$	order	dG(1) $\|\|(e_u, e_p)\|\|$	order	dG(2) $\|\|(e_u, e_p)\|\|$	order
1/10	1.102		2.641-1		1.528-1		2.926-2	
1/20	6.082-1	0.86	7.628-2	1.79	5.877-2	1.39	5.762-3	2.35
1/40	3.132-1	0.96	1.987-2	1.94	2.092-2	1.49	1.032-3	2.48
1/80	1.578-1	0.99	5.022-3	1.99	7.339-3	1.51	1.811-4	2.51
1/160	7.906-2	1.00	1.259-3	2.00	2.577-3	1.51	3.179-5	2.51
1/320	3.955-2	1.00	3.149-4	2.00	9.071-4	1.51	5.596-6	2.51
1/640	1.978-2	1.00	7.875-5	2.00	3.200-4	1.50		
theoret. order:		1		2		1.5		2.5

The errors and convergence orders for velocity and pressure in the discrete time points are presented in Tables 5.4 and 5.5. Comparing the absolute values of the error norms in Tables 5.4 and 5.5, we see that the super-convergence of order $2k$ for cGP(k)-method and of $2k+1$ for dG(k)-method are achieved. In [16], it was shown numerically that the cGP(k)-method is super-convergent of order $2k$ for transient Stokes problem, in [14, 15] for the heat equation and in [18] for Burgers equations. For dG(k)-method, it was proved in [26] that the dG-method is super-convergence of order $2k+1$ in the discrete time points for an abstract symmetric model problems like heat equations and in [16] for transient Stokes problem.

In the following, we consider how the results depend on the stabilization parameters μ_0. We consider the calculations which were carried out for cGP(1), cGP(2) and dG(1)-methods and the finite element triple $(V_h, Q_h, \mathcal{D}_h) = (Q_2^{\text{bubble}}, P_1^{\text{disc}}, P_1^{\text{disc}})$. The computations were carried out on level 3 (consisting of 266 degrees of freedom for pressure and both components of velocity), level 4 (978 degrees of freedom), level

Table 5.4
Errors and convergence orders in the discrete time points for the velocity and pressure.

	cGP(1)				cGP(2)			
τ	$\|e_u\|_\infty$	order	$\|e_p\|_\infty$	order	$\|e_u\|_\infty$	order	$\|e_p\|_\infty$	order
1/10	3.233-2		1.650-4		4.881-4		9.740-6	
1/20	8.200-3	1.98	4.729-5	1.80	3.777-5	3.69	7.324-7	3.73
1/40	2.050-3	2.00	1.188-5	1.99	2.457-6	3.94	5.310-8	3.79
1/80	5.124-4	2.00	2.916-6	2.00	1.543-7	3.99	1.882-9	4.82
1/160	1.281-4	2.00	7.234-7	2.00	9.653-9	4.00	3.99-10	2.34
1/320	3.202-5	2.00	1.805-7	2.00	6.086-10	3.99		
1/640	8.005-6	2.00	4.510-8	2.00	1.092-10	2.49		
theoret. order:		2		2		4		4

Table 5.5
Errors and convergence orders in the discrete time points for the velocity and pressure.

	dG(1)				dG(2)			
τ	$\|e_u\|_\infty$	order	$\|e_p\|_\infty$	order	$\|e_u\|_\infty$	order	$\|e_p\|_\infty$	order
1/10	2.793-3		4.055-5		1.604-4		2.257-6	
1/20	4.220-4	2.73	4.493-6	3.17	9.337-6	4.10	2.396-7	3.24
1/40	5.629-5	2.91	4.865-7	3.21	3.469-7	4.75	9.150-9	4.71
1/80	7.213-6	2.96	6.320-8	2.95	1.071-8	5.02	2.9743-10	4.94
1/160	9.094-7	2.99	8.483-9	2.90	3.266-10	5.04	9.195-12	5.02
1/320	1.140-7	3.00	1.106-9	2.94				
1/640	1.427-8	3.00	1.408-10	2.97				
theoret. order:		3		3		5		5

5 (3, 746 degrees of freedom), and level 6 (14, 658 degrees of freedom) and the time step length is set to be $\tau = 1/160$. Since the stabilization parameter should be chosen proportional to h_K, we have performed the calculations with $\mu_K = \mu_0 h_K$ where the constant varies from 10^{-6} to 10^4.

The graphs in Fig. 5.1 show the computed results on different levels for cGP(1)-method, in Fig. 5.2 for cGP(2)-method and in Fig. 5.3 for dG(1)-method. The graphs in Figs.5.1-5.3 indicate that the velocity error is much sensitive with respect to the stabilization parameter μ_0. If the stabilization parameter tends to zero in these calculations, then the errors in the integrated L^2-norm of velocity increase while the errors in the integrated triple and pressure norm remains bounded. Similarly if the stabilization parameter tends to infinity, then the errors in all integrated norms also increases. Note that the L^2-norm of pressure is also included in the triple norm. The behavior of the integrated pressure norm on level 5 and 6 for cGP(1) and dG(1)-methods is due to the influence of the error in time. Furthermore, for each norm the dependence of the error on the parameter μ_0 is very similar on different refinement levels. A suitable value for μ_0 lies in the range between 0.01 and 0.1.

Acknowledgments. The authors like to thank the German research foundation (DFG) for supporting the research under the grant MA 4713/2-1.

REFERENCES

[1] C. Johnson, J. Saranen, Streamline diffusion methods for the incompressible Euler and Navier-Stokes equations, Math. Comp. 47 (175) (1986) 1–18.

[2] T. J. R. Hughes, L. P. Franca, M. Balestra, A new finite element formulation for computational fluid dynamics. V. Circumventing the Babuška-Brezzi condition: a stable Petrov-Galerkin

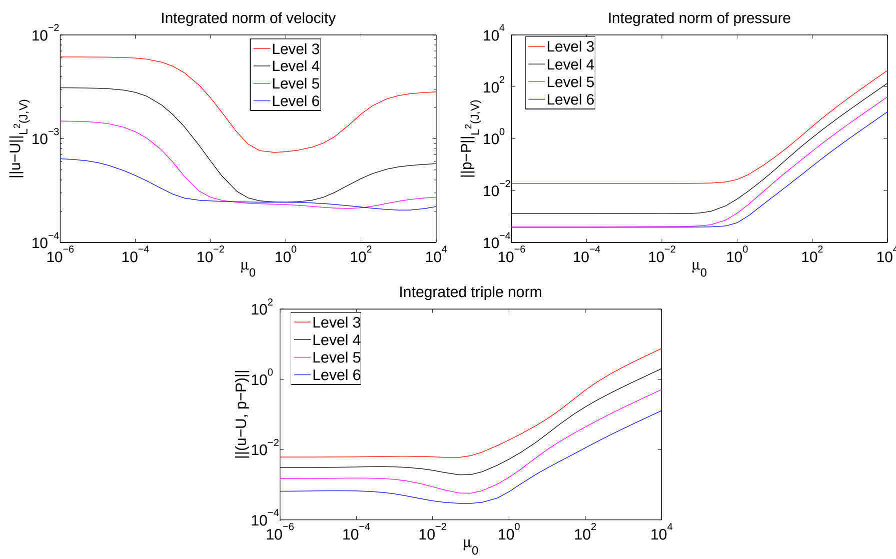

FIG. 5.1. *Dependence of norm on the stabilization parameter* μ_0 *for* $Q_2^{\text{bubble}}/P_1^{\text{disc}}$ *and cGP(1).*

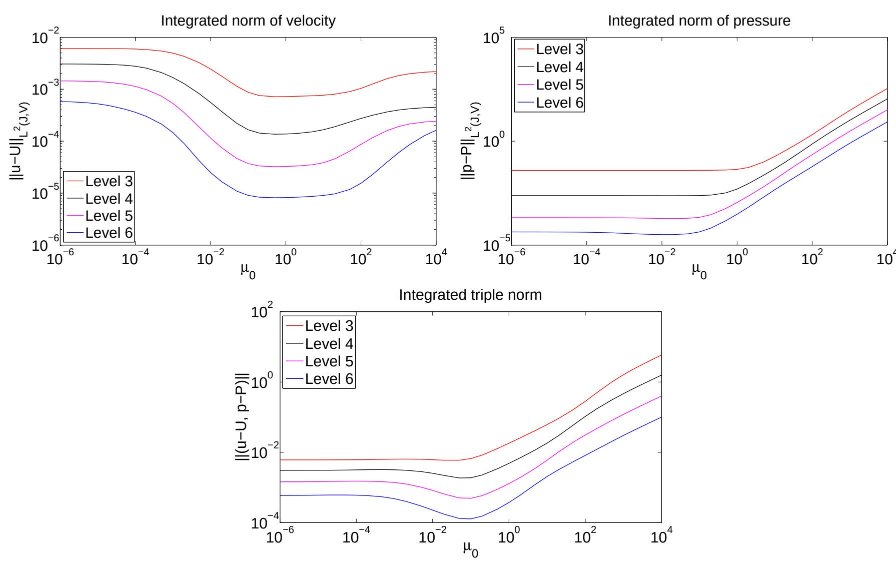

FIG. 5.2. *Dependence of norm on the stabilization parameter* μ_0 *for* $Q_2^{\text{bubble}}/P_1^{\text{disc}}$ *and cGP(2).*

formulation of the Stokes problem accommodating equal-order interpolations, Comput. Methods Appl. Mech. Engrg. 59 (1) (1986) 85–99.

[3] G. Matthies, G. Lube, and L. Röhe, Some remarks on residual-based stabilisation of inf-sup stable discretisations of the generalised Oseen problem, Comput. Methods Appl. Mech. Engrg. 9 (4) (2009) 368–390.

[4] T. Gelhard, G. Lube, M. A. Olshanskii, J.-H. Starcke, Stabilized finite element schemes with LBB-stable elements for incompressible flows, J. Comput. Appl. Math. 177 (2) (2005) 243–267.

[5] A. N. Brooks, T. J. R. Hughes, Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations,

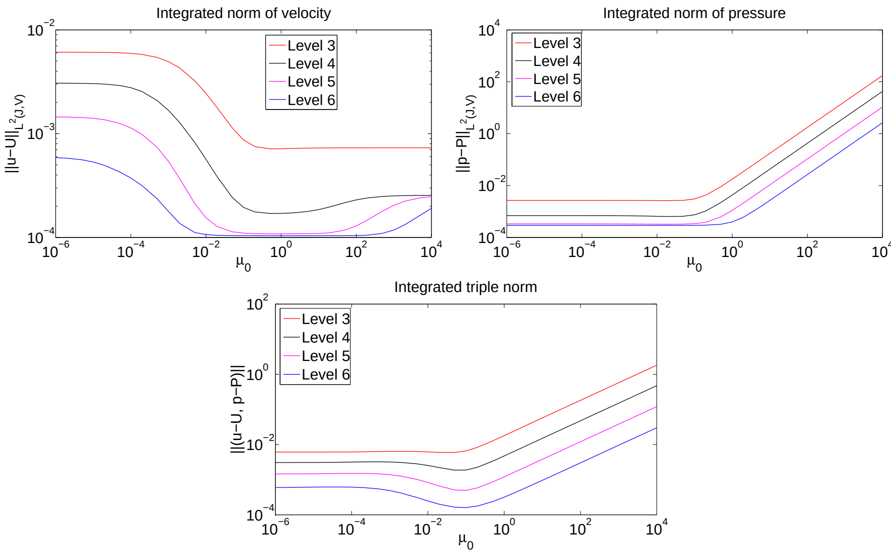

FIG. 5.3. *Dependence of norm on the stabilization parameter μ_0 for $Q_2^{\mathrm{bubble}}/P_1^{\mathrm{disc}}$ and dG(1).*

Comput. Methods Appl. Mech. Engrg. 32 (1-3) (1982) 199–259, fENOMECH '81, Part I (Stuttgart, 1981).

[6] P. Bochev, M. Gunzburger, An absolutely stable pressure-Poisson stabilized finite element method for the Stokes equations, SIAM J. Numer. Anal. 42 (3) (2004) 1189–1207 (electronic).

[7] P. B. Bochev, M. D. Gunzburger, R. B. Lehoucq, On stabilized finite element methods for the Stokes problem in the small time step limit, Internat. J. Numer. Methods Fluids 53 (4) (2007) 573–597.

[8] P. B. Bochev, M. D. Gunzburger, J. N. Shadid, On inf-sup stabilized finite element methods for transient problems, Comput. Methods Appl. Mech. Engrg. 193 (15-16) (2004) 1471–1489.

[9] E. Burman, P. Hansbo, Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems, Comput. Methods Appl. Mech. Engrg. 193 (15-16) (2004) 1437–1453.

[10] M. Braack, E. Burman, V. John, and G. Lube, Stabilized finite element methods for the generalized Oseen problem, Comput. Methods Appl. Mech. Engrg. 196 (4-6) (2007) 853–866.

[11] M. Braack, E. Burman, Local projection stabilization for the Oseen problem and its interpretation as a variational multiscale method, SIAM J. Numer. Anal. 43 (6) (2006) 2544–2566.

[12] G. Matthies, P. Skrzypacz, L. Tobiska, A unified convergence analysis for local projection stabilisations applied to the Oseen problem, M2AN Math. Model. Numer. Anal. 41 (4) (2007) 713–742.

[13] G. Matthies, L. Tobiska, Local projection type stabilization applied to inf-sup stable discretisations of the oseen problem, Preprint 47/2007, Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg (2007).

[14] A. K. Aziz, P. Monk, Continuous finite elements in space and time for the heat equation, Math. Comp. 52 (186) (1989) 255–274.

[15] S. Hussain, F. Schieweck, S. Turek, Higher order Galerkin time discretizations and fast multigrid solvers for the heat equation, J. Numer. Math 19 (1) (2011) 41–61.

[16] S. Hussain, F. Schieweck, S. Turek, A note on accurate and efficient higher order Galerkin time stepping schemes for the nonstationary Stokes equations, The Open Numerical Methods Journal, Accepted (2011).

[17] F. Schieweck, *A*-stable discontinuous Galerkin-Petrov time discretization of higher order, J. Numer. Math. 18 (1) (2010) 25–57.

[18] G. Matthies, F. Schieweck, Higher order variational time discretizations for nonlinear systems of ordinary differential equations, Preprint 23/2011, Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg (2011).

[19] E. Hairer, G. Wanner, Solving ordinary differential equations. II, 2nd Edition, Vol. 14 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 1996, stiff and differential-algebraic problems.

[20] M. Feistauer, J. Hájek, K. Svadlenka, Space-time discontinuous Galerkin method for solving nonstationary convection-diffusion-reaction problems, Appl. Math. 52 (3) (2007) 197–233.

[21] G. Lube, G. Rapin, J. Löwe, Local projection stabilization for incompressible flows: equal-order vs. inf-sup stable interpolation, Electron. Trans. Numer. Anal. 32 (2008) 106–122.

[22] P. Knobloch, On the application of local projection methods to convection-diffusion-reaction problems, in: BAIL 2008—boundary and interior layers, Vol. 69 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 2009, pp. 183–194.

[23] P. Knobloch, A generalization of the local projection stabilization for convection-diffusion-reaction equations, SIAM J. Numer. Anal. 48 (2) (2010) 659–680.

[24] V. John, G. Matthies, MooNMD—a program package based on mapped finite element methods, Comput. Vis. Sci. 6 (2-3) (2004) 163–169.

[25] V. John, W. J. Layton, Analysis of numerical errors in large eddy simulation, SIAM J. Numer. Anal. 40 (3) (2002) 995–1020.

[26] V. Thomée, Galerkin finite element methods for parabolic problems, 2nd Edition, Vol. 25 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2006.