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ON THE UNIFORM STABILITY OF THE SPACE-TIME DISCONTINUOUS GALERKIN METHOD FOR NONSTATIONARY PROBLEMS IN TIME-DEPENDENT DOMAINS

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Abstract. In this paper we investigate the stability of the space-time discontinuous Galerkin method (STDGM) for the solution of nonstationary, linear convection-diffusion-reaction problem in time-dependent domains formulated with the aid of the arbitrary Lagrangian-Eulerian (ALE) method. The stability is uniform with respect to the diffusion coefficient. The ALE method replaces the classical partial time derivative with the so called ALE-derivative and an additional convective term. In the second part of the paper we discretize our problem using the space-time discontinuous Galerkin method. In the formulation of the numerical scheme we use the nonsymmetric, symmetric and incomplete versions of the space discretization of diffusion terms and interior and boundary penalty. The space discretization uses piecewise polynomial approximations of degree $p \geq 1$, in time we use only piecewise linear discretization. Finally in the third part of the paper we present our results concerning the uniform unconditional stability of the method.

Key words. linear convection-diffusion-reaction problem, time-dependent domain, ALE method, space-time discontinuous Galerkin method, uniform unconditional stability

AMS subject classifications. 65M60, 65M99

1. Formulation of the continuous problem.

We consider an initial-boundary value nonstationary, linear convection-diffusion-reaction problem in a time-dependent bounded, polygonal domain $\Omega_t \subset \mathbb{R}^2$:

Find a function $u = u(x,t)$ with $x \in \Omega_t,\ t \in (0,T)$ such that

$$\frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u - \epsilon \triangle u + cu = g \quad \text{in} \quad \Omega_t,\, t \in (0,T), \tag{1.1}$$

$$u = u_D \quad \text{on} \quad \partial\Omega_t,\, t \in (0,T), \tag{1.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{1.3}$$

We assume that $\boldsymbol{v} = (v_1, v_2),\ c,\ g,\ u_D,\ u^0$ are given functions and $\epsilon > 0$ is a given constant. Moreover let $Q_T = \{(x,t);\ t \in (0,T),\ x \in \Omega_t\}$, and let us assume, that there exist constants $c_v,\ c_c > 0$, such that

$$\begin{aligned} &v \in C([0,T];\ W^{1,\infty}(\Omega_t)),\ |\nabla \boldsymbol{v}| \leq c_v,\ |\boldsymbol{v}| \leq c_v \quad \text{in} \quad Q_T, \\ &c \in C([0,T], L^\infty(\Omega_t)),\ |c(x,t)| \leq c_c \quad \text{in} \quad Q_T. \end{aligned}$$

Problem (1.1)–(1.3) will be reformulated using the so called arbitrary Lagrangian-Eulerian (ALE) method (see, e.g., [7]). It is based on a regular one-to-one ALE mapping of the reference domain Ω_0 onto the current configuration Ω_t:

$$\begin{aligned} &\mathcal{A}_t : \overline{\Omega}_0 \to \overline{\Omega}_t, \\ &X \in \overline{\Omega}_0 \to x = x(X,t) = \mathcal{A}_t(X) \in \overline{\Omega}_t, \quad t \in [0,T]. \end{aligned}$$

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We assume that $\mathcal{A}_t \in C^1([0,T]; W^{1,\infty}(\Omega_t))$, i.e. the mapping $\mathcal{A}_t$ belongs to the Bochner space of continuously differentiable functions in $[0,T]$ with values in the Sobolev space $W^{1,\infty}(\Omega_t)$. We define the ALE velocity by

$$\tilde{\boldsymbol{z}}(X,t) = \frac{\partial}{\partial t}\mathcal{A}_t(X), \quad t \in [0,T],\ X \in \Omega_0,$$
$$\boldsymbol{z}(x,t) = \tilde{\boldsymbol{z}}(\mathcal{A}_t^{-1}(x),t), \quad t \in [0,T],\ x \in \Omega_t.$$

Let $|\boldsymbol{z}(x,t)|,\ |\operatorname{div}\boldsymbol{z}(x,t)| \le c_z$ for $x \in \Omega_t,\ t \in (0,T)$. Further, we define the ALE derivative $D_t f = Df/Dt$ of a function $f = f(x,t)$ for $x \in \Omega_t$ and $t \in [0,T]$ as

$$D_t f(x,t) = \frac{D}{Dt} f(x,t) = \frac{\partial \tilde{f}}{\partial t}(X,t),$$

where $\tilde{f}(X,t) = f(\mathcal{A}_t(X),t)$, $X \in \Omega_0$, and $x = \mathcal{A}_t(X) \in \Omega_t$. The use of the chain rule yields the relation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{z} \cdot \nabla f, \tag{1.4}$$

which allows us to reformulate problem (1.1)–(1.3) in the ALE form:

Find $u = u(x,t)$ with $x \in \Omega_t,\ t \in (0,T)$ such that

$$D_t u + (\boldsymbol{v} - \boldsymbol{z}) \cdot \nabla u - \epsilon \triangle u + cu = g \quad \text{in} \quad \Omega_t,\ t \in (0,T), \tag{1.5}$$
$$u = u_D \quad \text{on} \quad \partial\Omega_t, \tag{1.6}$$
$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{1.7}$$

In what follows, we shall use the notation $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{z}$ for the ALE transport velocity.

In the case, when problem (1.1)–(1.3) is considered in a domain Ω independent of time, the space-time discontinuous Galerkin discretization was used and error estimates were derived in [6]. These results were generalized to the case of nonlinear convection and diffusion (cf. [4]). The paper [2] is devoted to the proof of unconditional stability of the space-time discontinuous Galerkin method (STDGM) applied to the nonlinear convection-diffusion problem in a fixed domain. The solution of initial-boundary value problems in time-dependent domains plays an important role, particularly in fluid-structure interaction. In [3], the stability of the time discontinuous Galerkin semi-discretization of problem (1.5)–(1.7) was analyzed. The work [1] deals with the stability of the complete space-time discontinuous Galerkin method for the nonlinear convection-diffusion problem in time-dependent domains. The stability was proved in the discrete $L^\infty(L^2)$-norm and the DG-H^1-norm, which is bounded by a constant multiplied by an expression containing norms of data. Because of the nonlinearity of the problem, the constant in the stability estimate blows up exponentially in dependence on the $1/\beta_0$, where β_0 is the lower bound of the diffusion coefficient. Here we are concerned with the investigation of the uniform stability independent of the arbitrarily small diffusion coefficient $\epsilon > 0$ of the complete STDGM applied to problem (1.5)–(1.7) in a time-dependent domain.

2. Derivation of the discrete problem. In the time interval $[0,T]$ we construct a partition formed by time instants $0 = t_0 < t_1 < \ldots < t_M = T$ and set $I_m = (t_{m-1}, t_m)$ and $\tau_m = t_m - t_{m-1}$ for $m = 1, \ldots, M$. Further we set

$\tau = \max_{m=1,\cdots,M} \tau_m$. For a function φ defined in $\bigcup_{m=1}^{M} I_m$ we denote one-sided limits at t_m as $\varphi_m^{\pm} = \varphi(t_m\pm) = \lim_{t\to t_m\pm} \varphi(t)$ and the jump as $\{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-)$.

For any $t \in [0,T]$ we denote by $\mathcal{T}_{h,t}$ a partition of the closure $\overline{\Omega}_t$ into a finite number of closed triangles with mutually disjoint interiors. We set $h_K = \operatorname{diam}(K)$ and denote by ρ_K the radius of the largest circle inscribed into $K \in \mathcal{T}_{h,t}$.

The boundary of the domain will be divided into to parts: $\partial\Omega_t = \partial\Omega_t^- \cup \partial\Omega_t^+$:

$$\begin{aligned} &\boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x) < 0 \text{ on } \partial\Omega_t^-, \forall t\in[0,T] \text{ (inflow boundary)}\\ &\boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x) \geq 0 \text{ on } \partial\Omega_t^+, \forall t\in[0,T] \text{ (outflow boundary),}\end{aligned}$$

where $\boldsymbol{n}$ denotes the unit outer normal to $\partial\Omega$. Similarly for each $K \in \mathcal{T}_{h,t}$ we set

$$\begin{aligned} \partial K^-(t) &= \{x\in\partial K;\, \boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x) < 0\},\\ \partial K^+(t) &= \{x\in\partial K;\, \boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x) \geq 0\}.\end{aligned}$$

Here $\boldsymbol{n}$ denotes the unit outer normal to ∂K.

By $\mathcal{F}_{h,t}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{h,t}$. It consists of the set of all inner faces $\mathcal{F}_{h,t}^I$ and the set of all boundary faces $\mathcal{F}_{h,t}^B$: $\mathcal{F}_{h,t} = \mathcal{F}_{h,t}^I \cup \mathcal{F}_{h,t}^B$. Each $\Gamma \in \mathcal{F}_{h,t}$ will be associated with a unit normal vector $\boldsymbol{n}_\Gamma$. By $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)} \in \mathcal{T}_{h,t}$ we denote the elements adjacent to the face $\Gamma \in \mathcal{F}_{h,t}$. We shall use the convention that $\boldsymbol{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$. Over a triangulation $\mathcal{T}_{h,t}$, for each positive integer k, we define the broken Sobolev space

$$H^k(\Omega_t, \mathcal{T}_{h,t}) = \{\varphi; \varphi|_K \in H^k(K) \quad \forall K \in \mathcal{T}_{h,t}\}.$$

If $\varphi \in H^1(\Omega_t, \mathcal{T}_{h,t})$ and $\Gamma \in \mathcal{F}_{h,t}$, then $\varphi|_\Gamma^{(L)}, \varphi|_\Gamma^{(R)}$ will denote the traces of φ on Γ from the side of elements $K_\Gamma^{(L)}, K_\Gamma^{(R)}$ adjacent to Γ. For $\Gamma \in \mathcal{F}_{h,t}^I$ we set

$$\begin{aligned} &\langle\varphi\rangle_\Gamma = \frac{1}{2}\left(\varphi|_\Gamma^{(L)} + \varphi|_\Gamma^{(R)}\right), \quad [\varphi]_\Gamma = \varphi|_\Gamma^{(L)} - \varphi|_\Gamma^{(R)},\\ &h(\Gamma) = \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2} \quad \text{for } \Gamma \in \mathcal{F}_{h,t}^I, \quad h(\Gamma) = h_{K_\Gamma^{(L)}} \quad \text{for } \Gamma \in \mathcal{F}_{h,t}^B.\end{aligned}$$

If $u, \varphi \in H^2(\Omega_t, \mathcal{T}_{h,t})$, $\theta \in \mathbb{R}$ and $c_W > 0$, we introduce the following forms (let us note that in integrals over faces we omit the subscript Γ).
Diffusion form:

$$\begin{aligned} a_h(u,\varphi,t) = &\sum_{K\in\mathcal{T}_{h,t}} \int_K \nabla u\cdot\nabla\varphi\,dx\\ &- \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \left(\langle\nabla u\rangle\cdot\boldsymbol{n}\,[\varphi] + \theta\,\langle\nabla\varphi\rangle\cdot\boldsymbol{n}\,[u]\right)dS\\ &- \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K^-\cap\partial\Omega_t} \left(\nabla u\cdot\boldsymbol{n}\,\varphi + \theta\nabla\varphi\cdot\boldsymbol{n}\,u\right)dS, \end{aligned} \tag{2.1}$$

Interior and boundary penalty:

$$J_h(u,\varphi,t) = c_W \sum_{\Gamma\in\mathcal{F}_{h,t}^I} h(\Gamma)^{-1} \int_\Gamma [u]\,[\varphi]\,dS \tag{2.2}$$

$$+c_W \sum_{K\in\mathcal{T}_{h,t}} h(\Gamma)^{-1} \int_{\partial K^-\cap\partial\Omega_t} u\,\varphi\,dS, \tag{2.3}$$

Complete diffusion form:

$$A_h(u,\varphi,t) = \epsilon a_h(u,\varphi,t) + \epsilon\, J_h(u,\varphi,t), \tag{2.4}$$

Reaction form:

$$c_h(u,\varphi,t) = \sum_{K\in\mathcal{T}_{h,t}} \int_K cu\varphi\, dx. \tag{2.5}$$

We consider $\theta = 1$, $\theta = 0$ and $\theta = -1$ and get the symmetric (SIPG), incomplete (IIPG) and nonsymmetric (NIPG) variants of the approximation of the diffusion terms, respectively.

It is important to discretize the expression $\int_K (\boldsymbol{w}\cdot\nabla u)\varphi\, dx$ in a suitable way. Applying Green's theorem, we get

$$\begin{aligned}\int_K (\boldsymbol{w}\cdot\nabla u)\varphi\, dx &= -\int_K u\,\nabla\cdot(\varphi\boldsymbol{w})\, dx + \int_{\partial K} (\boldsymbol{w}\cdot\boldsymbol{n})u\varphi\, dS\\ &= -\int_K u\,\nabla\cdot(\varphi\boldsymbol{w})\, dx + \int_{\partial K^-} (\boldsymbol{w}\cdot\boldsymbol{n})u\varphi\, dS + \int_{\partial K^+} (\boldsymbol{w}\cdot\boldsymbol{n})u\varphi\, dS.\end{aligned} \tag{2.6}$$

Then we approximate the expression $\int_K (\boldsymbol{w}\cdot\nabla u)\varphi\, dx$ in the form

$$\int_K (\boldsymbol{w}\cdot\nabla u)\varphi\, dx \approx -\int_K u\,\nabla\cdot(\varphi\boldsymbol{w})\, dx + \sum_{\Gamma\subset\partial K}\int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)\varphi\, dS, \tag{2.7}$$

where H is the numerical flux defined with the aid of upwinding:

$$H(u_1,u_2,\boldsymbol{n}) = \begin{cases} \boldsymbol{w}\cdot\boldsymbol{n}\; u_1, & \text{if } \boldsymbol{w}\cdot\boldsymbol{n} < 0,\\ \boldsymbol{w}\cdot\boldsymbol{n}\; u_2, & \text{if } \boldsymbol{w}\cdot\boldsymbol{n} \ge 0,\end{cases} \tag{2.8}$$

and $H(u_1,u_2,\boldsymbol{n}) = \boldsymbol{w}\cdot\boldsymbol{n}\, u_D$ on $\partial K^-\cap\partial\Omega_t$ (this numerical flux is Lipschitz continuous in u_1, u_2, consistent and conservative (see [5], Section 3.3.). We use the notation $[u] = u - u^-$, where by u^- we denote the value of u from outside of the element K on ∂K^-. Then substituting (2.8) into (2.7) we get the relation

$$\begin{aligned}&\int_K (\boldsymbol{w}\cdot\nabla u)\,\varphi dx\\ &= -\int_K u\nabla\cdot(\varphi\boldsymbol{w})\, dx + \int_{\partial K^-} (\boldsymbol{w}\cdot\boldsymbol{n})\, u^-\varphi dS + \int_{\partial K^+} (\boldsymbol{w}\cdot\boldsymbol{n})\, u\varphi dS\\ &= -\int_K u\nabla\cdot(\varphi\boldsymbol{w})\, dx + \int_{\partial K} (\boldsymbol{w}\cdot\boldsymbol{n})\, u\varphi dS - \int_{\partial K^+\cup\partial K^-} (\boldsymbol{w}\cdot\boldsymbol{n})\, u\varphi dS\\ &\quad + \int_{\partial K^-} (\boldsymbol{w}\cdot\boldsymbol{n})\, u^-\varphi dS + \int_{\partial K^+} (\boldsymbol{w}\cdot\boldsymbol{n})\, u\varphi dS.\end{aligned} \tag{2.9}$$

Applying the Green theorem to the first term on the right-hand side of (2.9), we find that

$$(2.10)\quad \int_K (\boldsymbol{w}\cdot\nabla u)\,\varphi dx$$

$$= \int_K (\boldsymbol{w}\cdot\nabla u)\,\varphi dx + \int_{\partial K^-} (\boldsymbol{w}\cdot\boldsymbol{n})\left(u^- - u\right)\varphi\, dS$$

$$= \int_K (\boldsymbol{w}\cdot\nabla u)\,\varphi dx - \int_{\partial K^-\setminus\partial\Omega_t} (\boldsymbol{w}\cdot\boldsymbol{n})\,[u]\,\varphi\, dS - \int_{\partial K^-\cap\partial\Omega_t} (\boldsymbol{w}\cdot\boldsymbol{n})\,(u-u_D)\,\varphi\, dS.$$

On the basis of this relation, we define the convection form

$$(2.11)\qquad b_h(u,\varphi,t) = \sum_{K\in\mathcal{T}_{h,t}} \int_K \boldsymbol{w}\cdot\nabla u\,\varphi\, dx - \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K^-\cap\partial\Omega_t} (\boldsymbol{w}\cdot\boldsymbol{n}) u\varphi\, dS - \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K^-\setminus\partial\Omega_t} (\boldsymbol{w}\cdot\boldsymbol{n})[u]\varphi\, dS.$$

and the right-hand side form

$$(2.12)\quad l_h(\varphi,t) = \sum_{K\in\mathcal{T}_{h,t}} \int_K g\varphi\, dx + \epsilon\, c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma u_D\,\varphi\, dS - \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K^-\cap\partial\Omega_t} \theta\nabla\varphi\cdot\boldsymbol{n}\, u_D\, dS - \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K^-\cap\partial\Omega_t} (\boldsymbol{w}\cdot\boldsymbol{n})\, u_D\varphi\, dS.$$

Further, we set

$$(\varphi,\psi)_\omega = \int_\omega \varphi\psi\, dx, \quad \|\varphi\|_\omega = \left(\int_\omega |\varphi|^2\, dx\right)^{1/2},$$
$$\|\eta\|_{\boldsymbol{w},\sigma} = \left\|\sqrt{|\boldsymbol{w}\cdot\boldsymbol{n}|}\,\eta\right\|_{L^2(\sigma)},$$

where $\omega\subset\mathbb{R}^2$, σ is either a subset of $\partial\Omega$ or ∂K and $\boldsymbol{n}$ denotes the corresponding outer unit normal to $\partial\Omega$ or ∂K, provided the integrals make sense.

Let $p,\ q\geq 1$ be integers. For any $m=1,\dots,M$ and $t\in[0,T]$ we define the finite-dimensional spaces

$$S^p_{h,t} = \left\{\varphi\in L^2(\Omega_t);\ \varphi|_K\in P^p(K),\ K\in\mathcal{T}_{h,t},\ t\in[0,T]\right\},$$
$$S^{p,q}_{h,\tau} = \Big\{\varphi\in L^2(Q_T);\ \varphi=\varphi(x,t),\ \text{for each } X\in\Omega_0 \text{ the function } \varphi(\mathcal{A}_t(X),t) \text{ is a polynomial of degree } \leq q \text{ in } t,\ \varphi(\cdot,t)\in S^p_{h,t} \text{ for every } t\in I_m,\ m=1,\dots,M\Big\}.$$

Definition 2.1. *We say that function U is an approximate solution of problem (1.5)–(1.7), if $U\in S^{p,q}_{h,\tau}$ and*

$$(2.13)\quad \int_{I_m} \left((D_tU,\varphi)_{\Omega_t} + A_h(U,\varphi,t) + b_h(U,\varphi,t) + c_h(U,\varphi,t)\right)\, dt + (\{U\}_{m-1},\varphi^+_{m-1})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi,t)\, dt \quad \forall\varphi\in S^{p,q}_{h,\tau}, \quad m=1,\dots,M,$$

$$(2.14)\quad U_0^-\in S^p_{h,0}, \quad (U_0^- - u^0, v_h) = 0 \quad \forall v_h\in S^p_{h,0}.$$

The discrete problem is constructed in such a way that it is consistent, which means that the exact solution $u \in H^2(\Omega)$ satisfies identity (2.13), when U is replaced by u.

3. Investigation of the stability. Our goal is to prove the uniform stability represented by an estimate of the approximate solution in suitable norms by data with constants independent of h, τ and the diffusion coefficient ϵ. Since this analysis is rather complicated, it is possible to give here only a brief description of results and of their proofs.

In our further considerations for each $t \in [0,T]$ we introduce a system of conforming triangulations $\{\mathcal{T}_{h,t}\}_{h\in(0,h_0)}$, where $h_0 > 0$. We assume that it is shape regular and locally quasiuniform. This means that there exist positive constants c_R and c_Q, independent of K, Γ, t and h such that for all $t \in [0,T]$ it holds

$$\frac{h_K}{\rho_K} \le c_R \quad \text{for all} \quad K \in \mathcal{T}_{h,t}, \tag{3.1}$$

$$h_{K_\Gamma^{(L)}} \le c_Q h_{K_\Gamma^{(R)}}, \quad h_{K_\Gamma^{(R)}} \le c_Q h_{K_\Gamma^{(L)}} \qquad \text{for all} \quad \Gamma \in \mathcal{F}_{h,t}^I. \tag{3.2}$$

Under these assumptions, the multiplicative trace inequality and the inverse inequality hold. Moreover, we assume that

$$\mathcal{T}_{h,t} = \{K_t = \mathcal{A}_t(K_0); K_0 \in \mathcal{T}_{h,0}\}.$$

This assumption is usually satisfied in practical computations, when the ALE mapping $\mathcal{A}_t$ is a continuous, piecewise affine mapping in $\overline{\Omega}_0$ for each $t \in [0,T]$.

In the space $H^1(\Omega, \mathcal{T}_{h,t})$ we define the norm

$$\|\varphi\|_{DG,t} = \left(\sum_{K\in\mathcal{T}_{h,t}} |\varphi|^2_{H^1(K)} + J_h(\varphi,\varphi,t) \right)^{1/2}.$$

Moreover, over $\partial\Omega$ we define the norm

$$\|u_D\|_{DGB,t} = \left(c_W \sum_{K\in\mathcal{T}_{h,t}} h^{-1}(\Gamma) \int_{\partial K^- \cap \partial\Omega_t} |u_D|^2 \, dS \right)^{1/2}.$$

If we use $\varphi := U$ as a test function in (2.13), we get the basic identity

$$\begin{aligned} &\int_{I_m} \big((D_t U, U)_{\Omega_t} + A_h(U,U,t) + b_h(U,U,t) + c_h(U,U,t) \big)\, dt \\ &+ (\{U\}_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(U,t)\, dt. \end{aligned} \tag{3.3}$$

Let us denote

$$\sigma(U) = \frac{1}{2} \sum_{K\in T_h} \left(\|U\|^2_{\boldsymbol{w},\partial K\cap\partial\Omega} + \|[U]\|^2_{\boldsymbol{w},\partial K^-\setminus\partial\Omega} \right). \tag{3.4}$$

For a sufficiently large constant c_W, whose lower bound is determined by the constants from the multiplicative trace inequality, inverse inequality and local quasiuniformity of the meshes, we can prove the coercivity of the diffusion and penalty terms:

$$\int_{I_m} A_h(U,U,t)\, dt \ge \frac{\epsilon}{2} \int_{I_m} \|U\|^2_{DG,t}\, dt. \tag{3.5}$$

Furthermore, for every $k_1 > 0$ the following inequalities for the convective term, reaction term and for the right-hand side form hold:

$$\int_{I_m} b_h(U,U,t)\,dt = \int_{I_m} \left(\sigma(U) - \frac{1}{2}\int_{\Omega_t} U^2 \nabla\cdot \boldsymbol{w}\,dx\right)\,dt, \tag{3.6}$$

$$\int_{I_m} |c_h(U,U,t)|\,dt \le c_c \int_{I_m} \|U\|^2_{\Omega_t}\,dt, \tag{3.7}$$

$$\int_{I_m} |l_h(U,t)|\,dt \le \frac{1}{2}\int_{I_m} \left(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}\right)\,dt + \epsilon k_1 \int_{I_m} \|u_D\|^2_{DGB,t}\,dt + \frac{\epsilon}{k_1}\int_{I_m} \|U\|^2_{DG,t}\,dt. \tag{3.8}$$

The proof of (3.7) follows immediately from the definition of the form c_h and the proof of (3.8) is a consequence of the definition of the form l_h and the Young inequality. However, the proof of (3.6) is rather complicated and technical and will be contained in a paper in progress.

In what follows, we are concerned with the derivation of inequalities based on estimating the expression $\int_{I_m}(D_tU,U)_{\Omega_t}\,dt$. By a simple manipulation we find that

$$\begin{aligned}&\int_{I_m} (D_tU,U)_{\Omega_t}\,dt + \left(\{U\}_{m-1}, U^+_{m-1}\right)\\ &\ge \frac{1}{2}\left(\|U^-_m\|^2_{\Omega_{t_m}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}}\right)\\ &\quad -\frac{1}{2}\int_{I_m} (U^2,\nabla\cdot \boldsymbol{z})_{\Omega_t}\,dt,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&\int_{I_m} (D_tU,U)_{\Omega_t}\,dt + (\{U\}_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}}\\ &\ge \frac{1}{2}(\|U^-_m\|^2_{\Omega_{t_m}} + \frac{1}{2}\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}) - (U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}}\\ &\quad -\frac{1}{2}\int_{I_m} (U^2,\nabla\cdot \boldsymbol{z})_{\Omega_t}\,dt.\end{aligned} \tag{3.10}$$

Taking into account that $\sigma(U)\ge 0$ and $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{z}$ and putting $k_1 = 4$, from (3.3) and (3.5)–(3.9), we get the relation

$$\begin{aligned}&\|U^-_m\|^2_{\Omega_{t_m}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} - \int_{I_m} (U^2,\nabla\cdot \boldsymbol{v})_{\Omega_t}\,dt\\ &\quad + \int_{I_m} (2c-1, U^2)_{\Omega_t} + \frac{\epsilon}{2}\int_{I_m} \|U\|^2_{DG,t}\,dt\\ &\le c_1 \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\,dt\end{aligned} \tag{3.11}$$

with a constant c_1 independent of data, h, τ and ϵ. First, let us assume that

$$2c(x,t) - \nabla\cdot \boldsymbol{v}(x,t) \ge 1, \quad x\in\Omega_t,\ t\in(0,T). \tag{3.12}$$

Then the summation of (3.11) over $m = 1, \dots, k \ \le M$ yields the estimate

$$\begin{aligned}(3.13) \qquad & \|U_k^-\|_{\Omega_{t_k}} + \frac{\epsilon}{2}\sum_{m=1}^{k}\int_{I_m}\|U\|^2_{DG,t}\,dt \\ & \le \|U_0^-\|^2_{\Omega_0} + c_1\sum_{m-1}^{k}\int_{I_m}\left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\,dt,\end{aligned}$$

which proves the stability.

If condition (3.12) is not valid, then the stability analysis is more complicated. In this case, instead of (3.11) we get the inequality

$$\begin{aligned}(3.14) \qquad & \|U_m^-\|^2_{\Omega_{t_m}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}} + \frac{\epsilon}{2}\int_{I_m}\|U\|^2_{DG,t}\,dt \\ & \le c_1\sum_{m-1}^{k}\int_{I_m}\left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\,dt + c_2\int_{I_m}\|U\|^2_{\Omega_t}\,dt,\end{aligned}$$

where the constants c_1 and c_2 are independent of h, τ, ϵ and of the data g, u_D.

It is necessary to estimate the term $\int_{I_m}\|U\|^2_{\Omega_t}\,dt$. It is rather technical and the proof has been carried out for $q = 1$, i.e., for piecewise linear time discretization. Then similarly as in [1] it is possible to show that there exist constants L_1 and M_1 such that

$$\begin{aligned}(3.15) \qquad \|U_{m-1}\|^2_{\Omega_{t_{m-1}}} + \|U_m\|^2_{\Omega_{t_m}} &\ge \frac{L_1}{\tau_m}\int_{I_m}\|U\|^2_{\Omega_t}\,dt, \\ \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} &\le \frac{M_1}{\tau_m}\int_{I_m}\|U\|^2_{\Omega_t}\,dt.\end{aligned}$$

This allows us to prove that there exists a constant $c^* > 0$ depending on c_2 and L_1 such that

$$(3.16) \quad \int_{I_m}\|U\|^2_{\Omega_t}dt \le \frac{2c_1}{L_1}\tau_m\int_{I_m}\left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\,dt + \frac{8M_1}{L_1^2}\tau_m\|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}}$$

holds, if $0 < \tau_m \le c^*$.

Now, by virtue of (3.14) and (3.16), the summation over $m = 1, \dots, k \ \le M$ and the application of the discrete Gronwall lemma we get the following result.

THEOREM 3.1. *Let $q = 1$ and $0 < \tau_m \le c^*$. Then there exists a constant $c_3 > 0$ independent of h, τ, ϵ such that*

$$\begin{aligned}(3.17) \quad & \|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^{m}\|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\epsilon}{2}\sum_{j=1}^{m}\int_{I_j}\|U\|^2_{DG,j}\,dt \\ & \le c_3\left(\|U_0^-\|^2_{\Omega_{t_0}} + c_1\left(1 + \frac{2c_2c^*}{L_1}\right)\sum_{j=1}^{m}\int_{I_j}\left(\|g\|^2_{\Omega_{t_j}} + \|u_D\|^2_{DGB,t}\right)\,dt\right), \\ & \qquad m = 1, \dots, M, \quad h \in (0, h_0).\end{aligned}$$

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