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RESIDUAL BASED ERROR ESTIMATES FOR THE SPACE-TIME DISCONTINUOUS GALERKIN METHOD APPLIED TO NONLINEAR HYPERBOLIC EQUATIONS

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Abstract. We present an adaptive numerical method for solving nonlinear hyperbolic equations. The method uses the space-time discontinuous Galerkin discretization, exploiting its high polynomial approximation degrees with respect to both space and time coordinates. We derive an residual-based a posteriori error estimator and propose an efficient strategy how to identify the parts of the computational error caused by the space and time discretization, respectively, as well as the errors arising from the linearization of the resultant algebraic system of equations. Further, an algorithm keeping all these three components of the computational error balanced is presented. The computational performance of the proposed method is demonstrated by numerical experiments.

Key words. nonlinear hyperbolic equation, space-time discontinuous Galerkin method, residual-based a posteriori error estimates

AMS subject classifications. 65M60, 65M70

1. Introduction.

Our main goal is to develop an efficient algorithm for solving unsteady nonlinear hyperbolic systems of partial differential equations. We employ the space discontinuous Galerkin method (DG), which is based on discontinuous piecewise polynomial aproximation. The space DG discretization leads to a system of stiff ordinary differential equation. We consider the time Discontinuous Galerkin method for discretizing this system. The resulting space-time discontinuous Galerkin method (STDG) employs discontinuous piecewise polynomial approximation with respect to both spatial and temporal variables. The approximate solution of the given system of partial differential equations is suffering from three types of errors:

- space error resulting from the space semi-discretization of the given system by the DG method
- time error resulting from the time DG discretization of the system of ordinary differential equations coming from the space DG method
- algebraic error (including rounding errors) resulting from the linearization and subsequent solution of the nonlinear algebraic system of equations arising from the STDG discretization.

All of these components of the total error need to be balanced in order to provide an efficient adaptive algorithm. In [5], we derived (rather heuristic) residual error estimators which are able to identify the *space*, *time* and *algebraic errors*. These estimates are based on the approximation of the errors in a dual norm similarly as in [6], where we dealt with steady nonlinear convection-diffusion problems. This approach is very fast and simple to implement since neither an additional problem is solved nor a finite element reconstruction is constructed. Based on these residual

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estimators we propose an algorithm, where the time and algebraic errors are controlled by the space error and therefore they do not essentially contribute to the total error. The resulting scheme enables local space and global time adaptation.

The content of this paper is the following. In Section 2 we recall the system of nonlinear hyperbolic equations and we shortly describe the space-time discretization and solution procedure. In Section 3 we present the residual estimators and the resulting adaptive algorithm. Section 4 contains a few numerical illustrations of the presented techniques.

2. Space-time DG discretization of the nonlinear hyperbolic equation. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a polygonal (polyhedral for $d = 3$) domain and $T > 0$. We set $Q_T = \Omega \times (0, T)$ and by $\partial\Omega$ we denote the boundary of Ω. The system of the nonlinear hyperbolic equations can be written in the following form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{i=1}^{d} \frac{\partial \boldsymbol{f}_i(\boldsymbol{w})}{\partial x_i} = 0 \quad \text{in } Q_T, \tag{2.1}$$

where $\boldsymbol{w} = \boldsymbol{w}(x,t) : Q_T \to \mathbb{R}^s$ is the unknown state vector and $\boldsymbol{f}_i : \mathbb{R}^s \to \mathbb{R}^s,\ i = 1, \dots, s$ represent the convective fluxes. The system (2.1) is equipped with the initial condition $\boldsymbol{w}(x, 0) = \boldsymbol{w}^0(x),\ x \in \Omega$ and suitable boundary conditions. We assume that the fluxes and boundary conditions are chosen such that the problem (2.1) has an unique solution, see [8].

2.1. Function spaces a discrete formulation of the problem. We use the standard notation for function spaces with usual norms and semi-norms (see, e. g., [13], [14]):

$L^2(M)$ denotes the Lebesgue space of square integrable functions over a set M, $H^k(M),\ k = 0, 1, \dots$ are the Sobolev spaces of functions with square integrable weak derivatives of order k over M. The bolted symbols $\boldsymbol{H}^k(M),\ k = 0, 1, \dots$ denote Sobolev spaces of vector-valued functions from M to $\mathbb{R}^s$. By $(\cdot,\cdot)_M$ we denote the L^2-scalar product over M.

Furthermore, $L^2(I; X)$ ($H^1(I; X)$) denotes the Bochner space of functions square integrable (square integrable first time derivative) over an interval $I \subset \mathbb{R}$ with values in a Banach space X.

Let $0 = t_0 < t_1 < \ldots < t_r = T$ be a partition of $(0, T)$ generating time intervals $I_m = (t_{m-1}, t_m],\ m = 1, \dots, r$ of the length $|I_m| = \tau_m$ and $\tau = \max_{m=1,\dots,r} \tau_m$. Moreover, we set $\mathcal{I}_\tau := \{I_m\}_{m=1}^r$. At every time level $t_m,\ m = 0, \dots, r$ we consider generally different space partition $\mathscr{T}_{h,m}$ consisting of a finite number of closed simplices K with mutually disjoint interiors covering $\overline{\Omega}$. Moreover, we set $\mathscr{T}_h := \{\mathscr{T}_{h,m}\}_{m=1}^r$ and $h := \max_{m=1,\dots,r} \max_{K \in \mathscr{T}_{h,m}} \mathrm{diam}(K)$. The pair $\{\mathscr{T}_h, \mathcal{I}_\tau\}$ we call the *space-time* partition of the domain Q_T.

Let $m = 0, \dots, r$ be arbitrary but fixed number denoting an index of a time interval. Let $\mathscr{T}_{h,m}$ be the corresponding triangulation. We define the so-called *broken Sobolev spaces*

$$H^1(\mathscr{T}_{h,m}) := \{v : \Omega \to \mathbb{R};\ v|_K \in H^1(K)\ \forall\, K \in \mathscr{T}_{h,m}\}, \quad \boldsymbol{H}^1(\mathscr{T}_{h,m}) := [H^1(\mathscr{T}_{h,m})]^s \tag{2.2}$$

of scalar and vector-valued functions, respectively.

Furthermore, we define the *broken space-time space* over $\{\mathscr{T}_h, \mathcal{I}_\tau\}$ by

$$H^1(\mathcal{I}_\tau, \boldsymbol{H}^1(\mathscr{T}_h)) := \left\{\boldsymbol{\psi}|_{K \times I_m} \in H^1(I_m; \boldsymbol{H}^1(K)),\ K \in \mathscr{T}_{h,m},\ I_m \in \mathcal{I}_\tau\right\}, \tag{2.3}$$

consisting of piecewise regular functions on space-time elements $K \times I_m$, $K \in \mathscr{T}_{h,m}$, $I_m \in \mathcal{I}_\tau$. These functions are in general discontinuous between every two neighbouring elements $K, K' \in \mathscr{T}_h$ as well as between any two consecutive time intervals $I_m, I_{m+1} \in \mathcal{I}_\tau$.

Moreover, we define the spaces of *discontinuous piecewise polynomial functions*. Even though the DG method allows using different polynomial degrees over mesh elements, we consider here only the fixed degrees of polynomial approximation for all $K \in \mathscr{T}_h$, for simplicity.

Let $\mathscr{T}_{h,m}$ be a triangulation on the time level $I_m,\ m = 0, \dots, r$. We put

$$S_{m,h,p} = \{\varphi : \Omega \to \mathbb{R};\ \varphi(x)|_K \in P_p(K)\ \forall K \in \mathscr{T}_{h,m}\}, \quad \boldsymbol{S}_{m,h,p} := [S_{m,h,p}]^s, \tag{2.4}$$

where $P_p(K)$ denotes the space of all polynomials on K of degree $\leq p$.

Furthermore, we define the spaces of functions on the space-time domain Q_T, for an integer $q \geq 0$ we put

$$P^q(\mathcal{I}_\tau) := \{\boldsymbol{v} : (0,T) \to \mathbb{R}^s,\ \boldsymbol{v}|_{I_m} \in [P^q(I_m)]^s,\ I_m \in \mathcal{I}_\tau\}, \tag{2.5}$$

where $P^q(I_m)$ is the space of vector-valued polynomials of order $\leq q$ on the interval I_m, $m = 1, \dots, r$.

For the purposes of the upcoming error measures we define three subspaces of $H^1(\mathcal{I}_\tau, \boldsymbol{H}^1(\mathscr{T}_h))$, namely

(2.6)
$$H^1(\mathcal{I}_\tau; \boldsymbol{S}_{h,p}) := \left\{\boldsymbol{\psi} \in H^1(\mathcal{I}_\tau, \boldsymbol{H}^1(\mathscr{T}_h));\ \boldsymbol{\psi}(\cdot, t) \in \boldsymbol{S}_{m,h,p} \ \text{ for a.e. } t \in I_m,\ m = 1, \dots, r\right\},$$

(2.7)
$$S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{H}^1(\mathscr{T}_h)) := \left\{\boldsymbol{\psi} \in H^1(\mathcal{I}_\tau, \boldsymbol{H}^1(\mathscr{T}_h));\ \boldsymbol{\psi}(x, \cdot) \in P^q(\mathcal{I}_\tau) \ \text{ for a.e. } x \in \Omega\right\},$$

(2.8)
$$S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p}) := \left\{\boldsymbol{\psi} \in H^1(\mathcal{I}_\tau, \boldsymbol{H}^1(\mathscr{T}_h));\ \boldsymbol{\psi}|_{K \times I_m} \in [P^p(K) \times P^q(I_m)]^s, K \in \mathscr{T}_{h,m},\ I_m \in \mathcal{I}_\tau\right\},$$

where $P^p(K) \times P^q(I_m)$ is the space of polynomials on $K \times I_m$ of the degree $\leq p$ with respect to $x \in K$ and the degree $\leq q$ with respect to $t \in I_m$ for $K \in \mathscr{T}_h$ and $I_m \in \mathcal{I}_\tau$.

All three spaces from (2.6)–(2.8) are piecewise regular on space time elements $K \times I_m,\ K \in \mathscr{T}_h,\ I_m \in \mathcal{I}_\tau$, but generally discontinuous on Q_T. Furthermore, we denote the space of piecewise polynomial functions on the concrete time interval I_m by

(2.9)
$$S^{\tau,q}(I_m; \boldsymbol{S}_{h,p}) := \{\boldsymbol{\psi} : \Omega \times I_m \to \mathbb{R}^s;\ \boldsymbol{\psi}|_{K \times I_m} \in [P^p(K) \times P^q(I_m)]^s, K \in \mathscr{T}_{h,m}\}.$$

Obviously, $\boldsymbol{\psi}|_{\Omega \times I_m} \in S^{\tau,q}(I_m; \boldsymbol{S}_{h,p})$ for all $\boldsymbol{\psi} \in S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p})$.

Finally, we introduce the jump of $\boldsymbol{\varphi} \in H^1(\mathcal{I}_\tau, \boldsymbol{H}^1(\mathscr{T}_h))$ with respect to time on the time level $t_m,\ m = 0, \dots, r$ by

$$\{\!\{\boldsymbol{\varphi}\}\!\}_m := \boldsymbol{\varphi}|_m^+ - \boldsymbol{\varphi}|_m^-, \qquad \text{where } \boldsymbol{\varphi}|_m^\pm := \lim_{\delta \to 0\pm} \boldsymbol{\varphi}(t_m + \delta). \tag{2.10}$$

We employ the discontinuous Galerkin method to discretize the system of equations (2.1). Since the specific definitions of the forms representing the DG discretization of convective fluxes of (2.1) are not particularly relevant for the purposes of this article, we introduce here only notation of the resulting forms

$$\boldsymbol{a}_{h,m} : \boldsymbol{H}^1(\mathscr{T}_{h,m}) \times \boldsymbol{H}^1(\mathscr{T}_{h,m}) \to \mathbb{R}, \quad m = 1, \dots, r. \tag{2.11}$$

More details about this methods can be found e.g. in [8, 5].

Further, we introduce the space-time discontinuous Galerkin (STDG) discretization of (2.1).

Definition 2.1. *We say that the function* $\boldsymbol{w}_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p})$ *is the* space-time discrete solution *of problem* (2.1) *if*

$$\begin{aligned} &\boldsymbol{A}_{h,m}(\boldsymbol{w}_{h\tau}, \boldsymbol{\psi}) = 0 \quad \forall \boldsymbol{\psi} \in S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p}),\ m = 1, \dots, r, \\ &\left(\boldsymbol{w}_{h\tau}|_0^-, \boldsymbol{\varphi}\right)_\Omega = (\boldsymbol{w}_0, \boldsymbol{\varphi})_\Omega \quad \forall \boldsymbol{\varphi} \in S_{0,h,p}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \boldsymbol{A}_{h,m}(\boldsymbol{w}, \boldsymbol{\psi}) := \int_{I_m} \{(\partial_t \boldsymbol{w}, \boldsymbol{\psi})_\Omega + \boldsymbol{a}_{h,m}(\boldsymbol{w}, \boldsymbol{\psi})\}\ \mathrm{d}t + \left(\{\!\{\boldsymbol{w}\}\!\}_{m-1}, \boldsymbol{\psi}|_{m-1}^+\right)_\Omega, \\ \boldsymbol{w}, \boldsymbol{\psi} \in H^1(\mathcal{I}_\tau, \boldsymbol{H}^1(\mathscr{T}_h)),\ m = 1, \dots, r \end{aligned} \tag{2.13}$$

and $\boldsymbol{w}_0$ *is the prescribed initial condition.*

2.2. Solution strategy. The definition of the space-time discrete solution (2.12) represents a nonlinear algebraic system for each time level $m = 1, \dots, r$. In the algebraic form the discrete problem (2.12) reads:

$$\text{find } \boldsymbol{\xi}_m \in \mathbb{R}^{N_m} \text{ such that } \boldsymbol{F}_{h,m}(\boldsymbol{\xi}_m) = \mathbf{0}, \quad m = 1, \dots, r, \tag{2.14}$$

where $\boldsymbol{\xi}_m$ is the vector of the coefficients of the discrete solution $\boldsymbol{w}_{h\tau}$ with respect to a basis of $S^{\tau,q}(I_m; \boldsymbol{S}_{h,p})$ and the vector-valued function $\boldsymbol{F}_{h,m}(\boldsymbol{\xi}_m)$ represents the form $\boldsymbol{A}_{h,m}(\boldsymbol{w}_{h\tau}^m, \cdot)$ tested by the basis functions of $S^{\tau,q}(I_m; \boldsymbol{S}_{h,p})$.

The system (2.14) is strongly nonlinear. We solve it by a damped Newton-like iterative method, see e.g. [15, 8], where the Jacobi matrix in the Newton method is replaced by the so-called *flux matrix* developed in the context of the semi-implicit DG method in [4, 7, 9].

In order to determine the solution $\boldsymbol{\xi}_m$ of the system (2.14), the employed damped Newton-like method [15] generates a sequence of approximations $\boldsymbol{\xi}_m^l$, $l = 0, 1, \dots$ of the actual numerical solution $\boldsymbol{\xi}_m$. The discrete solution $\boldsymbol{w}_{h\tau}^m(x,t) \longleftrightarrow \lim_{l\to\infty} \boldsymbol{\xi}_m^{(l)}$. Practically, we have to stop the iterative algorithm for some $l < \infty$. In the following, we denote by $\tilde{\boldsymbol{w}}_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p})$ the function corresponding to the output of finite number of steps of the damped Newton-like iterative algorithm. We call $\tilde{\boldsymbol{w}}_{h\tau}$ the *approximate solution* of problem (2.1). Obviously, due to inexact solution of (2.14), the approximate solution $\tilde{\boldsymbol{w}}_{h\tau}$ violates the relation (2.12).

The iterative algorithm is terminated when a suitable algebraic stopping criterion is achieved, i.e. $\left\|\boldsymbol{F}_{h,m}(\boldsymbol{\xi}_m^l)\right\| \leq \eta$, where $\|\cdot\|$ and η is a given norm and a given tolerance, respectively.

3. Error estimates. The main goal of our efforts is to find a robust strategy for adaptive choice of the time step, space mesh and for determining the stopping criterion for the iterative process. To achieve such goal we need to identify the time, space and algebraic errors. With this in mind we slightly reformulate the STDG method (2.12).

We define the form $\boldsymbol{A}_{h\tau}(\boldsymbol{z},\boldsymbol{\psi}) : H^1(\mathcal{I}_\tau,\boldsymbol{H}^1(\mathscr{T}_h)) \times H^1(\mathcal{I}_\tau,\boldsymbol{H}^1(\mathscr{T}_h)) \to \mathbb{R}$ by $\boldsymbol{A}_{h\tau}(\boldsymbol{z},\boldsymbol{\psi}) := \sum_{m=1}^r \boldsymbol{A}_{h,m}(\boldsymbol{z},\boldsymbol{\psi})$, where $\boldsymbol{A}_{h,m}$ is given by (2.13). The consistency of STDG scheme implies that the *exact* solution $\boldsymbol{w}$ of the problem (2.1) satisfies

$$\boldsymbol{A}_{h\tau}(\boldsymbol{w},\boldsymbol{\psi}) = 0 \qquad \forall \boldsymbol{\psi} \in H^1(\mathcal{I}_\tau,\boldsymbol{H}^1(\mathscr{T}_h)). \tag{3.1}$$

Moreover, let $\boldsymbol{w}_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau;\boldsymbol{S}_{h,p})$ be the *space-time discrete* solution given by (2.12). Then

$$\boldsymbol{A}_{h\tau}(\boldsymbol{w}_{h\tau},\boldsymbol{\varphi}_{h\tau}) = 0 \quad \forall \boldsymbol{\varphi}_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau;\boldsymbol{S}_{h,p}). \tag{3.2}$$

In addition we define the *time semi-discrete* solution $\boldsymbol{w}_\tau \in S^{\tau,q}(\mathcal{I}_\tau;\boldsymbol{H}^1(\mathscr{T}_h))$ (formally exact in space) solving

$$\boldsymbol{A}_{h\tau}(\boldsymbol{w}_\tau,\boldsymbol{\psi}) = 0 \quad \forall \boldsymbol{\psi} \in S^{\tau,q}(\mathcal{I}_\tau;\boldsymbol{H}^1(\mathscr{T}_h)). \tag{3.3}$$

Similarly the *space semi-discrete* solution $\boldsymbol{w}_h \in H^1(\mathcal{I}_\tau;\boldsymbol{S}_{h,p})$ (formally exact in time) solves

$$\boldsymbol{A}_{h\tau}(\boldsymbol{w}_h,\boldsymbol{\psi}) = 0 \quad \forall \boldsymbol{\psi} \in H^1(\mathcal{I}_\tau;\boldsymbol{S}_{h,p}). \tag{3.4}$$

Finally, we recall the approximate solution $\tilde{\boldsymbol{w}}_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau;\boldsymbol{S}_{h,p})$, which violates (3.2) due to the algebraic errors.

3.1. Dual error measures and error estimators. Similarly as in, e.g., [1, 2], we employ an error measure in the dual norm in the following way. Let V be a linear vector space with a norm $\|\cdot\|_V$, and let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a form linear with respect to its second argument and let V_h be a finite dimensional subspace of V. Moreover, let $u \in V$ and $u_h \in V_h$ be an exact and approximate solution of a fictitious problem defined by

$$a(u,\varphi) = 0 \quad \forall \varphi \in V \qquad \text{and} \qquad a(u_h,\varphi_h) = 0 \quad \forall \varphi_h \in V_h, \tag{3.5}$$

respectively. Then the *error measure in the dual norm* on the space V is given by

$$E(u_h) := \|Au_h - Au\|_{V'} := \sup_{\substack{\varphi\in V\\ \varphi\neq 0}} \frac{a(u_h,\varphi) - a(u,\varphi)}{\|\varphi\|_V} = \sup_{\substack{\varphi\in V\\ \varphi\neq 0}} \frac{a(u_h,\varphi)}{\|\varphi\|_V}, \tag{3.6}$$

where A is the operator from V to its dual space corresponding to $a(\cdot,\cdot)$ given by $\langle Au,\varphi\rangle := a(u,\varphi),\ u,\varphi \in V$, where $\langle\cdot,\cdot\rangle$ denotes the duality between V and V'. The last equality in (3.6) follows from (3.5).

Leaving the concrete specification to later discussion, we denote $\|\cdot\|_X$ a norm defined on $X := H^1(\mathcal{I}_\tau,\boldsymbol{H}^1(\mathscr{T}_h))$ and its subspaces. The only computable quantity of the above defined solutions is the approximate solution $\tilde{\boldsymbol{w}}_{h\tau}$. Therefore, we define the error measures in the following way using the general derivation of the error measure (3.6), see the diagram of the computational errors in Figure 3.1.

Space-time-algebraic error, i.e. $\boldsymbol{w} - \tilde{\boldsymbol{w}}_{h\tau}$, in the dual norm of the space $H^1(\mathcal{I}_\tau,\boldsymbol{H}^1(\mathscr{T}_h))$

$$\mathscr{E}_{STA}(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{\boldsymbol{\psi}\in H^1(\mathcal{I}_\tau,\boldsymbol{H}^1(\mathscr{T}_h))\\ \boldsymbol{\psi}\neq 0}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau},\boldsymbol{\psi}) - \boldsymbol{A}_{h\tau}(\boldsymbol{w},\boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_X}. \tag{3.7a}$$

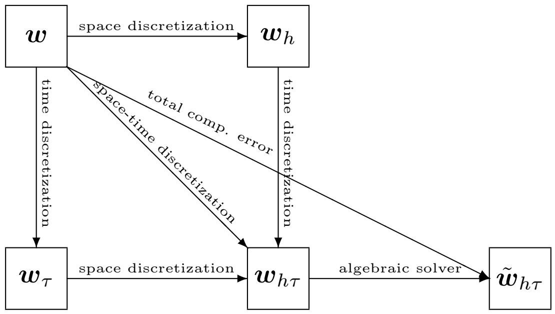


FIG. 3.1. *Types of the solutions and the errors*

Time-algebraic error, i.e. $\boldsymbol{w}_h - \tilde{\boldsymbol{w}}_{h\tau}$, in the dual norm of the space $H^1(\mathcal{I}_\tau; \boldsymbol{S}_{h,p})$

$$\mathscr{E}_{TA}(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{\boldsymbol{\psi} \in H^1(\mathcal{I}_\tau; \boldsymbol{S}_{h,p}) \\ \boldsymbol{\psi} \neq 0}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau}, \boldsymbol{\psi}) - \boldsymbol{A}_{h\tau}(\boldsymbol{w}_h, \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_X}. \tag{3.7b}$$

Space-algebraic error, i.e. $\boldsymbol{w}_\tau - \tilde{\boldsymbol{w}}_{h\tau}$, in the dual norm of the space $S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{H}^1(\mathscr{T}_h))$

$$\mathscr{E}_{SA}(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{\boldsymbol{\psi} \in S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{H}^1(\mathscr{T}_h)) \\ \boldsymbol{\psi} \neq 0}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau}, \boldsymbol{\psi}) - \boldsymbol{A}_{h\tau}(\boldsymbol{w}_\tau, \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_X}. \tag{3.7c}$$

Algebraic error, i.e. $\boldsymbol{w}_{h\tau} - \tilde{\boldsymbol{w}}_{h\tau}$, in the dual norm of the space $S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p})$

$$\mathscr{E}_{A}(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{\boldsymbol{\psi}_h \in S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p}) \\ \boldsymbol{\psi}_h \neq 0}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau}, \boldsymbol{\psi}_h) - \boldsymbol{A}_{h\tau}(\boldsymbol{w}_{h\tau}, \boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_X}, \tag{3.7d}$$

where the subtracted members vanish in all four cases due to (3.1) - (3.4).

Unfortunately, the error measures $\mathscr{E}_{STA}$, $\mathscr{E}_{TA}$ and $\mathscr{E}_{SA}$ are practically impossible to compute, since the suprema are taken over infinite-dimensional spaces. Therefore, we approximate these quantities by similar terms, where the spaces mentioned above are replaced by some finite dimensional subspaces. Indeed, these have to be chosen sufficiently large, e.g. the choice $S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p})$ would lead to equality between all these measures with the algebraic one.

Particularly, we employ the spaces

$$S^{\tau,q+1}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p}), \qquad S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p+1}), \qquad S^{\tau,q+1}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p+1}) \tag{3.8}$$

which extend the space $S^{\tau,q}(\mathcal{I}_\tau; \boldsymbol{S}_{h,p})$ by polynomials of one higher degree with respect to time, space and both space-time variables, respectively. Our choice is not the only possible one, e.g. one could enrich the space $S_{h,p}^{\tau,\mathbf{q}}$ by polynomials of even higher degree or introduce finer meshes, but numerical experiments show that this choice is sufficient and any further enrichment would lead to further computational costs.

Let $\tilde{\boldsymbol{w}}_{h\tau}$ be the computed approximate solution. Then based on (3.7a) – (3.7d), we define the *space-time-algebraic, time-algebraic, space-algebraic* and *algebraic residual* *error estimators*

$$\eta_{\mathrm{STA}}(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{\boldsymbol{\psi}_h \in S^{\tau,q+1}(\mathcal{I}_\tau;\boldsymbol{S}_{h,p+1})\\ \boldsymbol{\psi}_h \neq 0}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau},\boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_X}, \tag{3.9a}$$

$$\eta_{\mathrm{TA}}(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{\boldsymbol{\psi}_h \in S^{\tau,q+1}(\mathcal{I}_\tau;\boldsymbol{S}_{h,p})\\ \boldsymbol{\psi}_h \neq 0}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau},\boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_X}, \tag{3.9b}$$

$$\eta_{\mathrm{SA}}(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{\boldsymbol{\psi}_h \in S^{\tau,q}(\mathcal{I}_\tau;\boldsymbol{S}_{h,p+1})\\ \boldsymbol{\psi}_h \neq 0}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau},\boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_X}, \tag{3.9c}$$

$$\eta_{\mathrm{A}}(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{\boldsymbol{\psi}_h \in S^{\tau,q}(\mathcal{I}_\tau;\boldsymbol{S}_{h,p})\\ \boldsymbol{\psi}_h \neq 0}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau},\boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_X} = \mathscr{E}_A(\tilde{\boldsymbol{w}}_{h\tau}). \tag{3.9d}$$

3.2. Properties of the error estimators. Obviously, the exact solution $\boldsymbol{w} \in H^1(\mathcal{I}_\tau, \boldsymbol{H}^1(\mathscr{T}_h))$ satisfies $\eta_{\mathrm{STA}}(\boldsymbol{w}) = \eta_{\mathrm{TA}}(\boldsymbol{w}) = \eta_{\mathrm{SA}}(\boldsymbol{w}) = \eta_{\mathrm{A}}(\boldsymbol{w}) = 0$ due to the consistency of the above defined schemes. Simply from (3.9) we see that for any $\tilde{\boldsymbol{w}}_{h\tau} \in S^{\tau,q}(\mathcal{I}_\tau;\boldsymbol{S}_{h,p})$ it holds

$$\eta_{\mathrm{A}}(\tilde{\boldsymbol{w}}_{h\tau}) \le \eta_{\mathrm{TA}}(\tilde{\boldsymbol{w}}_{h\tau}) \le \eta_{\mathrm{STA}}(\tilde{\boldsymbol{w}}_{h\tau}), \quad \eta_{\mathrm{A}}(\tilde{\boldsymbol{w}}_{h\tau}) \le \eta_{\mathrm{SA}}(\tilde{\boldsymbol{w}}_{h\tau}) \le \eta_{\mathrm{STA}}(\tilde{\boldsymbol{w}}_{h\tau}). \tag{3.10}$$

Further, since the suprema in (3.9) are taken over subspaces of spaces in (3.7), we get the lower bounds $\eta_{\mathrm{STA}}(\tilde{\boldsymbol{w}}_{h\tau}) \le \mathscr{E}_{STA}(\tilde{\boldsymbol{w}}_{h\tau})$, $\eta_{\mathrm{TA}}(\tilde{\boldsymbol{w}}_{h\tau}) \le \mathscr{E}_{TA}(\tilde{\boldsymbol{w}}_{h\tau})$, $\eta_{\mathrm{SA}}(\tilde{\boldsymbol{w}}_{h\tau}) \le \mathscr{E}_{SA}(\tilde{\boldsymbol{w}}_{h\tau})$, $\eta_{\mathrm{A}}(\tilde{\boldsymbol{w}}_{h\tau}) = \mathscr{E}_A(\tilde{\boldsymbol{w}}_{h\tau})$. However, it is an open question, whether there there exist upper bound, i.e., $\mathscr{E}_*(\cdot) \le C\eta_*(\cdot)$.

Let $\tilde{\boldsymbol{w}}_{h\tau}$ be the computed approximate solution. In order to simplify the notation, we introduce a generic definition of the residual error estimators (3.9) by

$$\eta_\star(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{\boldsymbol{\psi}_h \in X_h\\ \boldsymbol{\psi}_h \neq 0}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau},\boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_X}, \qquad \star \in \{STA, TA, SA, A\}, \tag{3.11}$$

which formally represents any definition from (3.9a) – (3.9d), where X_h denotes the corresponding functional space.

3.3. Localization of the error estimators. Since we want to use the error estimators $\eta_\star(\tilde{\boldsymbol{w}}_{h\tau})$ to efficient adaptive algorithm, we need to localize these quantities. We define the *residual error estimators at time interval* I_m by

$$\eta_\star^m(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{0\neq\boldsymbol{\psi}_h \in X_h\\ \mathrm{supp}(\boldsymbol{\psi}_h)\subset\Omega\times I_m}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau},\boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_X}, \qquad m = 1,\dots,r \tag{3.12}$$

and the *element residual error estimators* by

$$\eta_\star^{m,K}(\tilde{\boldsymbol{w}}_{h\tau}) := \sup_{\substack{0\neq\boldsymbol{\psi}_h \in X_h\\ \mathrm{supp}(\boldsymbol{\psi}_h)\subset K\times I_m}} \frac{\boldsymbol{A}_{h\tau}(\tilde{\boldsymbol{w}}_{h\tau},\boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_X}, \qquad K \in \mathscr{T}_{h,m}, \quad m = 1,\dots,r, \tag{3.13}$$

which represent a "restriction" of the residual error estimators on the time interval I_m and the space-time element $K \times I_m$, respectively.

We need to define the X-norm in such a way that its evaluation is cheap. We present the following lemma, proved in [10].

LEMMA 3.1. *Let $(\cdot,\cdot)_X : X \times X \to \mathbb{R}$ be a scalar product generating the norm $\|\cdot\|_X$. Let $(\cdot,\cdot)_X$ satisfy the* element-orthogonality *condition, i.e.*

$$(\boldsymbol{\psi}_h, \boldsymbol{\psi}'_h)_X = 0 \quad \forall \boldsymbol{\psi}_h, \boldsymbol{\psi}'_h \in X \quad \text{such that } \mathrm{supp}(\boldsymbol{\psi}_h) \text{ and } \mathrm{supp}(\boldsymbol{\psi}'_h) \text{ have disjoint interiors.} \tag{3.14}$$

Then

$$\eta_\star(\boldsymbol{w}_{h\tau})^2 = \sum_{m=1}^{r} \eta_\star^m(\boldsymbol{w}_{h\tau})^2 = \sum_{m=1}^{r} \sum_{K \in \mathscr{T}_{h,m}} \eta_\star^{m,K}(\boldsymbol{w}_{h\tau})^2. \tag{3.15}$$

If the norm $\|\cdot\|_X$ is generated by a scalar product satisfying (3.14) then it is sufficient to evaluate the the element residual estimators $\eta_\star^{m,K}$, for all $m = 1, \dots, r$ and all $K \in \mathscr{T}_{h,m}$ and the global error estimators can be evaluated using the equality (3.15).

For the problem (2.1) it seems natural to measure the error in the $L^2(0,T,L^2(\Omega))$-norm. The equivalence between the error and the residual (3.6) was analysed in [18, 17]. Although, we are not able to prove such equivalence for problem (2.1), we use $X = H^1(\mathcal{I}_\tau, \boldsymbol{H}^1(\mathscr{T}_h))$ and

$$\|u\|_X := \int_0^T \left(\|\partial_t u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right)^{1/2} \mathrm{d}t, \quad u \in X. \tag{3.16}$$

This choice can be partly justified for linear hyperbolic equation, where integration by parts leads to this choice of the X-norm. This norm is clearly generated by a scalar product satisfying the favourable orthogonal property (3.14).

3.4. Application of error estimators in the solution strategy. In the solution process we control both the time and the algebraic error by the space error. For that reason, the time and the algebraic error do not essentially contribute to the total computational error in the proposed algorithm.

Termination of the iterative process. We stop the iterative process if the algebraic residual error estimate at time interval I_m is sufficiently small in comparison to the space residual error estimate at time interval I_m, i.e.,

$$\eta_{\mathrm{A}}^m(\tilde{\boldsymbol{w}}_{h\tau}^{m,(l)}) \le c_A \eta_{\mathrm{SA}}^m(\tilde{\boldsymbol{w}}_{h\tau}^{m,(l)}), \quad m = 1, \dots, r, \tag{3.17}$$

where $0 < c_A < 1$ is a suitable constant which controls the relative influence of the nonlinear algebraic error to the space discretization. It is reasonable to set $c_A \in [10^{-3}, 10^{-1}]$.

Choice of the time step in (2.12). Again, we choose the time step τ_m such that the time residual error estimator at time interval I_m is controlled by the space residual error estimator at time interval I_m, i.e.,

$$\eta_{\mathrm{TA}}^m(\tilde{\boldsymbol{w}}_{h\tau}^{m}) \le c_T \eta_{\mathrm{SA}}^m(\tilde{\boldsymbol{w}}_{h\tau}^{m}), \quad m = 1, \dots, r, \tag{3.18}$$

where $c_T > 0$ is a suitable constant representing a desired ratio of the time and space error. Therefore, at each time level $m = 1, \dots, r$, we evaluate estimates $\eta_{\mathrm{TA}}^m(\tilde{\boldsymbol{w}}_{h\tau}^{m})$ and $\eta_{\mathrm{SA}}^m(\tilde{\boldsymbol{w}}_{h\tau}^{m})$ and define the "optimal" time step

$$\tau_m^{\mathrm{opt}} := \tau_m \tilde{c}_T \left(\frac{c_T \eta_{\mathrm{SA}}^m(\tilde{\boldsymbol{w}}_{h\tau}^{m})}{\eta_{\mathrm{TA}}^m(\tilde{\boldsymbol{w}}_{h\tau}^{m})} \right)^{1/(q+1)}, \tag{3.19}$$

where $\tilde{c}_T \in (0,1)$ is an security factor (we use the value $\tilde{c}_T = 0.9$ in our experiments). This technique is standard, more details can be found in [12].

3.5. Adaptive space-time DG method. It is challenging to develop a full space-time adaptive technique for non-stationary problems. Although the residual error estimators described above do not give an upper error bound, we use them for an adaptive algorithm which adapts (locally) the mesh and (globally) the size of the time step.

Our aim is to adapt the mesh size and the time step in such a way that the *space-time-algebraic residual error estimator* η_{STA} is *under a given tolerance* $\omega > 0$, i.e.,

$$\eta_{\mathrm{STA}}(\tilde{\boldsymbol{w}}_{h\tau}) \leq \omega. \tag{3.20}$$

In the computational process, we prescribe the tolerance for the space-time-algebraic residual error estimates η^m_{STA} on the time interval $I_m,\ m = 1, \dots, r$, namely

$$\eta^m_{\mathrm{STA}}(\tilde{\boldsymbol{w}}_{h\tau}) \leq \omega_m, \quad \omega_m := \omega\sqrt{\tau_m/T}, \quad m = 1, \dots, r, \tag{3.21}$$

where ω_m is the tolerance for the time level $t_m,\ m = 1, \dots, r$. The condition (3.21) implies (3.20) due to (3.15). Then we define the following *space-time adaptive process*:

1) let $\omega > 0$ be a given tolerance, $\mathscr{T}_{h,0}$ the initial mesh and τ_0 the initial time step,
2) let $m = 1$,
3) we solve problem (2.14) by the damped Newton-like iterative method until the stopping criterion $\eta^m_{\mathrm{A}} \leq c_A \eta^m_{\mathrm{SA}}$ is satisfied,
4) if $\eta^m_{\mathrm{TA}} > c_T \eta^m_{\mathrm{SA}}$ we adapt the time step τ_m according to (3.19) and go to step 3) (repeat time step, time error is too high),
5) if $\eta^m_{\mathrm{STA}} > \omega_m$ then we adapt mesh $\mathscr{T}_{h,m}$ and go to step 3) (repeat time step, space error is too high),
6) if $t_m \geq T$ then the computation finishes,
   else we put $\mathscr{T}_{h,m+1} := \mathscr{T}_{h,m}$, $\tau_{m+1} := \tau^{\mathrm{opt}}_m$, $m := m + 1$ and go to step 3).

If condition (3.21) is violated for some $m = 1, \dots, r$, the mesh $\mathscr{T}_{h,m}$ has to be adapted (step 5 of the algorithm). More information about the used adaptation techniques can be found in [10, 3].

4. Numerical experiments. In this section, we illustrate the computional performance of the developed residual-based error estimation scheme in several situations.

4.1. Scalar nonlinear hyperbolic equation. Let us consider the scalar nonlinear hyperbolic equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x_1} + \frac{1}{2}\frac{\partial u^2}{\partial x_2} = g \quad \text{in } Q_T = (0,1)^2 \times (0,1), \tag{4.1}$$

with the function g chosen such that the exact solution has the form $u(x_1, x_2, t) = \sin(2\pi(x_1 + x_2 - 2t))$.

The purpose of this experiment is to show the same convergence orders of the above defined error estimators and the actual computational error. If the exact solution is regular enough, based on the a priori analysis of STDG methods, e.g. [11], the computational error satisfies

$$\|e_{h\tau}\|_{L^2(0,T;L^2(\Omega))} \leq C_1 h^{p+1} + C_2 \tau^{q+1} \tag{4.2}$$

where $e_{h\tau} := \boldsymbol{w} - \tilde{\boldsymbol{w}}_{h\tau}$, h denotes the size of the mesh step, τ is the size of the time step, p and q are the polynomial approximation degrees with respect to the space and

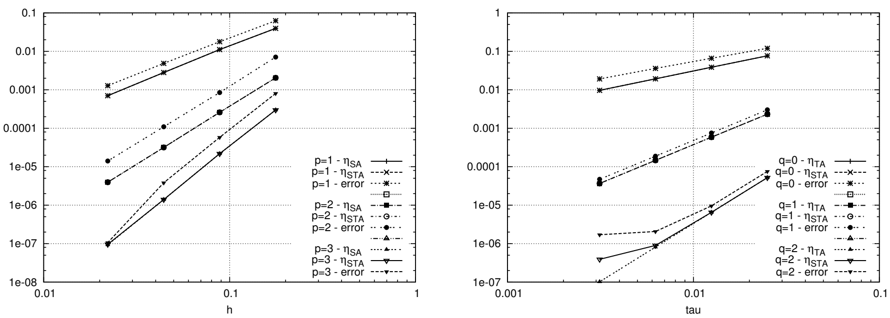


FIG. 4.1. *Scalar equation: convergence curves with respect to space (left) and time (right) for scalar nonlinear hyperbolic equation.*

the time coordinates, respectively, and $C_1 > 0$ and $C_2 > 0$ are constants independent of h and τ. We set the constant $C_A = 10^{-3}$ in (3.17) to suppress the algebraic errors. We present two examples. First, we oversolve the problem with respect to the time variables (q high and τ small) and we expect

$$\|e_h\|_{L^2(0,T;L^2(\Omega))} \approx \eta_{\mathrm{STA}} = O(h^{p+1}), \qquad \eta_{\mathrm{TA}} \ll \eta_{\mathrm{SA}} \approx \eta_{\mathrm{STA}}. \tag{4.3}$$

Second, we solve the problem on a very fine mesh with high order approximation degree p. In this case, we expect

$$\|e_h\|_{L^2(0,T;L^2(\Omega))} \approx \eta_{\mathrm{STA}} = O(h^{q+1}), \qquad \eta_{\mathrm{SA}} \ll \eta_{\mathrm{TA}} \approx \eta_{\mathrm{STA}}. \tag{4.4}$$

The resulting convergence curves are showed in Figure 4.1. We can see that the behaviour of the actual error is very similar to the error estimators in all cases.

4.2. Isentropic vortex propagation. We consider the propagation of an isentropic vortex in compressible inviscid flow, analysed numerically in [16]. The computational domain is taken as $[0, 10] \times [0, 10]$, extended periodically in both directions, and $T = 10$. The mean flow is $\rho = 1$, $\boldsymbol{v} = (1, 1)$ (diagonal flow) and $\mathtt{p} = 1$. To this mean flow we add an *isentropic vortex*, i.e. perturbation in $\boldsymbol{v}$ and the temperature $\theta = \mathtt{p}/\rho$, but no perturbation in the entropy $\eta = \mathtt{p}/\rho^{\gamma}$.

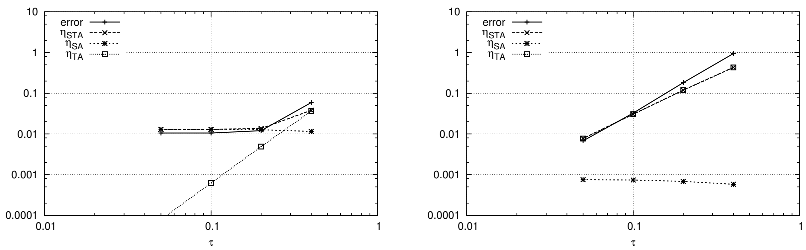
This example is suitable for the demonstration of the performance of the proposed residual error estimators since the regular exact solution is known and thus we can simply evaluate the computational error $e_{h\tau}$. Then we are able to identify the influence of the space and time discretization parameters h and τ, respectively, on the total computational error.

We use two unstructured quasi-uniform triangular meshes with $\#\mathscr{T}_h = 580$ and $\#\mathscr{T}_h = 2484$ triangles, i.e. $h = 0.894$ and $h = 0.448$, respectively. We fix the space polynomial degree $p = 3$ and the time polynomial degree $q = 2$ and $q = 1$, respectively, and we decrease the length of the time intervals τ several times. In the first case (Figure 4.2 - left) space-time error estimator decreases only in the first step, where $\eta_{\mathrm{STA}} \approx \eta_{\mathrm{TA}}$, and then it stagnates at the level 10^{-2}, while the time estimator continues descending even for the smaller time steps. This corresponds to the behaviour of the actual computational error $e_{h\tau}$, since for the smaller values of τ the space part of the error dominates over the time error. On the finer mesh (Figure 4.2 - left) the space error is suppressed enough and hence the $\eta_{\mathrm{TA}} \approx \eta_{\mathrm{STA}} \approx e_{h\tau}$.

Finally, in Figure 4.2 we present meshes and the isolines of the pressure computed by the fully adaptive algorithm presented in the Section 3.5 with $p = 3$ and $q = 2$. In

$\#\mathscr{T}_h$	$\|e_h\|_{L^2(0,T;L^2(\Omega))}$	CPU(s)
580	1.057×10^{-2}	8853
2484	5.368×10^{-3}	27954
adapt	2.953×10^{-3}	5045

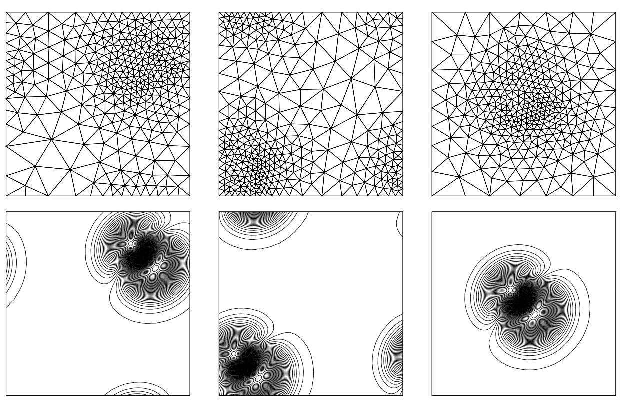
TABLE 4.1
Isentropic vortex: total errors and computational time



$\#\mathscr{T}_h = 580$, $p = 3$, $q = 2$ $\qquad$ $\#\mathscr{T}_h = 2484$, $p = 3$, $q = 1$

FIG. 4.2. *Isentropic vortex: error identification*

Table 4.2 we provide a comparison between the computation with fixed parameters and the adaptive algorithm. It is apparent that the adaptive procedure significantly fastens the computations.



$t = 3.0$ $\qquad$ $t = 7.0$ $\qquad$ $t = 10.0$

FIG. 4.3. *Isentropic vortex: adaptively refined meshes*

5. Conclusion. We introduced the residual based a posteriori error estimation technique for non-stationary nonlinear hyperbolic equations. The presented method uses the space-time discontinuous Galerkin method for discretizing the problem and the resulting nonlinear system is solved by the damped Newton-like method. Our scheme enables to identify the temporal, spatial and algebraic parts of the computational error, which leads to an efficient adaptive algorithm. The main drawback of this method is the missing theoretically guaranteed link between the error estimators used and the real size of the error. On the other hand, numerical experiments show that this method is able to approximate the actual error very accurately in many cases.

REFERENCES

[1] L. El Alaoui, A. Ern, and M. Vohralík. Guaranteed and robust a posteriori error estimates and balancing discretization and linearization errors for monotone nonlinear problems. *Comput. Methods Appl. Mech. Engrg*, 200:2782–2795, 2011.

[2] E. Burman and A. Ern. Discontinuous Galerkin approximation with discrete variational principle for the nonlinear Laplacian. *Comptes Rendus Mathematique*, 346(17-18):1013–1016, 2008.

[3] V. Dolejší. Anisotropic mesh adaptation for finite volume and finite element methods on triangular meshes. *Comput. Vis. Sci.*, 1(3):165–178, 1998.

[4] V. Dolejší. Semi-implicit Interior Penalty Discontinuous Galerkin Methods for Viscous Compressible Flows. *Commun. Comput. Phys.*, 4(2):231–274, 2008.

[5] V. Dolejší. A design of residual error estimates for a high order BDF-DGFE method applied to compressible flows. *Int. J. Numer. Meth. Fluids*, 73(6):523–559, 2013.

[6] V. Dolejší. *hp*-DGFEM for nonlinear convection-diffusion problems. *Math. Comput. Simul.*, 87:87–118, 2013.

[7] V. Dolejší and M. Feistauer. Semi-Implicit Discontinuous Galerkin Finite Element Method for the Numerical Solution of Inviscid Compressible Flow. *J. Comput. Phys.*, 198(2):727–746, 2004.

[8] V. Dolejší and M. Feistauer. *Discontinuous Galerkin Method*, volume 48. Springer International Publishing, 2015.

[9] V. Dolejší, M. Holík, and J. Hozman. Efficient solution strategy for the semi-implicit discontinuous Galerkin discretization of the Navier-Stokes equations. *J. Comput. Phys.*, 230:4176–4200, 2011.

[10] V. Dolejší, F. Roskovec, and M. Vlasák. Residual based error estimates for the space–time discontinuous Galerkin method applied to the compressible flows. *Computers & Fluids*, 117:304–324, 2015.

[11] M. Feistauer, V. Kučera, K. Najzar, and J. Prokopová. Analysis of space-time discontinuous Galerkin method for nonlinear convection-diffusion problems. *Numer. Math.*, 117:251–288, 2011.

[12] E. Hairer and G. Wanner. *Solving ordinary differential equations II, Stiff and differential-algebraic problems*. Springer Verlag, 2002.

[13] A. Kufner, O. John, and S. Fučí k. *Function Spaces*. Academia, Prague, 1977.

[14] J. Nečas. *Les Méthodes Directes en Thèorie des Equations Elliptiques*. Academia, Prague, 1967.

[15] Deuflhard P. *Newton Methods for Nonlinear Problems*. Springer Series in Computational Mathematics, Vol. 35, 2004.

[16] C.W. Shu. Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws. In A. Quarteroni et al, editor, *Advanced numerical approximation of nonlinear hyperbolic equations*, Lect. Notes Math. 1697, pages 325–432. Berlin: Springer, 1998.

[17] R. Verfürth. A Posteriori Error Estimates for Nonlinear Problems: $L^r(0,T;L^\rho(\Omega))$-Error Estimates for Finite Element Discretizations of Parabolic Equations. *Math. Comput.*, 67(224):1335–1360, 1998.

[18] R. Verfürth. A Posteriori Error Estimates for Nonlinear Problems: $L^r(0,T;W^{1,\rho}(\Omega))$-Error Estimates for Finite Element Discretizations of Parabolic Equations. *Numer. Meth. Part. Diff. Eqs*, 14:487–518, 1998.