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A FOURTH-ORDER COMPACT SCHEME FOR THE NAVIER-STOKES EQUATIONS IN IRREGULAR DOMAINS*

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Abstract. We present a high-order finite difference scheme for Navier-Stokes equations in irregular domains. The discretization offered here contains two types of interior points. The first is regular interior points, where all eight neighboring points of a grid point are inside the domain and not too close to the boundary. The second is interior points where at least one of the closest eight neighbors is outside the computational domain or too close to the boundary. In the second case we design discrete operators which approximate spatial derivatives of the streamfunction on irregular meshes, using discretizations of pure derivatives in the x, y and along the diagonals of the element.

Key words. fourth-Order Compact Scheme, Navier-Stokes Equations, irregular domains

AMS subject classifications. 65M06, 76D05, 65D25

1. Introduction. In this paper we are interested in high-order discretizations of the Navier-Stokes equations. The Navier-Stokes equations play a central role in modeling fluid flows. Here we focus on incompressible flows. It is well-known that this system may be represented in pure streamfunction formulation as follows (see [3]).

$$\left\{ \begin{array}{l} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \nu\Delta^2\psi = f(x,y,t), \quad (x,y) \in \Omega, \quad t > 0, \\ \psi = \frac{\partial\psi}{\partial n} = 0 \;\;,\;\; (x,y) \in \partial\Omega \;\;,\;\; t > 0 \\ \psi(x,y,0) = \psi_0(x,y), \quad (x,y) \in \Omega. \end{array} \right. \tag{1.1}$$

Recall that $\nabla^{\perp}\psi = (-\partial_y\psi, \partial_x\psi)$ is the velocity vector.

In this paper we extend the fourth-order scheme [2] to irregular domains. The strategy used here is to present the biharmonic operator $\partial_x^4 + 2\partial_x^2\partial_y^2 + \partial_y^4$ as a combination of pure fourth-order derivatives in the x, y and the diagonal directions $\eta = (x+y)/\sqrt{2}, \;\; \xi = (y-x)\sqrt{2}$. Then, the pure fourth-order derivatives may be approximated via a compact scheme using the values of the function and its directional derivatives (see also [5], [4]). An alternative approach is to construct a two-dimensional polynomial which collocates the values of the function and its directional derivatives at the corners of the irregular element and then approximate the biharmonic of the function by the biharmonic of this polynomial (see [1]).

2. Approximation of the Navier-Stokes equations on regular grids. Spatial derivatives in Equation (1.1) are discretized as we describe next. The domain Ω is embedded in a square $[a,b] \times [a,b]$. We lay out a grid $a = x_0 < x_1 < ... < x_N = b$, and $a = y_0 < y_1 < ... < y_N = b$, with $\Delta x = \Delta y = (b-a)/N = h$. To each grid point (x_i, y_j) we assign approximate values of the streamfunction ψ together with its first-order derivative ψ_x, ψ_y. These values are the unknowns to be determined by the scheme.

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The fourth order discrete Laplacian $\tilde{\Delta}_h\psi$ and biharmonic $\tilde{\Delta}_h^2\psi$ operators introduced in [2] are perturbations of the second order operators $\Delta_h\psi = (\delta_x^2 + \delta_y^2)\psi$ and $\Delta_h^2\psi = (\delta_x^4 + \delta_y^4 + 2\delta_x^2\delta_y^2)\psi$. They are designed as follows.

$$\tilde{\Delta}_h\psi_{i,j} = 2\delta_x^2\psi_{i,j} - \delta_x(\psi_x)_{i,j} + 2\delta_y^2\psi_{i,j} - \delta_y(\psi_y)_{i,j} = (\Delta\psi)_{i,j} + O(h^4). \tag{2.1}$$

Here, ψ_x, ψ_y are the fourth-order Hermitian approximations to $\partial_x\psi, \partial_y\psi$ described as

$$\begin{cases} \frac{1}{6}(\psi_x)_{i-1,j} + \frac{2}{3}(\psi_x)_{i,j} + \frac{1}{6}(\psi_x)_{i+1,j} = \delta_x\psi_{i,j} \quad , \quad 1 \le i,j \le N-1 \\ \frac{1}{6}(\psi_y)_{i,j-1} + \frac{2}{3}(\psi_y)_{i,j} + \frac{1}{6}(\psi_y)_{i,j+1} = \delta_y\psi_{i,j} \quad , \quad 1 \le i,j \le N-1. \end{cases} \tag{2.2}$$

We use the standard central difference operators δ_x, δ_y, δ_x^2, δ_y^2.

The fourth-order approximation to the biharmonic operator $\Delta^2\psi$ is

$$\tilde{\Delta}_h^2\psi_{i,j} = \delta_x^4\psi_{i,j} + \delta_y^4\psi_{i,j} + 2\delta_x^2\delta_y^2\psi_{i,j} - \frac{h^2}{6}(\delta_x^4\delta_y^2\psi_{i,j} + \delta_y^4\delta_x^2\psi_{i,j}), \tag{2.3}$$

where δ_x^4 (similarly δ_y^4) is the compact approximations of ∂_x^4 (and ∂_y^4).

$$\delta_x^4\psi_{i,j} = \frac{12}{h^2}\left((\delta_x\psi_x)_{i,j} - \delta_x^2\psi_{i,j}\right) \quad , \quad \delta_x^4\psi = \partial_x^4\psi - \frac{1}{720}h^4\partial_x^8\psi + O(h^6). \tag{2.4}$$

The Laplacian $\Delta\psi$ in (1.1) is $\Delta\psi = \partial_x^2\psi + \partial_y^2\psi$. Thus the Laplacian operator is already written with pure second order derivatives in x and y. Its approximation on a regular grid is given by

$$\tilde{\Delta}_h\psi_{i,j} = \tilde{\delta}_x^2\psi_{i,j} + \tilde{\delta}_y^2\psi_{i,j}, \tag{2.5}$$

where $\tilde{\delta}_x^2\psi_{i,j} = 2\delta_x^2\psi_{i,j} - \delta_x\psi_{x,i,j} = \partial_x^2\psi_{i,j} + O(h^4)$, and $\tilde{\delta}_y^2\psi_{i,j} = 2\delta_y^2\psi_{i,j} - \delta_y\psi_{y,i,j} = \partial_y^2\psi_{i,j} + O(h^4)$.

The convective term in (1.1) is $C(\psi) = -\partial_y\psi\Delta(\partial_x\psi) + \partial_x\psi\Delta(\partial_y\psi)$. Its fourth-order approximation needs special care. The mixed derivative $\partial_x\partial_y^2\psi$ may be approximated to fourth-order accuracy by $\tilde{\psi}_{yyx}$ using a suitable combination of lower order approximations.

$$(\tilde{\psi}_{yyx})_{i,j} = (\delta_y^2\psi_x + \delta_x\delta_y^2\psi - \delta_x\delta_y\psi_y)_{i,j} = (\partial_x\partial_y^2\psi)_{i,j} + O(h^4). \tag{2.6}$$

For the pure third order derivative $\partial_x^3\psi$ we note that if ψ is smooth then

$$(\psi_{xxx})_{i,j} = \frac{3}{2h^2}\left(10\delta_x\psi - h^2\delta_x^2\partial_x\psi - 10\partial_x\psi\right)_{i,j} + O(h^4). \tag{2.7}$$

One needs to approximate $\partial_x\psi$ to sixth-order accuracy in order to obtain from (2.7) a fourth-order approximation for $\partial_x^3\psi$. Denoting this approximation by $\tilde{\psi}_x$, we invoke the Pade formulation having the following form.

$$\frac{1}{3}(\tilde{\psi}_x)_{i+1,j} + (\tilde{\psi}_x)_{i,j} + \frac{1}{3}(\tilde{\psi}_x)_{i-1,j} = \frac{14}{9}\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h} + \frac{1}{9}\frac{\psi_{i+2,j} - \psi_{i-2,j}}{4h}. \tag{2.8}$$

Carrying out the same procedure for $\partial_y\psi$, which yields the approximate value $\tilde{\psi}_y$, and combining with all other mixed derivatives, a fourth order approximation of the convective term is

$$\tilde{C}_h(\psi) = -\psi_y \cdot \big(\Delta_h \tilde{\psi}_x + \frac{5}{2}\big(6\frac{\delta_x\psi - \tilde{\psi}_x}{h^2} - \delta_x^2\tilde{\psi}_x\big) + \delta_x\delta_y^2\psi - \delta_x\delta_y\tilde{\psi}_y\big)$$
$$+ \psi_x \cdot \big(\Delta_h \tilde{\psi}_y + \frac{5}{2}\big(6\frac{\delta_y\psi - \tilde{\psi}_y}{h^2} - \delta_y^2\tilde{\psi}_y\big) + \delta_y\delta_x^2\psi - \delta_y\delta_x\tilde{\psi}_x\big)$$
$$= C(\psi) + O(h^4). \tag{2.9}$$

Our implicit time-stepping scheme is of the Crank-Nicholson type as follows.

$$\frac{(\tilde{\Delta}_h\psi_{i,j})^{n+1/2} - (\tilde{\Delta}_h\psi_{i,j})^n}{\Delta t/2} = -\tilde{C}_h\psi^{(n)} + \frac{\nu}{2}[\tilde{\Delta}_h^2\psi_{i,j}^{n+1/2} + \tilde{\Delta}_h^2\psi_{i,j}^n] \tag{2.10}$$

$$\frac{(\tilde{\Delta}_h\psi_{i,j})^{n+1} - (\tilde{\Delta}_h\psi_{i,j})^n}{\Delta t} = -\tilde{C}_h\psi^{(n+1/2)} + \frac{\nu}{2}[\tilde{\Delta}_h^2\psi_{i,j}^{n+1} + \tilde{\Delta}_h^2\psi_{i,j}^n]. \tag{2.11}$$

3. Approximation of the Navier-Stokes equations on irregular domains.

In the previous section we described the approximation of the Navier-Stokes equations in streamfunction formulation in rectangular domains. If the domain is not a rectangular, one can either map the domain onto a rectangle or design an approximation of the equations on a cartesian grid embedded inside the domain Ω. In case we chose to map the domain onto a rectangle, then the differential equations take a new form, as the derivatives of the new coordinate system are involved in the equations, which complicates the equations. In addition, the transformation (such as a polar coordinate system) is sometimes singular at certain points and special treatment is needed near singular points.

In this paper we embed the domain Ω in a rectangle. Then, a uniform mesh is laid out inside the rectangle. Some of the mesh points are outside Ω, some are inside Ω and some may be on the boundary $\partial\Omega$.

If a mesh point is outside the computational domain Ω (flag=-1), then an arbitrary value, such as zero, is given to this point. Points which are outside the computational domain do not affect the values of the function at interior or at boundary points.

If a mesh point is on the boundary of the domain $\partial\Omega$ (flag=0), then the boundary values ψ of the function and its first-order normal derivative $\partial_n\psi$ are assigned to this point.

If a mesh point is inside the domain it may be labeled as follows.

Case 1: the point is in a center of a rectangle for which all the vertices are inside the domain (flag=1). In this case the differential operators for this point are approximated as in Section 2.

Case 2: the point is too close to the boundary (flag=2) then this point is not included in the set of computational points. Thus neither the differential equations nor the boundary conditions are imposed at this point. In our computations we have labeled a point with flag=2 if its distance to the boundary was less then βh, where h is the mesh size at the interior of the domain and $0 < \beta < 1$. In practice we have picked $\beta = 0.2$.

Case 3: the point is not too close to the boundary, but at least one of its eight nearest neighbors is outside the computational domain or at least one of its eight nearest neighbors is too close to the boundary (flag=3). In this case the point is the center of an irregular element. Thus, special discretization of the differential operator is needed.

We first have to describe how one constructs the element around such a computational point. Suppose the point under consideration is (x_i, y_j). If, for example

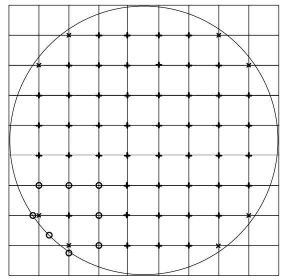

FIG. 3.1. *Grid: '+' computational point, 'o' eight neighbors of a computational point, 'x' point too close to the boundary.*

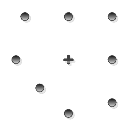

FIG. 3.2. *Single computational element: '+' computational point, 'o' eight neighbors of a computational point.*

(x_{i+1}, y_j) is outside the domain, then we define by (x_{east}, y_j) the point which is the closest on the right to (x_i, y_j) lying on the line $y = y_j$ and intersects with the boundary. We note by h_1 the distance from x_{east} to x_i, i.e., $h_1 = x_{east} - x_i$. Similarly in the case where (x_{i-1}, y_j) is outside Ω, for which we define $h_2 = x_i - x_{east}$. In the same fashion we treat the cases where (x_i, y_{j+1}) and (x_i, y_{j-1}) are outside the domain and define $h_3 = y_{north} - y_j$ and $h_4 = y_j - y_{south}$, respectively.

We also look at points along the line $x - x_i = y - y_j$. If (x_{i+1}, y_{j+1}) is outside the domain Ω, then we denote by $(x_{north-east}, y_{north-east})$ the intersection of the line $x - x_i = y - y_j$ going north-east of (x_i, y_j) with the boundary. We denote by h_5 the distance of $(x_{north-east}, y_{north-east})$ to (x_i, y_j), thus $(x_{north-east}, y_{north-east}) = (x_i + h_5/\sqrt{2}, y_j + h_5/\sqrt{2})$. Similarly $(x_{south-west}, y_{south-west}) = (x_i - h_6/\sqrt{2}, y_j - h_6/\sqrt{2})$. We also treat the points along the line $x - x_i = y_j - y$, thus defining h_7 as the distance of the point $(x_{north-west}, y_{north-west})$ to (x_i, y_j) and h_8 is the distance from $(x_{south-east}, y_{south-east})$ to (x_i, y_j) .

Now we have to approximate $\Delta^2\psi$ at (x_i, y_j) in case where (x_i, y_j) is a computational point which is in the center of an irregular element. Define a new coordinate system $\eta = (x+y)/\sqrt{2}, \quad \xi = (y-x)/\sqrt{2}$. This yields $y = (\eta+\xi)/\sqrt{2}, \quad x = (\eta-\xi)/\sqrt{2}$.

Expressing $\psi_{\eta\eta\eta\eta}$ and $\psi_{\xi\xi\xi\xi}$ in terms of ψ_{xxxx}, ψ_{xxyy} and ψ_{yyyy}, we have

$$
(3.1) \qquad \begin{cases}
\psi_\eta = \frac{1}{\sqrt{2}}(\psi_x + \psi_y), \\
\psi_{\eta\eta} = \frac{1}{2}(\psi_{xx} + 2\psi_{xy} + \psi_{yy}) \\
\psi_{\eta\eta\eta} = \frac{1}{2\sqrt{2}}(\psi_{xxx} + 3\psi_{xxy} + 3\psi_{xyy} + \psi_{yyy}), \\
\psi_{\eta\eta\eta\eta} = \frac{1}{4}(\psi_{xxxx} + 4\psi_{xxxy} + 6\psi_{xxyy} + 4\psi_{xyyy} + \psi_{yyyy}).
\end{cases}
$$

$$(3.2)\qquad \begin{cases} \psi_\xi = \frac{1}{\sqrt{2}}(\psi_y - \psi_x), \\ \psi_{\xi\xi} = \frac{1}{2}(\psi_{xx} - 2\psi_{xy} + \psi_{yy}), \\ \psi_{\xi\xi\xi} = \frac{1}{2\sqrt{2}}(-\psi_{xxx} + 3\psi_{xxy} - 3\psi_{xyy} + \psi_{yyy}), \\ \psi_{\xi\xi\xi\xi} = \frac{1}{4}(\psi_{xxxx} - 4\psi_{xxxy} + 6\psi_{xxyy} - 4\psi_{xyyy} + \psi_{yyyy}). \end{cases}$$

Therefore,

$$(3.3)\qquad 2(\psi_{\eta\eta\eta\eta} + \psi_{\xi\xi\xi\xi}) = \psi_{xxxx} + 6\psi_{xxyy} + \psi_{yyyy}.$$

Thus,

$$(3.4)\qquad 2\psi_{xxyy} = \frac{2}{3}(\psi_{\eta\eta\eta\eta} + \psi_{\xi\xi\xi\xi}) - \frac{1}{3}(\psi_{xxxx} + \psi_{yyyy}).$$

This yields

$$(3.5)\qquad \begin{aligned} \Delta^2\psi &= \psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy} \\ &= \frac{2}{3}(\psi_{\eta\eta\eta\eta} + \psi_{\xi\xi\xi\xi} + \psi_{xxxx} + \psi_{yyyy}). \end{aligned}$$

Thus, the operator Δ^2 can be expressed via pure fourth-order derivatives in the directions of x, y and η, ξ.

We can therefore approximate $\Delta^2\psi$ by $\tilde{\Delta}_h^2\psi$, where

$$(3.6)\qquad \tilde{\Delta}_h^2\psi = \frac{2}{3}(\delta_\eta^4\psi + \delta_\xi^4\psi + \delta_x^4\psi + \delta_y^4\psi).$$

The discretizations of ψ_{xxxx}, ψ_{yyyy} by $\delta_x^4\psi, \delta_y^4\psi$ and those of $\psi_{\eta\eta\eta\eta}, \psi_{\xi\xi\xi\xi}$ by $\delta_\eta^4\psi, \delta_\xi^4\psi$, respectively, are carried out via one-dimensional approximations of pure fourth-order derivatives.

We describe now the approximation of the convective term $C(\psi) = \nabla^\perp\psi \cdot \nabla\Delta\psi$. This may be written as

$$(3.7)\qquad C(\psi) = -(\partial_y\psi)\cdot(\partial_{xxx}\psi + \partial_{xyy}\psi) + (\partial_x\psi)\cdot(\partial_{xxy}\psi + \partial_{yyy}\psi).$$

The pure third-order derivatives (for example $\partial_{xxx}\psi$) may be approximated to fourth-order accuracy by the interpolation of a one-dimensional fifth-order polynomial using the data of ψ and $\partial_x\psi$ at the three points $x_{i-1,j}, x_{i,j}, x_{i+1,j}$. The mixed third-order derivatives ψ_{xxy} and ψ_{xyy} may be approximated using (3.1-3.2) by

$$(3.8)\qquad \psi_{xxy} = \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} + \psi_{\xi\xi\xi}) - \frac{1}{3}\psi_{yyy}$$

and

$$(3.9)\qquad \psi_{xyy} = \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} - \psi_{\xi\xi\xi}) - \frac{1}{3}\psi_{xxx}.$$

Inserting Equations (3.8)-(3.9) in Equation (3.7), we obtain

$$(3.10)\qquad C(\psi) = -\psi_y\Big(\frac{2}{3}\psi_{xxx} + \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} - \psi_{\xi\xi\xi})\Big) + \psi_x\Big(\frac{2}{3}\psi_{yyy} + \frac{\sqrt{2}}{3}(\psi_{\eta\eta\eta} + \psi_{\xi\xi\xi})\Big).$$

Note that in order to approximate ψ_{xxx} (similarly to other pure third-order derivatives) to fourth-order accuracy, we need to use an interpolating polynomial of degree five. In this case we need a sixth-order approximation to ψ_x (see [2]). In Section 4 we concentrate on the approximation of the biharmonic and the Laplacian operators. We discuss the truncation error due to the various discretizations. In Section 5 we describe the approximation of the convective term at near-boundary points.

Notice that the values of ψ, ψ_x and ψ_y on the boundary points may be computed from the boundary conditions $\psi(x,y,t) = g_1(x,y,t)$ and $\frac{\partial\psi}{\partial n}(x,y,t) = g_2(x,y,t)$, for $(x,y) \in \partial\Omega$. Indeed, ψ is given on the boundary. In addition, from the given ψ on the boundary one may compute the tangential derivative of ψ. The tangential derivative together with the given normal derivative determine ψ_x and ψ_y on the boundary points.

4. Approximation of $\partial_x\psi$, $\partial_x^4\psi$ and $\partial_x^2\psi$ on an irregular mesh. We describe how to approximate $\partial_x\psi$, $\partial_x^4\psi$ and $\partial_x^2\psi$ in case the mesh is irregular.

Let (x_i, y_j) be a grid point where at least one of its neighbors to the right or to the left is inside the domain or on the boundary but its distance to (x_i, y_j) is not h. Define the neighbor of (x_i, y_j) to the right by (x_{east}, y_j) and its neighbor to the left by (x_{west}, y_j). Let $h_1 = x_{east} - x_i$ and $h_2 = x_i - x_{west}$.

By the requirements we set in Section 2 on a computational point, we find that there exist positive constants, which does not depend on the mesh size, such that

$$C_1 \le h_1/h \le C_2, \qquad C_1 \le h_2/h \le C_2. \tag{4.1}$$

Let $Q(x)$ be a polynomial of degree less or equal 4.

$$Q(x) = a_0 + a_1(x - x_i) + a_2(x - x_i)^2 + a_3(x - x_i)^3 + a_4(x - x_i)^4. \tag{4.2}$$

The interpolating data is

$$\psi(x_{west}, y_j), \psi(x_i, y_j), \psi(x_{east}, y_j), \psi_x(x_{west}, y_j), \psi_x(x_{east}, y_j). \tag{4.3}$$

Then, a_1 is set as an approximation to $\partial_x\psi$ at (x_i, y_j) and is denoted by $\psi_{x,i,j}$. We also set $24a_4$ as an approximation to $\partial_x^4\psi$ at (x_i, y_j) and denote it by $\delta_x^4\psi_{i,j}$. In a similar manner one approximates other first and fourth-order pure derivatives with respect to y.

We now describe in detail the approximation of $\partial_x\psi$ at (x_i, y_j). Define

$$\begin{cases} c = \frac{4h_1h_2^3 - 4h_1^3h_2 + 2h_2^4 - 2h_1^4}{h_1h_2(h_1+h_2)^3}, \\ c_p = \frac{2h_2^4 + 4h_1h_2^3}{h_1h_2(h_1+h_1)^3}, \\ c_m = \frac{2h_1^4 + 4h_2h_1^3}{h_1h_2(h_1+h_2)^3}, \\ cx_p = \frac{h_2^3h_1^2 + h_2^4h_1}{h_1h_2(h_1+h_2)^3}, \\ cx_m = \frac{h_2^2h_1^3 + h_1^4h_2}{h_1h_2(h_1+h_2)^3}. \end{cases} \tag{4.4}$$

Then, the approximation $\psi_{x,i,j}$ to $\partial_x\psi_{i,j}$ is given by

$$\begin{aligned} &\psi_{x,i,j} + cx_p \cdot \psi_x(x_{east}, y_j) + cx_m \cdot \psi_x(x_{west}, y_j) \\ &\quad = c_p \cdot \psi(x_{east}, y_j) - c_m \cdot \psi(x_{west}, y_j) - c \cdot \psi_{i,j}. \end{aligned} \tag{4.5}$$

Note that in case $h_1 = h_2 = h$ then (4.5) is equivalent to (2.2). Similar representations are valid to $\partial_y \psi$.

One can show, using Taylor expansions that the truncation error for ψ_x for an irregular element is bounded as follows.

$$|(\psi_x)_{i,j} - \partial_x \psi| \le Ch^4 \|\psi^{(5)}\|_{L^\infty}. \tag{4.6}$$

The derivatives along the diagonals $\partial_\eta \psi$ and $\partial_\eta^4 \psi$ and $\partial_\xi \psi$ and $\partial_\xi^4 \psi$ are approximated using the chain rule $\psi_\eta = \frac{1}{\sqrt{2}}(\psi_x + \psi_y)$, and $\psi_\xi = \frac{1}{\sqrt{2}}(\psi_y - \psi_x)$.

For the approximation of $\partial_x^4 \psi$ at an irregular point we define

$$\begin{cases} b = 24\frac{(h_1+h_2)^3}{h_1^2 h_2^2 (h_1+h_2)^3}, \\ b_p = 24\frac{h_2+3h_1}{h_1^2(h_1+h_2)^3}, \\ b_m = 24\frac{h_1+3h_2}{h_2^2(h_1+h_2)^3}, \\ bx_p = 24\frac{h_1+h_2}{h_1(h_1+h_2)^3}, \\ bx_m = 24\frac{h_1+h_2}{h_2(h_1+h_2)^3}. \end{cases} \tag{4.7}$$

Then, the approximation $\bar{\delta}_x^4 \psi_{i,j}$ to $\partial_x^4 \psi$ is given by

$$\begin{aligned} \bar{\delta}_x^4 \psi_{i,j} &= bx_p \cdot \psi_x(x_{east}, y_j) - bx_m \cdot \psi_x(x_{west}, y_j) \\ &\quad -(b_p \cdot \psi(x_{east}, y_j) + b_m \cdot \psi(x_{west}, y_j) - b \cdot \psi_{i,j}). \end{aligned} \tag{4.8}$$

Note that in case $h_1 = h_2 = h$ then (4.8) is equivalent to (2.4). Similar representations are valid to $\partial_y^4 \psi$, and to derivatives along the diagonals $\partial_\eta^4 \psi$ and $\partial_\xi^4 \psi$, given that $\partial_\eta \psi$ and $\partial_\xi \psi$ are approximated using the chain rule.

It is possible to show, using Taylor expansions and (4.1), that if the values of ψ_x are chosen as the exact values of the first-order derivative, then the truncation error for the approximation of the $\partial_x^4 \psi$ for an irregular element is bounded as follows.

$$|\bar{\delta}_x^4 \psi_{i,j} - \partial_x^4 \psi| \le Ch \|\psi^{(5)}\|_{L^\infty}. \tag{4.9}$$

For the approximation of $\partial_x^2 \psi$ at an irregular point we define

$$\begin{cases} d = 2\frac{8(h_2^2 h_1^3 + h_2^3 h_1^2) + h_2 h_1^4 + h_2^4 h_1 - h_2^5 - h_1^5}{h_2^2 h_1^2 (h_1+h_2)^3}, \\ d_p = 2\frac{h_2(-h_2^2 + 8h_1^2 + h_2 h_1)}{h_1^2(h_1+h_2)^3}, \\ d_m = 2\frac{h_1(-h_1^2 + 8h_2^2 + h_2 h_1)}{h_2^2(h_1+h_2)^3}, \\ dx_p = 2\frac{h_2(2h_1^2 + h_2 h_1 - h_2^2)}{h_1(h_1+h_2)^3}, \\ dx_m = 2\frac{h_1(2h_2^2 + h_2 h_1 - h_1^2)}{h_2(h_1+h_2)^3}. \end{cases} \tag{4.10}$$

Then, the approximation $\bar{\delta}_x^2\psi_{i,j}$ to $\partial_x^2\psi$ is given by

$$\begin{aligned}\bar{\delta}_x^2\psi_{i,j} &= d_p\cdot\psi(x_{east},y_j)+d_m\cdot\psi(x_{west},y_j)-d\cdot\psi_{i,j}\\ &\quad -(dx_p\cdot\psi_x(x_{east},y_j)-dx_m\cdot\psi_x(x_{west},y_j)).\end{aligned}\tag{4.11}$$

Note that in case $h_1=h_2=h$ then (4.11) is equivalent to the approximation of $\delta_x^2\psi$ in (2.1). Similar representations are valid for $\partial_y^2\psi$.

One can show, using Taylor expansions and (4.1), that if the values of ψ_x are chosen as the exact values of the first-order derivative, then the truncation error for the approximation of the $\partial_x^2\psi$ for an irregular element is bounded as follows.

$$|\bar{\delta}_x^2\psi_{i,j}-\partial_x^2\psi|\le Ch^3\|\psi^{(5)}\|_{L^\infty}.\tag{4.12}$$

5. Approximation of convective term on an irregular mesh. In order to approximate the convective term (3.7) (or its equivalent form (3.10)), we have to discretize pure third-order derivatives of ψ in x,y and in ξ,η. Note that we have already obtained fourth-order approximations to $\partial_x\psi$ and $\partial_y\psi$ (see (4.5)).

In [2] we have constructed a sixth-order approximation to the first-order derivative, using a sixth-order interpolating polynomial based on the interpolating values $\psi_{i-2,j},\psi_{i-1,j},\psi_{i,j},\psi_{i+1,j},\psi_{i+1,j}$ and $\psi_{x,i-1,j},\psi_{x,i,j},\psi_{x,i+1,j}$. Then we inserted these values into an approximation of $\partial_x^3\psi$, based on a fifth-order polynomial. The latter interpolates the values $\psi_{i-1,j},\psi_{x,i-1,j},\psi_{i,j},\psi_{x,i,j},\psi_{i+1,j},\psi_{x,i+1,j}$ and the resulting approximation was fourth-order accurate for $\partial_x^3\psi$.

We first describe the approximation to $\partial_x^3\psi$ and then show how to obtain a higher-order approximation to the first-order derivative. Let (x_i,y_j) be a grid point where two of its neighbors to the right (x_{west},y_j) and to the left (x_{east},y_j) are inside the domain or on the boundary. Define $h_1=x_{east}-x_i$ and $h_2=x_i-x_{west}$. Define

$$\begin{cases} q=12\frac{h_2^3-4h_2^2h_1+4h1^2h_2-h_1^3}{h_1^3h_2^3},\\[2ex] q_p=12\frac{h_2(h_1h_2-h_2^2+5h_1^2)}{h_1^3(h_1+h_2)^3},\\[2ex] q_m=-12\frac{h_1(h_1h_2-h_1^2+5h_2^2)}{h_2^3(h_1+h_2)^3},\\[2ex] qx=6\frac{h_2^2-4h_1h_2+h_1^2}{h_1^2h_2^2},\\[2ex] qx_p=-6\frac{h_2(2h_1-h_2)}{h_1^2(h_1+h_2)^2},\\[2ex] qx_m=-6\frac{h_1(2h_2-h_1)}{h_2^2(h_1+h_2)^2}.\end{cases}\tag{5.1}$$

Then, the approximation $\bar{\delta}_x^3\psi_{i,j}$ to $\partial_x^3\psi$ is given by

$$\begin{aligned}\bar{\delta}_x^3\psi_{i,j} &= q\cdot\psi_{i,j}+q_p\cdot\psi(x_{east},y_j)+q_m\cdot\psi(x_{west},y_j)\\ &\quad +qx\cdot\psi_x(x_i,y_j)+qx_p\cdot\psi_x(x_{east},y_j)+qx_m\cdot\psi_x(x_{west},y_j)).\end{aligned}\tag{5.2}$$

Other pure third-order derivatives may be similarly discretized. Note that in case $h_1=h_2=h$ then (5.2) is equivalent to the approximation of $\delta_x^3\psi$ in Equation (3.29) of [2].

One may show, using Taylor expansions and (4.1), that if the values of ψ_x are chosen as the exact values of the first-order derivative, then the truncation error for the approximation of the $\partial_x^3\psi$ for an irregular element is bounded as follows.

$$|\bar{\delta}_x^3\psi_{i,j} - \partial_x^3\psi| \le Ch^3\|\psi^{(6)}\|_{L^\infty}. \tag{5.3}$$

Note that when we approximated to $\partial_x^3\psi$ to fourth-order accuracy on a uniform mesh we needed to to approximate $\partial_x\psi$ to sixth-order accuracy, so that by using theses values for the first-order derivative, one can obtain a fourth-order approximation to $\partial_x^3\psi$. The construct analogue of the sixth-order approximation to ψ_x, derived this time for a non-uniform grid may be constructed as well.

6. Numerical accuracy of the the scheme in irregular domains. In order to verify the spatial fourth order accuracy of the scheme, we performed several numerical tests. The time-step was set to $dt = Ch^2$.

In the Tables below we present the error, e, and the relative error, where

$$e_2 = \|\psi_{comp} - \psi_{exact}\|_{l_2},$$

$$e_\infty = \|\psi_{comp} - \psi_{exact}\|_{l_\infty}.$$

Similarly,

$$(e_x)_2 = \|(\psi_x)_{comp} - (\psi_x)_{exact}\|_{l_2},$$

$$(e_x)_\infty = \|(\psi_x)_{comp} - (\psi_x)_{exact}\|_{l_\infty},$$

Here, ψ_{comp},$(\psi_x)_{comp}$ and ψ_{exact},$(\psi_x)_{exact}$ are the computed and the exact streamfunction and of ψ and its $x-$ derivative, respectively.

6.0.1. Case 1: Navier-Stokes with exact solution $\psi(x,y,t) = e^{x+y-t}$ in a unit circle. Here

$$f(x,y,t) = \partial_t\Delta\psi + \nabla^\perp\psi\cdot\nabla\Delta\psi - \Delta^2\psi, \tag{6.1}$$

where $\psi(x,y,t) = e^{x+y-t}$.

Our aim is to recover $\psi(x,y,t)$ from $f(x,y,t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t\Delta\psi + \nabla^\perp\psi\cdot\nabla\Delta\psi - \Delta^2\psi = f(x,y,t), \quad (x,y)\in\Omega \\ \psi(x,y,0) = e^{x+y}, \quad (x,y)\in\Omega \\ \psi(x,y,t) = e^{x+y-t}, \quad (x,y)\in\partial\Omega \\ \frac{\partial\psi(x,y,t)}{\partial n} = \frac{\partial e^{x+y-t}}{\partial n}, \quad (x,y)\in\partial\Omega. \end{cases} \tag{6.2}$$

mesh	9×9	Rate	17×17	Rate	33×33
e_2	9.955E-05	4.31	5.0042E-06	4.01	2.998E-07
e_∞	1.6792E-04	4.70	6.4755E-06	3.55	5.5991E-07
$(e_x)_2$	3.0959E-04	4.40	1.4634E-05	3.80	1.0508E-06
$(e_x)_\infty$	6.6237E-04	4.09	3.8936E-05	3.11	4.5138E-06

Table 1: Compact scheme for the Navier-Stokes equation with exact solution: $\psi = e^{x+y-t}$ on $x^2+y^2 \le 1$. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x\psi$. Here $\Delta t = 0.25h^2$ and $t = 0.25$.

6.0.2. Case 2: Navier-Stokes equation with exact solution $\psi(x,y,t) = (1-x^2)^3(1-y^2)^3e^{-t}$ on a unit circle. Here

$$f(x,y,t) = \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi, \tag{6.3}$$

where $\psi(x,y,t) = (1-x^2)^3(1-y^2)^3e^{-t}$. Our aim is to recover $\psi(x,y,t)$ from $f(x,y,t)$. Thus, we resolve numerically

$$\begin{cases} \partial_t \Delta\psi + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \Delta^2\psi = f(x,y,t), \quad (x,y) \in \Omega \\ \psi(x,y,0) = (1-x^2)^3(1-y^2)^3, \quad (x,y) \in \Omega \\ \psi(x,y,t) = (1-x^2)^3(1-y^2)^3e^{-t}, \quad (x,y) \in \partial\Omega \\ \frac{\partial\psi(x,y,t)}{\partial n} = \frac{\partial(1-x^2)^3(1-y^2)^2e^{-t}}{\partial n}, \quad (x,y) \in \partial\Omega. \end{cases} \tag{6.4}$$

mesh	9×9	Rate	17×17	Rate	33×33
e_2	1.1040E-02	4.56	4.6817E-04	4.06	2.8002E-05
e_∞	1.2010E-02	4.57	5.0701E-04	4.30	2.5781E-05
$(e_x)_2$	2.7300E-02	4.33	1.3530E-03	4.60	5.5872E-05
$(e_x)_\infty$	3.7950E-02	4.12	2.1861E-03	4.10	1.2750E-04

Table 2: Compact scheme for Navier-Stokes equation with exact solution: $\psi = (1-x^2)^3(1-y^2)^3e^{-t}$ on $x^2+y^2 \le 1$. We present e and e_x, the l_2 errors for the streamfunction and for $\partial_x\psi$. Here $\Delta t = 0.25h^2$ and $t = 0.25$.

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