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DISCRETE DUALITY FINITE VOLUME SCHEME FOR SOLVING HESTON MODEL *

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Abstract. New numerical scheme for tensor diffusion equation based on discrete duality finite volume (DDFV) method is derived. Tensor diffusion equation represents an important model in many fields of science. We focused our attention to the problem which arises in financial mathematics and is known as 2D Heston model see [5]. Existence and uniqueness of numerical solution is derived and numerical experiment using proposed scheme are included.

Key words. Heston model, tensor diffusion, discrete duality finite volume method, existence and uniqueness of the numerical solution

AMS subject classifications. 35K20, 65M08, 65M22, 35Q91

1. Introduction. Heston model in transformed form can be represent by

$$\frac{\partial u}{\partial \tau} + \overrightarrow{A} \cdot \nabla u = \nabla \cdot (\mathbf{B} \nabla u) - ru, \text{in } \Omega \times [t_1, t_2], \tag{1.1}$$

where unknown function is $u = u(x, y, \tau)$ and

$$B = \frac{1}{2} y \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}, \quad \overrightarrow{A} = -\begin{pmatrix} r - \frac{1}{2}y - \frac{1}{2}\rho\sigma \\ \kappa(\theta - y) - \lambda y - \frac{1}{2}\sigma^2 \end{pmatrix}. \tag{1.2}$$

Ω is rectangular 2D domain $\Omega = (X_r, X_l) \times (0, Y)$ and $[t_1, t_2]$ is time interval: $0 < t_1 < t_2 < \infty$. The financial parameters of the model have the following properties: $\rho \in \langle -1, 1, \rangle$, $\sigma > 0$, $r > 0$, $\kappa > 0$, $\theta > 0$, $\lambda > 0$. For further details see [5] and references therein.

The above problem can be endowed with the initial condition and boundary conditions of the type

$$u(x, y, 0) = max\left(0, e^x - 1\right), \tag{1.3}$$

$$\begin{gathered} u(X_a, y, \tau) = 0, \quad u(X_b, y, \tau) = e^{X_b} - e^{-r\tau}, \\ \frac{\partial u(x, Y, \tau)}{\partial \mathbf{n}} = 0, \quad \frac{\partial u(x, 0, \tau)}{\partial \mathbf{n}} = 0, \end{gathered} \tag{1.4}$$

where $\mathbf{n}$ is the outward normal to the boundary $\partial\Omega$.

2. Discretization. We present here fully implicit scheme. For discretization in time we use uniform discrete time step $k = \frac{t_2 - t_1}{N}$ and $t_n = nk$ for $n = 0, 1, \ldots N$. The time derivative is approximated using the backward difference. We denote by u^n the piecewise constant function on each time interval $[t_{n-1}, t_n]$, $n = 1, \ldots N$. We have

$$\frac{u^n - u^{n-1}}{k} = \nabla . \left(B \nabla u^n\right) - \nabla \cdot \left(\overrightarrow{A} u^n\right) + \left(\nabla \cdot \overrightarrow{A}\right) u^n - ru^n. \tag{2.1}$$

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2.1. Classical Finite volume method in 2D. For the space discretization we use method based on finite volumes. That means our numerical solution will be piecewise constant function on each finite volume and each time interval $[t_{n-1}, t_n]$. For the sake of simplicity in subsections 2.1 and 2.2 we will use the notation: $x := (x, y) \in \Omega$. Moreover in financial mathematics (Heston model) and in image processing problems (anisotropic tensor diffusion model) we often use the rectangular domain in 2D. Following [4] our finite volume mesh will consist of cells $V_{ij} \in \mathcal{T}_h$ with the

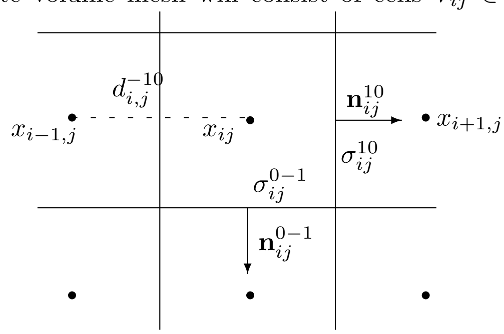

Fig. 2.1. *Finite volume discretization*

measure $m(V_{ij})$, associated with DF nodes $x_{ij} := (x_{ij}, y_{ij}) \in V_{ij}$, say $i = 1, \ldots, N_1$, $j = 1, \ldots, N_2$ in such a way that $\overline{\Omega} = \bigcup_{V_{ij} \in \mathcal{T}_h}$. Further we denote σ_{ij}^{pq} with the 1D measure $m(\sigma_{ij}^{pq})$ the edges of finite volume V_{ij} and the distance between the neighbouring representative points we denote by $|x_{ij} - x_{i+p,j+q}| = d_{ij}^{pq},\ |p| + |q| = 1$. Especially we denote $d_{1j}^{-10} = dist(x_{1j}, \partial\Omega)$ and $d_{N_1 j}^{10} = dist(x_{N_1 j}, \partial\Omega)$ due to Dirichlet boundary conditions. Unit outward normal vector to the edge σ_{ij}^{pq} we denote by $\mathbf{n}_{ij}^{pq},\ |p| + |q| = 1$ as is shown in Figure 2.1

Integrate the equation (2.1) over finite volume V_{ij}, using the same form for advection term as in [5], use divergence theorem and the fact that u_{ij}^n is a piecewise constant function on finite volume V_{ij}, and time interval $[t_{n-1}, t_n]$, we obtain

$$\frac{u_{ij}^n - u_{ij}^{n-1}}{k} m(V_{ij}) = \sum_{|p|+|q|=1} \int_{\sigma_{ij}^{pq}} B \nabla u^n \mathbf{n_{ij}^{pq}} ds - \tag{2.2}$$
$$\sum_{|p|+|q|=1} \int_{\sigma_{ij}^{pq}} \overrightarrow{A} u^n \mathbf{n_{ij}^{pq}} ds + u_{ij}^n \int_{V_{ij}} \left(\nabla \cdot \overrightarrow{A} \right) dx - r u_{ij}^n m(V_{ij}).$$

We further denote the coefficient for the tensor and advection term for each finite volume V_{ij} and each edge σ_{ij}^{pq} in the form

$$B_{ij}^{pq} = \begin{pmatrix} b_{ij,pq}^{11} & b_{ij,pq}^{12} \\ b_{ij,pq}^{21} & b_{ij,pq}^{22} \end{pmatrix}, \quad \overrightarrow{A}_{ij}^{pq} = \begin{pmatrix} a_{ij,pq}^{1} \\ a_{ij,pq}^{2} \end{pmatrix}. \tag{2.3}$$

Remark 2.1. For our numerical approximation we can express all coefficients as a constant function on the whole edge for example by a value at the central point of an edge, which will be very useful for our purpose as we will see in the next section.

So the question now is to approximate the gradients on edges of the finite volume properly. As we can see in Figure 2.2, the natural way is to approximate the gradient as a piecewise constant function on diamond around the edges. The only problem is how to approximate the values in corner of the finite volume. One possibility is used for example in [5] where the average values of the neighbouring values in representative points are used. Another possibility will be described below.

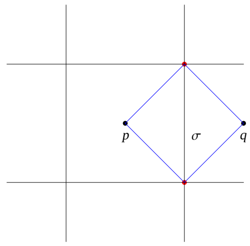

FIG. 2.2. *Rectangular mesh (black lines) and one diamond around the edge σ (blue lines) with representative point p, q of neighbouring finite volumes*

2.2. Discrete duality finite volume method. The main idea is derived in [3] and [1] for elliptic equations and for parabolic type equation in [4]. For much more general problems very important results are in [2] but they do not cover the tensor diffusion term as in our model.

In this case we have two meshes first one as in classical finite volumes and second one, we called it dual mesh is shifted to the north-east direction, consists of cells $\overline{V}_{ij} \in \overline{\mathcal{T}}_h$ with measure $m(\overline{V}_{ij})$ associated only with DF nodes $\overline{x}_{ij}$, say $i = 0, \ldots, N_1$, $j = 0, \ldots, N_2$ in such a way that $\overline{x}_{ij}$ is the right top corner for the volume V_{ij} of the original mesh - see Figure 2.3. Again, all inner dual finite volumes are rectangles and boundary volumes are created in such a way that $\overline{\Omega} = \bigcup_{\overline{V}_{ij} \in \overline{\mathcal{T}}_h}$. All other entities are denote in similar way as for primal mesh but "barred". The new unknown function $\overline{u}_{h,k}(t, x)$ will be given by discrete values in the corners of original rectangles and is again piecewise constant function in space and time. We can define constant

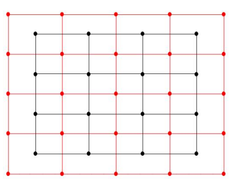

FIG. 2.3. *Example of sets of finite volumes for primal (red) and dual (black) meshes*

gradients on diamond cells as we can see in Figure 2.4, which is the union of $\mathcal{D}_h$ and $\overline{\mathcal{D}}_h$, where

$$\mathcal{D}_h = \bigcup_{(i,j)=(0,0),\ldots,(N_1,N_2)} D_{ij},$$

where D_{ij} has the vertices $\{x_{ij}, \overline{x}_{i,j-1}, x_{i+1,j}, \overline{x}_{ij}, \}$ with degenerated (triangles) diamonds on the boundaries (for $i = 0$, or $i = N_1$) and

$$\overline{\mathcal{D}}_h = \bigcup_{(i,j)=(0,0),\ldots,(N_1,N_2)} \overline{D}_{ij},$$

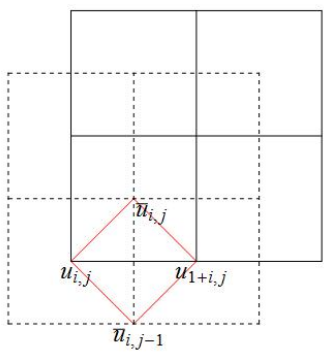

FIG. 2.4. *Diamond cell (red lines), where the gradient is constant, primal mesh (dashed lines), dual mesh (solid lines)*

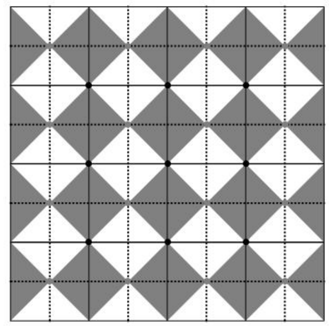

FIG. 2.5. *Diamonds D_{ij} (gray) and diamonds $\overline{D}_{ij}$ (white) with primal-dual mesh and unknowns u (gray points) and $\overline{u}$ (black points)*

where $\overline{D}_{ij}$ has the vertices $\{x_{ij}, \overline{x}_{ij}, x_{i,j+1}, \overline{x}_{i-1,j}\}$ with degenerated (triangles) diamonds on the boundaries (for $j = 0$, or $j = N_2$). Under this notation it is clear that $\overline{\Omega} = \mathcal{D}_h \cup \overline{\mathcal{D}}_h$ (see Figure 2.5).

Then the gradient can be define in the form:

$$\nabla u_{ij} = \left(\frac{u_{i+1,j} - u_{ij}}{d_{ij}^{10}}, \frac{\overline{u}_{ij} - \overline{u}_{i,j-1}}{m(\sigma_{ij}^{10})} \right) = (u_x^{ij}, u_y^{ij}) \text{ on } D_{ij}, \tag{2.4}$$

$$\nabla \overline{u}_{ij} = \left(\frac{\overline{u}_{ij} - \overline{u}_{i-1,j}}{m(\sigma_{ij}^{01})}, \frac{u_{i,j+1} - u_{ij}}{d_{ij}^{01}} \right) = (\overline{u}_x^{ij}, \overline{u}_y^{ij}) \text{ on } \overline{D}_{ij}.$$

and similarly for the time dependent gradient at time level n.

To obtain numerical scheme for both primal and dual meshes, we must approximate all terms on the right hand side of (2.2). Notice that when at least one edge of the finite volume belongs to ∂D then in these points boundary conditions are take into account, that means for the Dirichlet boundary condition the unknowns $u_{0,j}^n = 0$ for $j = 1, \ldots, N2$, $\overline{u}_{0,j}^n = 0$ for $j = 1, \ldots, N2-1$, $u_{N1+1,j}^n = e^{X_b} - e^{-rkn}$ for $j = 1, \ldots, N2$, $\overline{u}_{N1,j}^n = e^{X_b} - e^{-rkn}$ for $j = 1, \ldots, N2-1$. For zero Neumann boundary condition we pose at all additional so called ghost points values $u_{i0} = u_{i1},\ u_{i,N2+1} = u_{i,N2}$ for $1 = 1, \ldots, N1$, and $\overline{u}_{i0} = \overline{u}_{i1},\ \overline{u}_{i,N2} = \overline{u}_{i,N2+1}$ for $1 = 1, \ldots, N1-1$. We will derive it for the primal mesh precisely. It can be done for the dual mesh analogously. Our computation domain is rectangle, so we can construct both meshes as super admissible mesh consisting of rectangles only with edges h_x in x-direction and h_y in y direction. In this case $m(V_{ij}) = m(\overline{V}_{ij}) = h_x h_y,\ m(\sigma_{ij}) = \overline{d}_{ij} = h_y, d_{ij} = m(\overline{\sigma}_{ij}) = h_x$. Under this simplification we can approximate first term on the right hand side of (2.2) in the form

$$\sum_{|p|+|q|=1} \int_{\sigma_{ij}^{pq}} B \nabla u^n \mathbf{n}_{\mathbf{ij}}^{\mathbf{pq}} ds \approx h_y[b_{ij,10}^{11} u_x^{ij,n} + b_{ij,10}^{12} u_y^{ij,n}] + h_x[b_{ij,01}^{21} \overline{u}_x^{ij,n} + b_{ij,01}^{22} \overline{u}_y^{ij,n}] -$$
$$h_y[b_{ij,-10}^{11} u_x^{i-1j,n} + b_{ij,-10}^{12} u_y^{i-1j}] - h_x[b_{ij,0-1}^{21} \overline{u}_x^{ij-1,n} + b_{ij,0-1}^{22} \overline{u}_y^{ij-1,n}].$$

Second term can be express

$$\sum_{|p|+|q|=1} \int_{\sigma_{ij}^{pq}} \overrightarrow{A} u^n \mathbf{n_{ij}^{pq}} ds \approx h_y a^1_{ij,10} \frac{u^n_{ij} + u^n_{i+1,j}}{2} + h_x a^2_{ij,01} \frac{u^n_{ij} + u^n_{i,j+1}}{2} -$$

$$h_y a^1_{ij,-10} \frac{u^n_{ij} + u^n_{i-1,j}}{2} - h_x a^2_{ij,0-1} \frac{u^n_{ij} + u^n_{i,j-1}}{2}.$$

and third term

$$u^n_{ij} \int_{V_{ij}} \left(\nabla \cdot \overrightarrow{A}\right) dx \approx u^n_{ij} \left(h_y a^1_{ij,10} + h_x a^2_{ij,01} - h_y a^1_{ij,-10} - h_x a^2_{ij,0-1}\right).$$

We can write numerical scheme for unknown value u_{ij}

$$\begin{aligned}
&\frac{u^n_{ij} - u^{n-1}_{ij}}{k} h_x h_y + r u^n_{ij} h_x h_y - \\
&h_y[b^{11}_{ij,10} u_x^{ij,n} + b^{12}_{ij,10} u_y^{ij,n}] - h_x[b^{21}_{ij,01} \overline{u}_x^{ij,n} + b^{22}_{ij,01} \overline{u}_y^{ij,n}] + \\
&h_y[b^{11}_{ij,-10} u_x^{i-1j,n} + b^{12}_{ij,-10} u_y^{i-1j,n}] + h_x[b^{21}_{ij,0-1} \overline{u}_x^{ij-1,n} + b^{22}_{ij,0-1} \overline{u}_y^{ij-1,n}] + \\
&h_y a^1_{ij,10} \frac{u^n_{i+1j} - u^n_{ij}}{2} + h_x a^2_{ij,01} \frac{u^n_{ij+1} - u^n_{ij}}{2} - \\
&h_y a^1_{ij,-10} \frac{u^n_{i-1j} - u^n_{ij}}{2} - h_x a^2_{ij,0-1} \frac{u^n_{ij-1} - u^n_{ij}}{2} = 0.
\end{aligned} \tag{2.5}$$

For unknown value $\overline{u}_{ij}$ we now use the Remark 2.1. If we denote for a moment the coefficients for advection vector and diffusion tensor with "bars" we can derive that for all coefficients of the matrix $\mathbf{B}$ from (2.3) we have $\overline{b}_{ij,10} = b_{i+1,j+1,0-1}$ $\overline{b}_{ij,01} = b_{i+1,j+1,-10}$ $\overline{b}_{ij,-10} = b_{i,j,01}$ $\overline{b}_{ij,0-1} = b_{i,j,10}$. And the same is true for the vector $\overrightarrow{A}$. We have

$$\begin{aligned}
&\frac{\overline{u}^n_{ij} - \overline{u}^{n-1}_{ij}}{k} h_x h_y + r \overline{u}^n_{ij} h_x h_y - \\
&h_y[b^{11}_{i+1j+1,0-1} \overline{u}_x^{i+1,j} + b^{12}_{i+1j+1,0-1} \overline{u}_y^{i+1,j}] - h_x[b^{21}_{i+1j+1,-10} u_x^{i,j+1} + b^{22}_{i+1j+1,-10} u_y^{i,j+1}] \\
&+h_y[b^{11}_{ij,01} \overline{u}_x^{i,j} + b^{12}_{ij,01} \overline{u}_y^{i,j}] + h_x[b^{21}_{ij,11} u_x^{i,j} + b^{22}_{ij,10} u_y^{i,j}] + \\
&+h_y a^1_{i+1j+1,0-1} \frac{\overline{u}^n_{i+1j} - \overline{u}^n_{i,j}}{2} + h_x a^2_{i+1j+1,-10} \frac{\overline{u}^n_{ij+1} - \overline{u}^n_{i,j}}{2} - \\
&h_y a^1_{ij,01} \frac{\overline{u}^n_{i-1j} - \overline{u}^n_{i,j}}{2} - h_x a^2_{ij,10} \frac{\overline{u}^n_{ij-1} - \overline{u}^n_{i,j}}{2} = 0.
\end{aligned} \tag{2.6}$$

3. Existence and uniqueness of the numerical solution. At each time step $t_n = nk$ we must solve linear system of equations for unknown values $u^n_{ij}, i = 1, \ldots N1,\ j = 1, \ldots N2$ and $\overline{u}^n_{ij}, i = 1, N1-1,\ j-1, \ldots N2-1$ described in (2.5) and (2.6). For further considerations we must use the properties of our data, that means the coefficients of matrix $\mathbf{B}$ and advection vector $\overrightarrow{A}$ and a special construction of our primal and dual mesh. Moreover we use $a^1_{ij} = \frac{1}{2} y_{ij} + \frac{1}{2}\rho\sigma - r$ and $a^2_{ij} = \frac{1}{2}\sigma^2 - \kappa\theta + \kappa y_{ij}$.

THEOREM 3.1. *Let the discretization mesh has the properties described in section 2 and* $k,\ h_x,\ h_y$ *have the same meaning. We denote by* $C = \max\{\frac{Y}{2}, r, \frac{\sigma^2}{2}, \kappa\theta, \kappa + \lambda\}$. *Let us suppose*

$$\sigma \geq \rho, \quad 1 \geq \rho\sigma. \tag{3.1}$$

Let the time step k fulfils the condition

$$k \leq \frac{\max\{h_x, h_y\}}{C}. \tag{3.2}$$

Then for each $n = 1, \ldots N$ there exists unique solution to the numerical scheme (2.5), (2.6).

Proof. The existence and the uniqueness of numerical solution can be proved in such a way that for homogeneous linear system there exists unique zero solution. For this purpose we can rewrite our linear algebraic system in the form: $\mathbf{Lu_h^n} = \mathbf{f}$, where $u_h^n = \left(u_{11}^n, \overline{u}_{11}^n, \ldots, \overline{u}_{N1-1,N2-1}^n, u_{N1,N2}^n\right)$. The equation for unknown value u_{ij}^n for an interior point is of the form

$$\begin{aligned}
u_{ij}^n h_x h_y - k h_y [b_{ij,10}^{11} u_x^{ij,n} + b_{ij,10}^{12} u_y^{ij,n}] - k h_x [b_{ij,01}^{21} \overline{u}_x^{ij,n} + b_{ij,01}^{22} \overline{u}_y^{ij,n}] + \\
k h_y [b_{ij,-10}^{11} u_x^{i-1j,n} + b_{ij,-10}^{12} u_y^{i-1j,n}] + k h_x [b_{ij,0-1}^{21} \overline{u}_x^{ij-1,n} + b_{ij,0-1}^{22} \overline{u}_y^{ij-1,n}] + \\
k h_y a_{ij,10}^1 \frac{u_{i+1j}^n - u_{ij}^n}{2} + k h_x a_{ij,01}^2 \frac{u_{ij+1}^n - u_{ij}^n}{2} - \\
k h_y a_{ij,-10}^1 \frac{u_{i-1j}^n - u_{ij}^n}{2} - k h_x a_{ij,0-1}^2 \frac{u_{ij-1}^n - u_{ij}^n}{2} + \\
k r u_{ij}^n h_x h_y = u_{ij}^{n-1} h_x h_y.
\end{aligned} \tag{3.3}$$

For the unknown value u_{ij}^n for any point neighbouring to the non zero Dirichlet boundary that means points $x_{N1,j}$ and for the unknown value u_{ij}^n for any point neighbouring to the zero Dirichlet boundary that means points $x_{1,j}$ we have little bit another situation, because we must split appropriate approximation of space derivatives where their members are the nodes on the boundary. For the page limit reasons we omit it here. This can be done analogously for unknown value $\overline{u}_{ij}$.

We notice that h_y is the size of vertical diagonal of each D_{ij} and h_x the size of horizontal diagonal of D_{ij}. We denote by b_{ij}, a_{ij} the coefficients of the tensor and convection terms from the equation evaluated at the barycentre of D_{ij}. Analogously h_x is the size horizontal diagonal of $\overline{D}_{ij}$, h_y the size of vertical diagonal of $\overline{D}_{ij}$ and $\overline{b}_{ij}, \overline{a}_{ij}$ the coefficients of the tensor and convection terms from the equation evaluated at the barycentre of $\overline{D}_{ij}$. Finally we denote by $\mathcal{D}_{h,int}$ the set of all diamonds D_{ij} for $i = 1, \ldots N1 - 1,\ j = 1, \ldots N2$ and by $\overline{\mathcal{D}}_{h,int}$ the set of all diamonds $\overline{D}_{ij}$ for $i = 2, \ldots N1 - 1,\ j = 1, \ldots N2 - 1$. That means the terms for diamonds that have at least one point belonging to the Dirichlet boundary condition we will treat extra.

Let us now pose the right hand side of all rows as zeros. First we multiply all equations (3.3) for zero right hand side by u_{ij}^n and sum over all finite volumes of the primal mesh. Using the usual finite volume property we have

$$\begin{aligned}
&\sum_{V_{ij} \in \mathcal{T}} (1 + kr)(u_{ij}^n)^2 h_x h_y + \\
&k \sum_{D_{ij} \in \mathcal{D}_{h,int}} \left(h_x h_y \left(b_{ij}^{11} (u_x^{ij,n})^2 + b_{ij}^{12} u_y^{ij,n} u_x^{ij,n} \right) + h_y \frac{a_{ij}^1}{2} ((u_{i+1j}^n)^2 - (u_{ij}^n)^2) \right) + \\
&k \sum_{\overline{D}_{ij} \in \overline{\mathcal{D}}_{h,int}} \left(h_x h_y \left(\overline{b}_{ij}^{22} (\overline{u}_y^{ij,n})^2 + \overline{b}_{ij}^{21} \overline{u}_x^{ij,n} \overline{u}_y^{ij,n} \right) + h_x \frac{\overline{a}_{ij}^2}{2} ((u_{ij+1}^n)^2 - (u_{ij}^n)^2) \right) + \\
&\qquad k h_y \sum_{j=1}^{N2} \left(\left(\frac{2 b_{N1,j}^{11}}{h_x} - \frac{a_{N1j}^1}{2} \right) (u_{N1j}^n)^2 + \left(\frac{2 b_{1,j}^{11}}{h_x} + \frac{a_{1j}^1}{2} \right) (u_{1j}^n)^2 \right) +
\end{aligned} \tag{3.4}$$

$$kh_x \sum_{j=1}^{N2-1} \left(\overline{b}^{22}_{N1,j} h_y (\overline{u}^{N1j,n}_y)^2 - \frac{\overline{b}^{21}_{N1,j} h_y}{h_x} \overline{u}^n_{N1-1,j} \overline{u}^{N1,j,n}_y \right) +$$

$$kh_x \sum_{j=1}^{N2-1} \left(\overline{b}^{22}_{1,j} h_y (\overline{u}^{1j,n}_y)^2 - \frac{\overline{b}^{21}_{1,j} h_y}{h_x} \overline{u}^n_{1,j} \overline{u}^{1,j,n}_y \right) +$$

$$kh_x \sum_{j=1}^{N2-1} \left(\frac{\overline{a}^2_{N1j}}{2} ((u^n_{N1j+1})^2 - (u^n_{N1j})^2) + \frac{\overline{a}^2_{1j}}{2} ((u^n_{1j+1})^2 - (u^n_{1j})^2) \right) = 0$$

and analogously for the unknowns of the dual mesh:

$$\sum_{\overline{V}_{ij} \in \overline{\mathcal{T}}} (1 + kr)(\overline{u}^n_{ij})^2) h_x h_y + \tag{3.5}$$

$$k \sum_{D_{ij} \in \mathcal{D}_{h,int}} \left(h_x h_y \left(b^{22}_{ij} (u^{ij,n}_y)^2 + b^{12}_{ij} u^{ij,n}_y u^{ij,n}_x \right) + h_y \frac{a^2_{ij}}{2} ((\overline{u}^n_{ij})^2 - (\overline{u}^n_{ij-1})^2) \right) +$$

$$k \sum_{\overline{D}_{ij} \in \overline{\mathcal{D}}_h, int} \left(h_x h_y \left(\overline{b}^{11}_{ij} (\overline{u}^{ij,n}_x)^2 + \overline{b}^{21}_{ij} \overline{u}^{ij,n}_x \overline{u}^{ij,n}_y \right) + h_x \frac{\overline{a}^1_{ij}}{2} ((\overline{u}^n_{ij})^2 - (\overline{u}^n_{i-1,j})^2) \right) +$$

$$kh_y \sum_{j=1}^{N2-1} \left((\frac{\overline{b}^{11}_{N1,j}}{h_x} - \frac{\overline{a}^1_{N1_1 j}}{2})(\overline{u}^n_{N1-1j})^2 - \overline{b}^{12}_{N1,j} \overline{u}^{N1,j,n}_y \overline{u}^n_{N1-1j} \right)$$

$$+ kh_y \sum_{j=1}^{N2-1} \left((\frac{\overline{b}^{11}_{1,j}}{h_x} + \frac{\overline{a}^1_{1_1 j}}{2})(\overline{u}^n_{1j})^2 + \overline{b}^{12}_{1,j} \overline{u}^{1,j,n}_y \overline{u}^n_{1j} \right) = 0.$$

Putting both equations (3.4), (3.5) together we obtain

$$\sum_{V_{ij} \in \mathcal{T}} (1 + kr)(u^n_{ij})^2 h_x h_y + \sum_{\overline{V}_{ij} \in \overline{\mathcal{T}}} (1 + kr)(\overline{u}^n_{ij})^2) h_x h_y + \tag{3.6}$$

$$k \sum_{D_{ij} \in \mathcal{D}_{h,int}} \left(b^{11}_{ij} (u^{ij,n}_x)^2 + b^{22}_{ij} (u^{ij,n}_y)^2 + 2 b^{12}_{ij} u^{ij,n}_y u^{ij,n}_x \right) h_x h_y +$$

$$k \sum_{\overline{D}_{ij} \in \overline{\mathcal{D}}_{h,int}} \left(\overline{b}^{22}_{ij} (\overline{u}^{ij,n}_y)^2 + \overline{b}^{11}_{ij} (\overline{u}^{ij,n}_x)^2 + 2 \overline{b}^{21}_{ij} \overline{u}^{ij,n}_x \overline{u}^{ij,n}_y \right) h_x h_y + A + B + C = 0,$$

where

$$A = k \sum_{D_{ij} \in \mathcal{D}_{h,int}} h_y \frac{a^1_{ij}}{2} ((u^n_{i+1j})^2 - (u^n_{ij})^2) + h_x \frac{a^2_{ij}}{2} ((\overline{u}^n_{ij})^2 - (\overline{u}^n_{ij-1})^2) +$$

$$k \sum_{\overline{D}_{ij} \in \overline{\mathcal{D}}_{h,int}} h_x \frac{\overline{a}^2_{ij}}{2} ((u^n_{ij+1})^2 - (u^n_{ij})^2) + h_y \frac{\overline{a}^1_{ij}}{2} ((\overline{u}^n_{ij})^2 - (\overline{u}^n_{i-1,j})^2) +$$

$$+ kh_x \sum_{j=1}^{N2-1} \left(\frac{\overline{a}^2_{N1j}}{2} ((u^n_{N1j+1})^2 - (u^n_{N1j})^2) + \frac{\overline{a}^2_{1j}}{2} ((u^n_{1j+1})^2 - (u^n_{1j})^2) \right),$$

$$B = kh_y \sum_{j=1}^{N2} \left(\left(\frac{2 b^{11}_{N1,j}}{h_x} - \frac{a^1_{N1j}}{2} \right) (u^n_{N1j})^2 + \left(\frac{2 b^{11}_{1,j}}{h_x} + \frac{a^1_{1j}}{2} \right) (u^n_{1j})^2 \right),$$

$$C = kh_x \sum_{j=1}^{N2-1} \left(\overline{b}_{N1,j}^{22} h_y (\overline{u}_y^{N1j,n})^2 - \frac{\overline{b}_{N1,j}^{21} h_y}{h_x} \overline{u}_{N1-1,j}^{n} \overline{u}_y^{N1,j,n} \right) +$$

$$kh_x \sum_{j=1}^{N2-1} \left(+\overline{b}_{1,j}^{22} h_y (\overline{u}_y^{1j,n})^2 - \frac{\overline{b}_{1,j}^{21} h_y}{h_x} \overline{u}_{1,j}^{n} \overline{u}_y^{1,j,n} \right) +$$

$$kh_y \sum_{j=1}^{N2-1} \left((\frac{\overline{b}_{N1,j}^{11}}{h_x} - \frac{\overline{a}_{N1_1 j}^{1}}{2})(\overline{u}_{N1-1j}^{n})^2 - \overline{b}_{N1,j}^{12} \overline{u}_y^{N1,j,n} \overline{u}_{N1-1j}^{n} \right) +$$

$$kh_y \sum_{j=1}^{N2-1} \left((\frac{\overline{b}_{1,j}^{11}}{h_x} + \frac{\overline{a}_{1_1 j}^{1}}{2})(\overline{u}_{1j}^{n})^2 + \overline{b}_{1,j}^{12} \overline{u}_y^{1,j,n} \overline{u}_{1j}^{n} \right).$$

We estimate term A, B, C using (1.2)

$$A = k\frac{h_y}{2} \sum_{j=1}^{N2} (\frac{1}{4} h_y (2j-1) + \frac{1}{2}\rho\sigma - r) \left((u_{N1,j}^n)^2 - (u_{1,j}^n)^2 \right) +$$

$$k\frac{h_x}{2}(\frac{1}{2}\sigma^2 - \kappa\theta) \left(\sum_{i=1}^{N1-1} \left((\overline{u}_{i,N2-1}^n)^2 - (\overline{u}_{i1}^n)^2 \right) + \sum_{i=1}^{N1} \left((u_{i,N2}^n)^2 - (u_{i1}^n)^2 \right) \right) -$$

$$k(\kappa+\lambda)\frac{h_x h_y}{4} \sum_{i=1}^{N1} \left(\sum_{j=1}^{N2-1} (\overline{u}_{ij}^n)^2 + \frac{(\overline{u}_{i1}^n)^2}{2} - \frac{2N2-3}{2} (\overline{u}_{i,N2-1}^n)^2 \right) -$$

$$k(\kappa+\lambda)\frac{h_y h_x}{2} \sum_{i=1}^{N1} \left(\sum_{j=1}^{N2-1} (u_{ij}^n)^2 - (N2-1)(u_{iN2}^n)^2 \right)$$

$$+k\frac{h_y}{2} \sum_{j=1}^{N2} (\frac{1}{2} j h_y + \frac{1}{2}\rho\sigma - r) \left((\overline{u}_{N1-1,j}^n)^2 - (\overline{u}_{1,j}^n)^2 \right),$$

$$B = \frac{kh_y}{2} \sum_{j=1}^{N2} \frac{h_y(2j-1)}{h_x} \left((u_{N1j}^n)^2 + (u_{1j}^n)^2 \right) +$$

$$\frac{kh_y}{2} \sum_{j=1}^{N2} \left(\frac{(2j-1)h_y}{4} + \frac{\rho\sigma}{2} - r \right) \left((u_{1j}^n)^2 - (u_{N1j}^n)^2 \right),$$

$$C = \frac{kh_x h_y}{2} \sum_{j=1}^{N2-1} \sigma^2 j h_y \left((\overline{u}_y^{N1,j,n})^2 + (\overline{u}_y^{1,jn})^2 \right) -$$

$$\frac{kh_x h_y}{2} \sum_{j=1}^{N2-1} \frac{j h_y \rho\sigma}{h_x} \left(\overline{u}_{1,j}^n \overline{u}_y^{1,n} + \overline{u}_{N1-1,j}^n \overline{u}_y^{N1j,n} \right) +$$

$$\sum_{j=1}^{N2-1} k\frac{h_y}{2h_x} j h_y \left((\overline{u}_{N1-1,j}^n)^2 + (\overline{u}_{1,j}^n)^2 \right) +$$

$$\sum_{j=1}^{N2-1} k\frac{h_y}{4} (j h_y + \rho\sigma - 2r) \left((\overline{u}_{1,j}^n)^2 - (\overline{u}_{N1-1,j}^n)^2 \right) +$$

$$\sum_{j=1}^{N2-1} k\frac{h_y}{2}\rho\sigma j h_y\left(\overline{u}^n_{1,j}\overline{u}_y^{1,j,n}-\overline{u}^n_{N1-1,j}\overline{u}_y^{N1,j,n}\right).$$

Collect all these expressions together and using boundary conditions, we obtain

$$A+B+C\geq -D=$$
$$-k\frac{h_y}{2}\sum_{j=1}^{N2}(\frac{(2j-1)h_y}{4}+r)\left((u^n_{1,j})^2+(u^n_{N1,j})^2\right)+$$
$$-k\frac{h_y}{2}\sum_{j=1}^{N2}(\frac{jh_y}{2}+r)\left((\overline{u}^n_{1,j})^2+(\overline{u}^n_{N1-1,j})^2\right)-$$
$$k\frac{h_x}{2}\sum_{i=1}^{N1-1}\left(\frac{\sigma^2}{2}\left((u^n_{i1})^2+(\overline{u}^n_{i1})^2\right)+\kappa\theta\left((u^n_{i,N2})^2+(\overline{u}^n_{i,N2-1})^2\right)\right)-$$
$$k(\kappa+\lambda)\frac{h_yh_x}{2}\sum_{i=2}^{N1-1}\sum_{j=1}^{N2-1}((u^n_{ij})^2+(\overline{u}^n_{ij})^2)).$$

Now we can estimate for all finite volumes

$$\begin{aligned}2b^{12}_{ij}u_y^{ij,n}u_x^{ij,n}&\leq|b^{12}_{ij}|(u_x^{ij,n})^2+|b^{12}_{ij}|(u_y^{ij,n})^2,\\ 2\overline{b}^{21}_{ij}\overline{u}_x^{ij,n}\overline{u}_y^{ij,n}&\leq|\overline{b}^{21}_{ij}|(\overline{u}_x^{ij,n})^2+|\overline{b}^{21}_{ij}|(\overline{u}_y^{ij,n})^2.\end{aligned}\tag{3.7}$$

Involving these estimations to the (3.6) we get

$$\sum_{V_{ij}\in\mathcal{T}}(1+kr)(u^n_{ij})^2h_xh_y+\sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}}(1+kr)(\overline{u}^n_{ij})^2)h_xh_y+\tag{3.8}$$
$$kh_xh_y\sum_{D_{ij}\in\mathcal{D}_{h,int}}\left(\left(b^{11}_{ij}-|b^{12}_{ij}|\right)(u_x^{ij,n})^2+\left(b^{22}_{ij}-|b^{12}_{ij}|\right)(u_y^{ij,n})^2\right)+$$
$$kh_xh_y\sum_{\overline{D}_{ij}\in\overline{\mathcal{D}}_{h,int}}\left(\left(\overline{b}^{22}_{ij}-|\overline{b}^{2}_{ij}|\right)\overline{u}_y^{ij,n})^2+\left(\overline{b}^{11}_{ij}-|\overline{b}^{2}_{ij}|\right)(\overline{u}_x^{ij,n})^2\right)-D\leq 0.$$

Now we can use the assumptions (3.1) of the theorem. We have (the same for "overlined" coefficients)

$$b^{11}_{ij}-|b^{12}_{ij}|=\frac{y_{ij}}{2}(1-\rho\sigma)>0\quad b^{22}_{ij}-|b^{12}_{ij}|=\frac{y_{ij}}{2}(\sigma^2-\rho\sigma)>0.$$

Thus we obtain

$$\sum_{V_{ij}\in\mathcal{T}}(1+kr)(u^n_{ij})^2h_xh_y+\sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}}(1+kr)(\overline{u}^n_{ij})^2)h_xh_y-$$
$$k\sum_{j=1}^{N2}\frac{h_y}{4}(jh_y+2r)\left((u^n_{1,j})^2+(u^n_{N1,j})^2+(\overline{u}^n_{1,j})^2+(\overline{u}^n_{N1-1,j})^2\right)-$$
$$k\frac{h_x}{2}\sum_{i=1}^{N1-1}\left(\frac{\sigma^2}{2}\left((u^n_{i1})^2+(\overline{u}^n_{i1})^2\right)+\kappa\theta\left((u^n_{i,N2})^2+(\overline{u}^n_{i,N2-1})^2\right)\right)-$$
$$k(\kappa+\lambda)\frac{h_yh_x}{2}\sum_{i=2}^{N1-1}\sum_{j=1}^{N2-1}((u^n_{ij})^2+(\overline{u}^n_{ij})^2))\leq 0.\tag{3.9}$$

Now if we pose the time step k as in (3.1) we obtain

$$kr \sum_{V_{ij}\in\mathcal{T}} (u_{ij}^n)^2 h_x h_y + kr \sum_{\overline{V}_{ij}\in\overline{\mathcal{T}}} (\overline{u}_{ij}^n)^2) h_x h_y \leq 0, \tag{3.10}$$

which conclude the proof. $\square$

4. Numerical experiment. Here we focused on computing the problem described by Heston model of the form, where we can compute the solution in closed form. For details see [5]. The results using DDFV scheme were computed in collaboration with Mária Zboranová, author's diploma student in [6]. They can be compared with those obtained by classical finite volume method presented in [5]. The data of the experiment have the following values:

$\rho = -0.5,\ \sigma = 0.5,\ r = 0.1,\ \kappa = 5.,\ \theta = 0.07,\ \lambda = 0,\ E = 100.$

Computational domain is of the form:

$$\Omega = \left\{(x,y) \in R^2 | -7 \leq x \leq 3, 0 \leq y \leq 1\right\}.$$

We compute the problem on the time interval $[0, 0.05]$ and the initial and boundary conditions are :

$$u\,(x,y,0) = max\,(0, e^x - 1)\,,$$
$$u\,(-7,y,\tau) = 0, \quad u\,(3,y,\tau) = e^3 - e^{-r\tau},$$
$$\frac{\partial u}{\partial y}\,(x,0,\tau) = 0, \quad \frac{\partial u}{\partial y}\,(x,1,\tau) = 0.$$

For comparing results obtained by proposed method and the method used in [5] we use the same definition of L_2 error. That means we compute the error only on the sub domain $< -1, 1 > \times < 0, 1 >$. When we take into account the used transformation $x = \ln\frac{S}{E}$ in Heston model we obtain for variable S which represent price of the underlying asset the interval $< 36, 272 >$ which represent the usual interval for underlying asset prices. For more detail see [5]. Results are presented in the Table 4.1 , where $N_x,\ N_y$ is number of finite volumes of primal mesh along the horizontal boundary respectively vertical boundary. N_{ts} is number of computed time steps.

$\mathbf{N_x}$	$\mathbf{N_y}$	$\mathbf{N_{ts}}$	**k**	$\mathbf{L_2C}$	$\mathbf{L_2D}$
20	10	1	0.05	0.00368077	0.00318228
40	20	4	0.0125	0.00249468	0.00205695
80	40	16	0.003125	0.00188673	0.00150821
160	80	64	0.000781	0.00157042	0.00124239

Table 4.1

Results for Heston model, classical (C) and dual (D) finite volume methods

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