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FULLY AUTOMATIC AFFINE REGISTRATION OF PLANAR PARAMETRIC CURVES

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Abstract. A new, fast and fully automatic algorithm for registration of 2D parametric curves is proposed in this paper. Two functionals, expressing difference between given curves are defined and minimized. The first one is based on difference in signed curvature of curves. Its optimization leads to optimal curve parametrization offset. The second one is based on distances between corresponding points and leads to optimal affine transformation parameters. Optimization of the parametrization offset is necessary in order to identify points correspondence of given curves. Numerical experiments on real data are presented and discussed.

Key words. affine registration, parametric planar curves, uniform redistribution, parametrization offset, reparametrization

AMS subject classifications. 53C44, 65K10, 68T10

1. Introduction. The registration problem appears in many application areas nowadays. It is used in order to process given shapes or images in medicine, robotics, security, etc.. Our goal is to solve the registration problem of planar curves, in this paper. It is necessary, e.g. in 2D atlas based image segmentations. They are based on set of precisely prepared templates, representing the class of shapes, similar to a segmented object.

There are different ways of shape representation. The most common is a representation by a zero iso-surface of distance function [10, 7]. However, the most natural is a representation by ordered set of points, discrete curve. This representation offers fast segmentations (Active Contours [1, 3]), compared with the one based on level-set method (Geodesic Active Contours [5, 8]), thanks to significantly smaller dimensionality of computational domain. However, this approach had some drawbacks. The first difficulty is to overcome problem of topological changes, solved in [14, 15]. The second one is coming from necessity of registration of atlas shapes. Shapes in atlas must be aligned to each other. In case of Geodesic Active Contours, there are several approaches how to register shapes [9, 11]. In case of Active Contours, the problem is solved using labeled "landmark" points [2, 6]. Each labeled point represents an important feature of the shape. The success of registration depends on precise labeling and correct determination of correspondence of the labeled points. It is therefore done manually, in general. This problem is solved originally and efficiently in case of planar curves in this paper.However, we assume that having two curves, the ratio of lengths of important segments corresponding to each other on these two curves and the overall lengths of the curves are similar to each other.

The correspondence problem between curves points is solved in the second section. In the third section we deal with problem of affine registration. The obtained results are presented and discussed in the last section.

2. Correspondence problem. The main problem of curves registration is how to identify corresponding parts of each curve in practice. If one delineates two curves

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representing the boundary of the same object, they would have same geometrical properties, but different parametrizations in general. The reason is that the starting points can be different. A curves registration, described in the next section, is based on a sum of distances between corresponding curves points. Therefore, the result of the registration algorithm depends on correctly defined correspondence. If there are significant inaccuracies, it fails. The main objective of this paper is to introduce an algorithm for solving the registration problem of discrete curves. For better understanding, we present also continuous formulation of the problems.

Let us consider two closed, plane curves $\Gamma_1, \Gamma_2 : S_1 \to R^2$. Let they be naturally parametrized by $u_1, u_2 \in S_1$, where S_1 is a periodic circle with unit length. These parametric curves represent a boundary of the same object, but they may have different parametrizations. The parameter u_1 can be shifted with respect to the parameter u_2. Than we can say that $u_1 \in [0,1]$, $\Gamma_1 = \{\mathbf{r}_1(u_1); u_1 \in S_1\}$ and $u_2 = u_2(u_1, o) = (u_1 + o) \bmod 1$, $\Gamma_2 = \{\mathbf{r}_2(u_2), u_2 \in S_1\}$, where $\mathbf{r}_1(u_1)$ and $\mathbf{r}_2(u_2)$ are position vectors of the curve Γ_1 resp. Γ_2, mod is a modulo operator and o denotes the offset. The problem is how to find this offset.

If we find the optimal offset, under assumption of similarity of curve segments ratio and thanks to natural parametrization, we obtain correct correspondence of curve points automatically. In other words we can navigate on particular curves easily. Let us consider two curves representing some shapes of hand. If $\mathbf{r}_1(0)$ and $\mathbf{r}_2(o)$ represent the point of same feature, e.g. a tip of a forefinger, then $\mathbf{r}_1(\delta)$ and $\mathbf{r}_2(\delta + o)$ give another common feature, e.g. a tip of the first finger.

Let us consider a representation of the curve, invariant with respect to transformations preserving a shape of the curve. A signed curvature seems to be suitable such representation [4]. It can be computed as

$$\kappa_\Gamma(u) = \partial_u \mathbf{T}(u).\mathbf{N}(u), \tag{2.1}$$

where $\mathbf{T}$ is unit tangent and $\mathbf{N}$ unit normal vector to the curve Γ, and $\mathbf{T} \wedge \mathbf{N} = 1$, where $\mathbf{T} \wedge \mathbf{N}$ denotes the determinant of the matrix with columns $\mathbf{T}$ and $\mathbf{N}$. An optimal offset causes that graphs of signed curvature will be the same, or very similar. A functional describing the difference between graphs of signed curvature of two curves can be written as

$$F(o) = \int_0^1 (\kappa_{\Gamma_1}(u_1) - \kappa_{\Gamma_2}(u_2(u_1, o)))^2 du_1. \tag{2.2}$$

The optimal offset can be found by minimizing functional (2.2), using gradient descent method in direction of its derivative

$$\partial_o F = -2 \int_0^1 (\kappa_{\Gamma_1}(u_1) - \kappa_{\Gamma_2}(u_2(u_1, o)))\partial_o \kappa_{\Gamma_2}(u_2(u_1, o)) du_1. \tag{2.3}$$

Discrete case is straightforward. Let us consider two discrete curves given by ordered sets of position vectors $\Gamma_1 = \{\mathbf{r}_{1,i}; i \in \langle 1, N\rangle, i \in Z\}$ and $\Gamma_2 = \{\mathbf{r}_{2,i}; i \in \langle 1, N\rangle, i \in Z\}$. Let us reparametrize curves uniformly by solving equation

$$\partial_t \mathbf{r} = \beta \mathbf{N} + \alpha \mathbf{T}, \tag{2.4}$$

according to [12, 13]. Functions β and α are given as follows

$$\beta = \varepsilon\kappa_\Gamma \tag{2.5}$$

$$\partial_s\alpha = \kappa_\Gamma\beta - \langle\kappa_\Gamma\beta\rangle_\Gamma + \omega\left(\frac{L}{g} - 1\right), \tag{2.6}$$

where ε is a parameter, s is the unit arc-length of a curve, $\langle\kappa_\Gamma\beta\rangle_\Gamma$ is an average of $\kappa_\Gamma\beta$ along the curve, L denotes a total curve length and $g = |\mathbf{r}_u|$. We obtain new reparametrized curves $\tilde\Gamma_1$ and $\tilde\Gamma_2$, where $\mathbf{r}_{1,i}$ corresponds to $\mathbf{r}_{2,i+o}$, $i = 1, ..., N$. We use a discrete form of (2.2)

$$F_D = \frac{1}{N}\sum_{i=1}^{N}\left(k_{\tilde\Gamma_1,i} - k_{\tilde\Gamma_2,i+o}\right)^2, \tag{2.7}$$

where $k_{\tilde\Gamma_1}$ and $k_{\tilde\Gamma_2}$ denote discrete signed curvature in i-th point, computed according to [14] as

$$k_{\Gamma,i} = \operatorname{sign}(\mathbf{R}_{i-1}\wedge\mathbf{R}_{i+1})\frac{1}{2h_i}\arccos\left(\frac{\mathbf{R}_{i+1}\cdot\mathbf{R}_{i-1}}{h_{i+1}h_{i-1}}\right), \tag{2.8}$$

where $\mathbf{R}_i = \mathbf{r}_i - \mathbf{r}_{i-1}$, $\operatorname{sign}(x) = 1$ if $x \geq 0$ and -1 otherwise and $h_i = |\mathbf{R}_i|$. Approximation of derivative (2.3) is given by

$$\partial_o F_D \approx -\sum_{i=1}^{N}(k_{1,i} - k_{2,i+o})(k_{2,i+o+1} - k_{2,i+o-1}), \tag{2.9}$$

Since o is an integer index, its value is increased or decreased by one according to $\operatorname{sign}(\partial_o F_D)$, in the optimization process, until optimum is reached.

The gradient methods find a local minimum, which need not to be a global extreme in general. This problem can be solved efficiently by using so-called multi-scale approach, which helps to ignore irrelevant features, while capturing strong ones. In our case, we consider the signed curvatures at different levels of smoothing obtained by the so-called mean filter. The smoothing is combined with the offset optimization. Offset obtained on higher smoothing level is used as initial condition for optimization at lower smoothing level. Obtained results are presented in section 4.

3. Affine registration of reparametrized curves. In this section we deal with a registration of two curves with same parameter domain. In other words, we minimize difference between two shapes, given by curves Γ_1 and Γ_2, caused by a transformation $\mathbf{A}$

$$\mathbf{A} = \mathbf{A}(\mathbf{T},\theta,\mathbf{s}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{T}_x & \mathbf{T}_y & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\mathbf{s}_x} & 0 & 0 \\ 0 & \frac{1}{\mathbf{s}_y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.1}$$

where $\mathbf{T} = \begin{bmatrix} \mathbf{T}_x \\ \mathbf{T}_y \end{bmatrix}$ is translation vector, θ is rotation angle and $\mathbf{s} = \begin{bmatrix} \mathbf{s}_x \\ \mathbf{s}_y \end{bmatrix}$ is scaling vector. For optimization purposes we define a functional

$$F(\mathbf{A}) = \int_0^1 (\mathbf{r}_1(u) - \mathbf{A}\mathbf{r}_2(u)) \cdot (\mathbf{r}_1(u) - \mathbf{A}\mathbf{r}_2(u))du \tag{3.2}$$

where $\mathbf{r}_1 \approx \mathbf{A}\mathbf{r}_2$. For better understanding of the formulas in the sequel, we can rewrite functional (3.2) into the form

$$F(\mathbf{A}) = \int_0^1 (\mathbf{r}_{1,x} - (\mathbf{A}\mathbf{r}_2)_x)^2 + (\mathbf{r}_{1,y} - (\mathbf{A}\mathbf{r}_2)_y)^2 \, du, \tag{3.3}$$

where

$$(\mathbf{A}\mathbf{r})_x = \mathbf{T}_x + \mathbf{s}_x(\mathbf{r}_x \cos\theta - \mathbf{r}_y \sin\theta) \tag{3.4}$$

$$(\mathbf{A}\mathbf{r})_y = \mathbf{T}_y + \mathbf{s}_y(\mathbf{r}_x \sin\theta - \mathbf{r}_y \cos\theta) \tag{3.5}$$

Transformation $\mathbf{A}$ can be found by minimizing (3.3) by using gradient descent method. The gradient components can be found as follows

$$\begin{aligned}
\partial_{\mathbf{T}} F &= \int_0^1 -2(\mathbf{r}_1 - (\mathbf{A}\mathbf{r}_2))du \\
\partial_\theta F &= \int_0^1 -2(\mathbf{r}_1 - \mathbf{A}\mathbf{r}_2) \cdot \begin{bmatrix} -\mathbf{s}_x(\mathbf{r}_{2,x} \sin\theta + \mathbf{r}_{2,y}\cos\theta) \\ \mathbf{s}_y(\mathbf{r}_{2,x}\cos\theta + \mathbf{r}_{2,y}\sin\theta) \end{bmatrix} du \\
\partial_{\mathbf{s}} F &= \int_0^1 -2(\mathbf{r}_1 - (\mathbf{A}\mathbf{r}_2)) \begin{bmatrix} \mathbf{r}_{2,x}\cos\theta - \mathbf{r}_{2,y}\sin\theta & 0 \\ 0 & \mathbf{r}_{2,x}\sin\theta + \mathbf{r}_{2,y}\cos\theta \end{bmatrix} du.
\end{aligned} \tag{3.6}$$

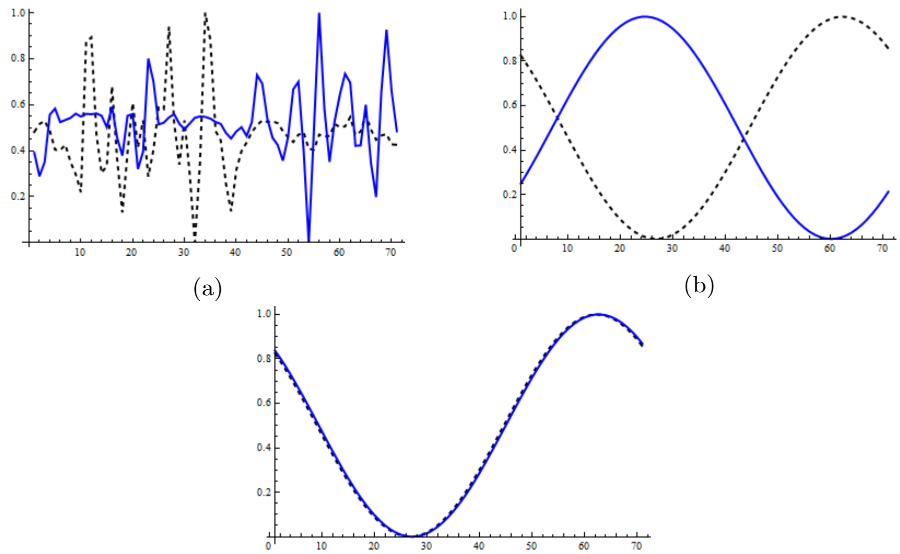
In discrete case, we can obtain optimal affine transformation similarly. Let us consider two ordered sets of points Γ_1 and Γ_2 representing uniformly discretized curves with N points. Let $\mathbf{r}_{1,i}$ and $\mathbf{r}_{2,i}$ are corresponding points. Functional (3.3) can be rewritten in discrete form as

$$F_D = \frac{1}{N}\sum_{i=1}^{N}((\mathbf{r}_{1,i} - \mathbf{A}\mathbf{r}_{2,i}) \cdot (\mathbf{r}_{1,i} - \mathbf{A}\mathbf{r}_{2,i})). \tag{3.7}$$

Approximations of gradient components in (3.6) are given as

$$\begin{aligned}
\partial_{\mathbf{T}} F_D &= \frac{1}{N}\sum_{i=1}^{N} -2(\mathbf{r}_{1,i} - (\mathbf{A}\mathbf{r}_{2,i})) \\
\partial_\theta F_D &= \frac{1}{N}\sum_{i=1}^{N} -2(\mathbf{r}_{1,i} - \mathbf{A}\mathbf{r}_{2,i}) \cdot \begin{bmatrix} -\mathbf{s}_x(\mathbf{r}_{2,i,x}\sin\theta + \mathbf{r}_{2,i,y}\cos\theta) \\ \mathbf{s}_y(\mathbf{r}_{2,i,x}\cos\theta + \mathbf{r}_{2,i,y}\sin\theta) \end{bmatrix} \\
\partial_{\mathbf{s}} F_{AD} &= \frac{1}{N}\frac{1}{N}\sum_{i=1}^{N} -2(\mathbf{r}_{1,i} - (\mathbf{A}\mathbf{r}_{2,i})) \begin{bmatrix} \mathbf{r}_{2,i,x}\cos\theta - \mathbf{r}_{2,i,y}\sin\theta & 0 \\ 0 & \mathbf{r}_{2,i,x}\sin\theta + \mathbf{r}_{2,i,y}\cos\theta \end{bmatrix}
\end{aligned}$$

Optimization process starts with alignment of centers of gravity $\mathbf{C}_{\Gamma_1}$ and $\mathbf{C}_{\Gamma_2}$, computed as average of all curve points. So initial transformation parameters can be set as $\mathbf{T} = \mathbf{C}_{\Gamma_1} - \mathbf{C}_{\Gamma_2}, \mathbf{s} = \mathbf{1}, \theta = 0$. Then transformation parameters can be optimized iteratively, by using gradient descent method.



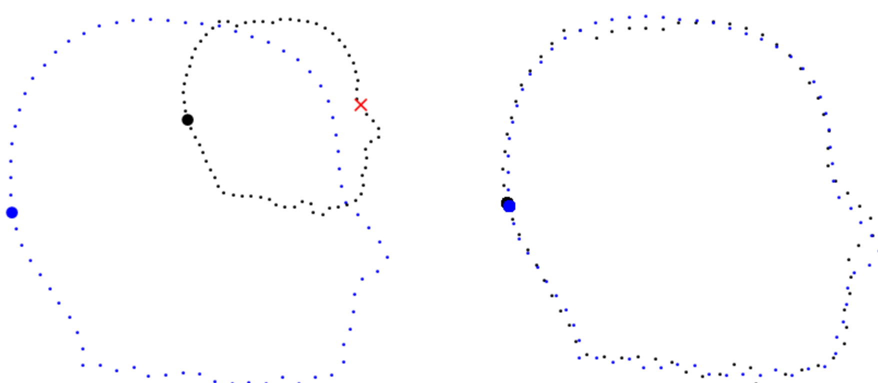
(a) (b)

(c)

Fig. 4.1: Graphs of normalized signed curvature computed for curves representing the human head shapes in Figure 4.2. a) Original graphs. b) Smoothed graphs. c) Aligned graphs.

4. Results. In this section we present results obtained by proposed fast and fully automatic algorithm for affine registration of discrete planar curves. In the first two examples we register curves representing the shape of human head.

In Figure 4.1 and Figure 4.3 we can see normalized graphs of discrete signed curvature (a), their smoothing by 1000 steps of mean filter (b) (which is used also in all further examples) and result of alignment of the smoothed graphs (c). Obtained offsets are used for curves registrations. Results are presented in Figure 4.2 and Figure 4.4



(a) (b)

Fig. 4.2: Registration of planar curves representing human heads. a) Original curves, where the points $\mathbf{r}_{1,1}$ and $\mathbf{r}_{2,1+o}$ are marked by large point and $\mathbf{r}_{2,1}$ is marked by cross. b) Registered curves.

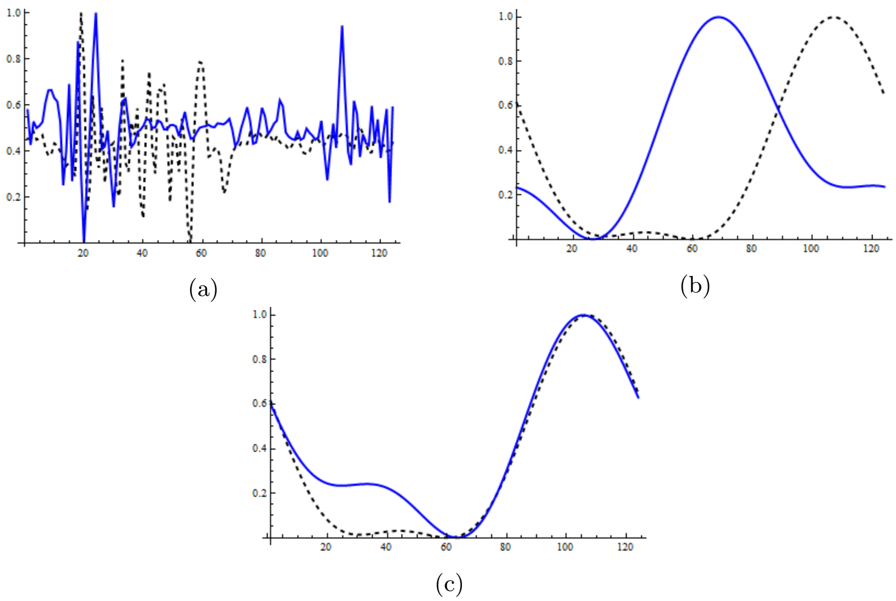


Fig. 4.3: Graphs of normalized signed curvature computed for curves representing the human head shapes in Figure 4.4. a) Original graphs. b) Smoothed graphs. c) Aligned graphs.

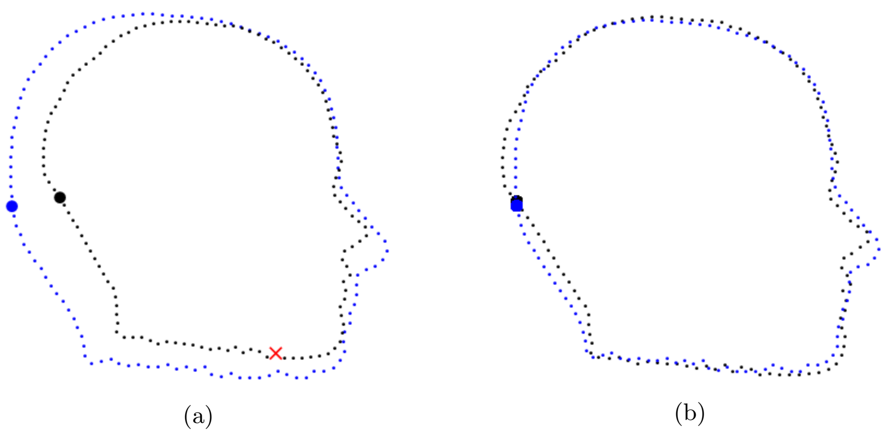
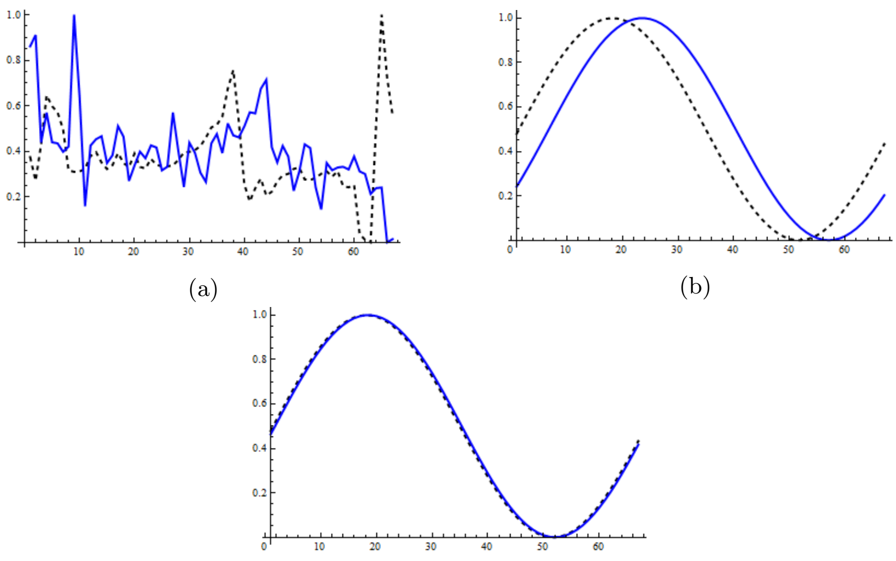


Fig. 4.4: Registration of planar curves representing human heads. a) Original curves, where the points $\mathbf{r}_{1,1}$ and $\mathbf{r}_{2,1+o}$ are marked by large point and $\mathbf{r}_{2,1}$ is marked by cross. b) Registered curves.

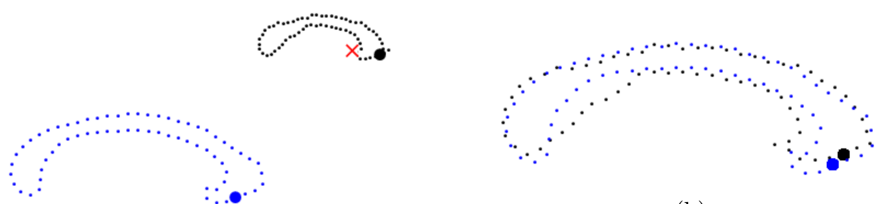
Next examples represent registration of two pairs of shapes of corpus callosum. In Figure 4.5 and Figure 4.7 we can see alignment of curvature graphs and in Figure 4.6 and in Figure 4.8 the shapes registration results.

Next two experiments are dealing with registration of human hands and fighter jets. Since the proportions of fingers and wings are similar, our method gives successful registrations. If this is not fulfilled, we do not find a correct offset and thus the points correspondence and the registration fails. This is an open problem, worth to be studied in future. The proposed method is very fast. CPU time for each presented experiment is far less than a second (tens of miliseconds). It has computational complexity $O(n)$,



(a) (b) (c)

Fig. 4.5: Graphs of normalized signed curvature computed for curves representing the corpus callosum shapes in Figure 4.6. a) Original graphs. b) Smoothed graphs. c) Aligned graphs.



(a) (b)

Fig. 4.6: Registration of planar curves representing corpus callosum. a) Original curves, where the points $\mathbf{r}_{1,1}$ and $\mathbf{r}_{2,1+o}$ are marked by large point and $\mathbf{r}_{2,1}$ is marked by cross. b) Registered curves.

where n denotes a number of curve points. Moreover, almost the whole algorithm can be parallelized, which may again to increase the efficiency significantly.

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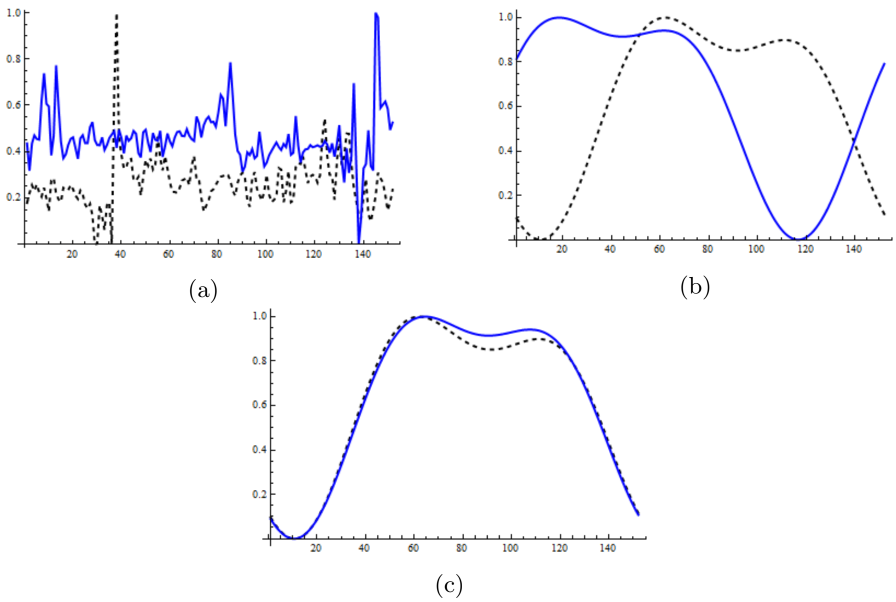

Fig. 4.7: Graphs of normalized signed curvature computed for curves representing the corpus callosum shapes in Figure 4.8. a) Original graphs. b) Smoothed graphs. c) Aligned graphs.

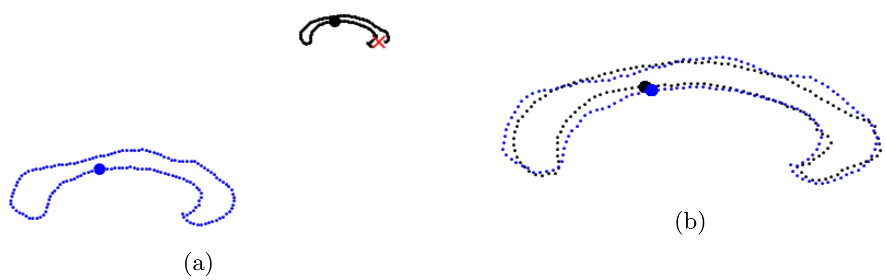

Fig. 4.8: Registration of planar curves representing corpus callosum. a) Original curves, where the points $\mathbf{r}_{1,1}$ and $\mathbf{r}_{2,1+o}$ are marked by large point and $\mathbf{r}_{2,1}$ is marked by cross. b) Registered curves.

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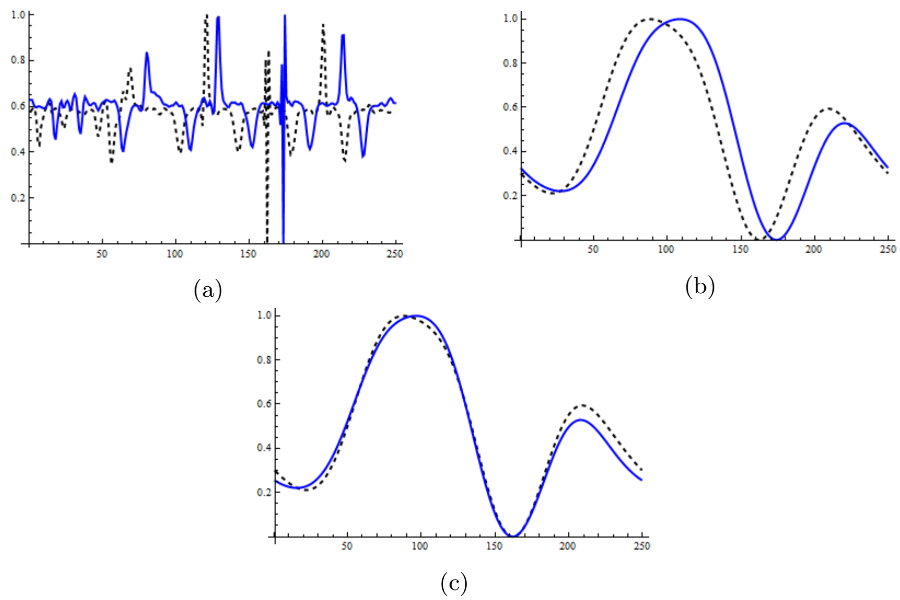

Fig. 4.9: Graphs of normalized signed curvature computed for curves representing the hand shapes in Figure 4.10. a) Original graphs. b) Smoothed graphs. c) Aligned graphs.

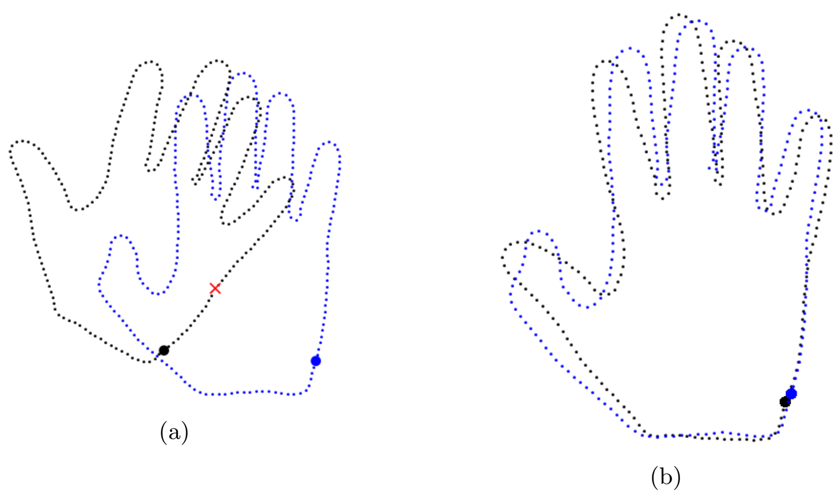

Fig. 4.10: Registration of planar curves representing hands. a) Original curves, where the points $\mathbf{r}_{1,1}$ and $\mathbf{r}_{2,1+o}$ are marked by large point and $\mathbf{r}_{2,1}$ is marked by cross. b) Registered curves.

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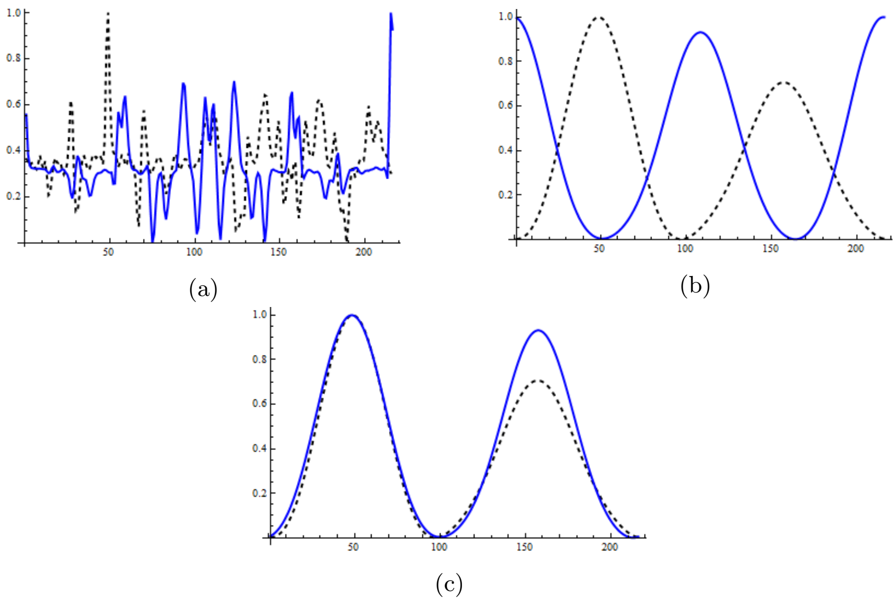

Fig. 4.11: Graphs of normalized signed curvature computed for curves representing the fighter jet shapes in Figure 4.12. a) Original graphs. b) Smoothed graphs. c) Aligned graphs.

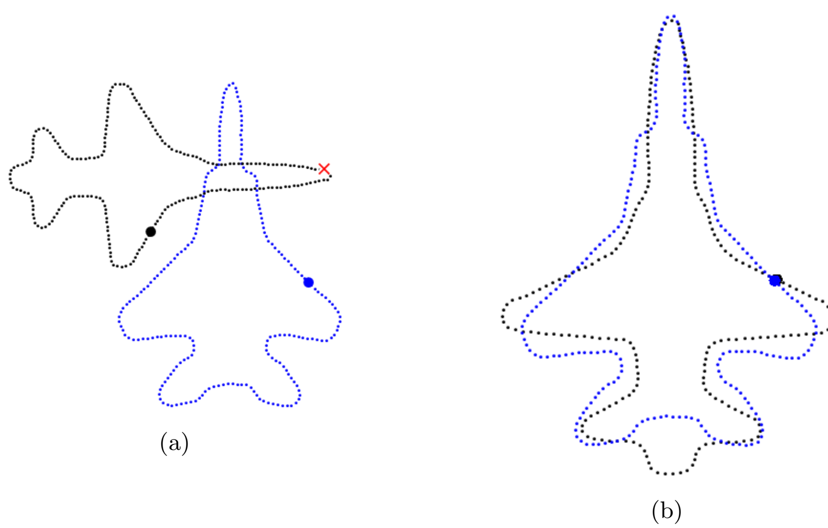

Fig. 4.12: Registration of planar curves representing fighter jets. a) Original curves, where the points $\mathbf{r}_{1,1}$ and $\mathbf{r}_{2,1+o}$ are marked by large point and $\mathbf{r}_{2,1}$ is marked by cross. b) Registered curves.

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