Acta Math. Univ. Comenianae
Vol. LXXXIX, 1 (2020), pp. 75–86

ON LEONARDO NUMBERS

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Abstract. In this paper, we consider the sequence of Leonardo numbers and we present some properties involving this sequence, including the Binet formula, and the generating function. Furthermore, Cassini's identity, Catalan's identity and d'Ocagne's identity for this sequence are given. Also some expressions of sums and products involving terms of this sequence are established.

1. Introduction and background

In the existing literature, there has been a great interest in the study of sequences of integers and their applications in various scientific domains. One of the sequences that has been extensively studied is the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ defined by the following recurrence relation

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2, \tag{1}$$

with $F_0 = 0$ and $F_1 = 1$.

Such sequence has been presented in several math articles inserted in several areas of mathematics such as group theory, calculus, applied mathematics, algebra, statistics, and also in physics and computer science articles (see the works [**1**], [**2**], [**3**], [**4**], [**7**], [**12**], [**15**], [**18**], [**19**], [**20**], [**21**], [**22**], [**23**], among others).

The first thirty Fibonacci Numbers are

$$0, 1, 1, 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 223,\ 377,\ 610,\ 987, 1597, 2584, 4181,$$
$$6765,\ 10496,\ 17711,\ 28657,\ 46368,\ 75025,\ 121393,\ 196418,\ 317811,\ 514229$$

and such sequence corresponds to the sequence A000045 in the on-line encyclopedia of integers sequences in [**24**].

For $n \geq 0$, the Binet formula of Fibonacci numbers is given by

$$F_n = \frac{\Phi^n - \Psi^n}{\Phi - \Psi}, \tag{2}$$

Received September 2, 2018; revised October 18, 2018.
2010 *Mathematics Subject Classification.* Primary 11B37, 11B39, 11Y55.
Key words and phrases. Fibonacci sequence; Binet's formula; recurrence relation; generating function.
This research was financed by Portuguese Funds through FCT – Fundação para a Ciência e a Tecnologia, within the Project UID/MAT/00013/2013 and the project UID/CED/00194/2013.

where $\Phi=\frac{1+\sqrt{5}}{2}$ and $\Psi=\frac{1-\sqrt{5}}{2}$ are the roots of the quadratic equation $r^2-r-1=0$.

A variety of generalizations of this sequence has been chosen for the research of scientists such as [**5**], [**6**], [**8**], [**10**], [**11**], [**13**], [**14**], [**16**], [**17**], [**25**], among others.

The sequence of Lucas numbers $\{L_n\}_{n=0}^{\infty}$ is also one of the integers sequence of great interest in this area. It is a sequence that satisfies the same researchers' recurrence relation of the Fibonacci sequence

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2, \tag{3}$$

but with initial conditions $L_0 = 2$ and $L_1 = 1$.

The first thirty Lucas Numbers are

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778,

9349, 15127, 24476, 39603, 64079, 103682, 167761, 271443, 439204, 710647, 1149851

and such sequence corresponds to the sequence A000032 in the on-line encyclopedia of integers sequences in [**24**].

There are several relationships between this sequence and Fibonacci sequence. The next result shows two of these relations which are used in the Section 2.

Lemma 1.1. *For $n \geq 1$,*

$$F_n = \frac{L_{n-1} + L_{n+1}}{5}, \tag{4}$$

$$F_{n+2} = \frac{L_{n-1} + L_{n+5}}{10}, \tag{5}$$

$$L_n = F_n + 2F_{n-1}. \tag{6}$$

Proof. We show the equality of (4) by induction on n. For $n = 1$, $F_1 = \frac{L_0+L_2}{5} = \frac{2+3}{5} = 1$, and thus the equality holds. Now, suppose that the equality holds for $1 < k \leq n$. Then we have

$$\begin{aligned} F_{n+1} = F_n + F_{n-1} &= \left(\frac{L_{n-1} + L_{n+1}}{5}\right) + \left(\frac{L_{n-2} + L_n}{5}\right) \\ &= \frac{(L_{n-1} + L_{n-2}) + (L_{n+1} + L_n)}{5} = \frac{L_n + L_{n+2}}{5}, \end{aligned}$$

and so the equality is true for $n + 1$, as required.
The identities (5) and (6) can be both proved in a similar way. □

In this paper, we consider Leonardo sequence that is also an integers sequence which is related to the Fibonacci and also to the Lucas sequences. The structure adopted in this paper is the following: after this introdutory section, in Section 2, we present the Leonardo sequence and the statement of the respective Binet formula. Section 3 is dedicated to sums and products formulae for this sequence. Several identities are presented in Section 4, while in the Section 5, a generating function is constructed. The paper ends with some concluding notes.

2. The Leonardo sequence

This section is devoted to the introduction of Leonardo sequence. In order to be not confused with Lucas number, throughout this paper, we adopt the expression Le_n to denote the nth Leonardo number and consequently the Leonardo sequence is denoted by $\{Le_n\}_{n=0}^{\infty}$. This sequence is defined by the following recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1,\ n \geq 2, \tag{7}$$

with initial conditions $Le_0 = Le_1 = 1$.

Such sequence is the sequence A001595 of the on-line encyclopedia of integers sequences [**24**]. The first thirty Leonardo numbers are

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, 753, 1219, 1973, 3193, 5167, 8361, 13529,$$
$$21891, 35421, 57323, 92745, 150069, 242815, 392885, 635701, 1028587, 1664289.$$

It can be easily verified that every Leonard number is odd as we can see in the following result

Lemma 2.1. *For $n \geq 0$, Le_n is an odd number.*

Proof. We prove by induction on n. The statement is true for $n = 0,\ 1$. Now suppose that the statement is true for $2 < k \leq n$. Then, we can verify it for $n+1$. Since by (7) we have $Le_{n+1} = Le_n + Le_{n-1} + 1$ and as the sum of two odd numbers which are Le_n and Le_{n-1} by induction hypothesis is even and, in turn, the sum of an even number with the number 1 is an odd number, the proof is complete. □

Moreover, we can also observe that the sequence of the units digits of Leonardo numbers is periodic with period 20.

Note that from the relation (7) and since this recurrence relation is inhomogeneous, substituting n by $n+1$ in (7), we obtain the new form

$$Le_{n+1} = Le_n + Le_{n-1} + 1. \tag{8}$$

Now, subtracting (7) to (8), we obtain $Le_n - Le_{n+1} = Le_{n-2} - Le_n$, and then

$$Le_{n+1} = 2Le_n - Le_{n-2},\ n \geq 2. \tag{9}$$

The relation between Leonardo and Fibonacci numbers is expressed in the following proposition.

Proposition 2.2. *For $n \geq 0$,*

$$Le_n = 2F_{n+1} - 1. \tag{10}$$

Proof. We prove by induction on n. For $n = 0$ and $n = 1$ the identity (7) is easily verified. Now, we suppose that (10) is true for all $1 < k \leq n$, and we prove that equality (10) remains valid for $k = n + 1$.

In fact, from the recurrence relation (7), using the induction hypothesis and the recurrence relation (1) we can successively write

$$\begin{aligned} Le_{n+1} &= Le_n + Le_{n-1} + 1 = (2F_{n+1} - 1) + (2F_n - 1) + 1 \\ &= 2\left(F_{n+1} + F_n\right) - 1 = 2F_{n+2} - 1, \end{aligned}$$

and thus the result is verified. □

According to this proposition and using (4), (5) and (6), we can establish the next result, which can be easily proved and we omit the proof that shows possible relationships between Leonardo, Lucas, and Fibonacci numbers.

Proposition 2.3. *For the nth Leonardo number Le_n, the following identities hold:*

$$Le_n = 2\Big(\frac{L_n + L_{n+2}}{5}\Big) - 1, \qquad n \geq 0, \tag{11}$$

$$Le_{n+3} = \frac{L_{n+1} + L_{n+7}}{5} - 1, \qquad n \geq 0, \tag{12}$$

$$Le_n = L_{n+2} - F_{n+2} - 1, \qquad n \geq 0, \tag{13}$$

where L_n is the nth Lucas number and F_n is the nth Fibonacci number.

Now, using the Binet formula of Fibonacci numbers (2) and Proposition 2.2, the Binet formula for Leonardo numbers can be easily established as we can see in the following result.

Proposition 2.4 (Binet's formula)**.** *For $n \geq 0$,*

$$Le_n = 2\Big(\frac{\Phi^{n+1} - \Psi^{n+1}}{\Phi - \Psi}\Big) - 1 = \frac{\Phi\left(2\Phi^n - 1\right) - \Psi\left(2\Psi^n - 1\right)}{\Phi - \Psi}, \tag{14}$$

where Le_n is the nth Leonardo number, $\Phi = \frac{1+\sqrt{5}}{2}$ and $\Psi = \frac{1-\sqrt{5}}{2}$.

3. Sums and products

In this section, we present some results concerning sums and products of terms of the Leonardo sequence by using some results of Fibonacci and Lucas sequences. From (1), Proposition 2.2, and some results for Fibonacci and Lucas numbers in [**9**], we can obtain the following results for sums of Leonardo Numbers.

Proposition 3.1 (Sums formulae)**.** *For $n \geq 0$,*

1. $\sum_{n} j = 0^n Le_j = Le_{n+2} - (n+2)$,
2. $\sum_{j=0}^{n} Le_{2j} = Le_{2n+1} - n$,
3. $\sum_{j=0}^{n} Le_{2j+1} = Le_{2n+2} - (n+2)$.

Proof. The proof of i) follows from (1), Proposition 2.1 i) with $k = 1$ of the work [**9**] and Proposition 2.2, by performing some calculations. In fact, we have

$$\begin{aligned}\sum_{j=0}^{n} Le_j &= \sum_{j=0}^{n} (2F_{j+1} - 1) = 2\sum_{j=0}^{n} F_{j+1} - (n+1)\\ &= 2\sum_{j=0}^{n+1} F_j - (n+1) = 2\left(F_{n+2} + F_{n+1} - 1\right) - (n+1)\\ &= (2F_{n+3} - 1) - (n+2) = Le_{n+2} - (n+2)\,.\end{aligned}$$

To prove ii), we use again (1), Proposition 2.2, and Proposition 2.3 i) with $k = 1$ [**9**] in the following way

$$\begin{aligned}\sum_{j=0}^{n} Le_{2j} &= \sum_{j=0}^{n} (2F_{2j+1} - 1) = 2\sum_{j=0}^{n} F_{2j+1} - (n+1)\\ &= 2\sum_{j=1}^{n+1} F_{2j-1} - (n+1) = 2F_{2n+2} - (n+1)\\ &= \big(2F_{(2n+1)+1} - 1\big) - n = Le_{2n+1} - n.\end{aligned}$$

Finally, to prove iii), we use once more (1), the Proposition 2.2, and the Proposition 2.2 i) with $k = 1$ from [**9**]

$$\begin{aligned}\sum_{j=0}^{n} Le_{2j+1} &= \sum_{j=0}^{n} (2F_{2j+2} - 1) = 2\sum_{j=0}^{n} F_{2j+2} - (n+1)\\ &= 2\sum_{j=0}^{n+1} F_{2j} - (n+1) = 2\left(F_{2n+3} - 1\right) - (n+1)\\ &= \big(2F_{(2n+2)+1} - 1\big) - (n+2) = Le_{2n+2} - (n+2)\end{aligned}$$

and the proof is completed. □

For the sum of the homologous terms of Fibonacci, Lucas, and Leonardo sequences, we can state the result bellow.

Proposition 3.2. *For $n \geq 0$, the following identities hold:*

1. $\displaystyle\sum_{j=0}^{n} (F_j + Le_j) = F_{n+2} + Le_{n+2} - (n+3)$,
2. $\displaystyle\sum_{j=0}^{n} (F_j + Le_j) = F_{n+5} - (n+4)$,
3. $\displaystyle\sum_{j=0}^{n} (L_j + Le_j) = L_{n+2} + Le_{n+2} - (n+3)$,
4. $\displaystyle\sum_{j=0}^{n} (L_j + Le_j) = \frac{7L_{n+2} + 2L_{n+4}}{5} - (n+4)$,

5. $\sum_{j=0}^{n}(L_j+Le_j)=2F_{n+3}+L_{n+2}-(n+4)$,

where L_j is the jth Lucas number, F_j is the jth Fibonacci number, and Le_j is the jth Leonardo number.

Proof. 1. The result easily follows from (1), Proposition 2.1 i) with $k=1$ from [**9**], and the item 1 of Proposition 3.1 as follows:

$$\begin{aligned}\sum_{j=0}^{n}(F_j+Le_j)&=\Sigma_{j=0}^{n}F_j+\Sigma_{j=0}^{n}Le_j=(F_{n+1}+F_n-1)+(Le_{n+2}-(n+2))\\&=F_{n+2}+Le_{n+2}-(n+3)\,.\end{aligned}$$

2. For the proof of this identity, we use Proposition 2.2, the relation (1), and also the previous identity of this proposition

$$\begin{aligned}\sum_{j=0}^{n}(F_j+Le_j)&=F_{n+2}+Le_{n+2}-(n+3)=F_{n+2}+(2F_{n+3}-1)-(n+3)\\&=F_{n+4}+F_{n+3}-1-(n+3)=F_{n+5}-(n+4)\,.\end{aligned}$$

3. We use (ii) [**9**, Proposition 2.1] with $k=1$, the item 1. of Proposition 3.1, the relation (3), and we obtain

$$\begin{aligned}\sum_{j=0}^{n}(L_j+Le_j)&=\sum_{j=0}^{n}L_j+\Sigma_{j=0}^{n}Le_j=(L_{n+1}+L_n-1)+(Le_{n+2}-(n+2))\\&=L_{n+2}+Le_{n+2}-(n+3)\,.\end{aligned}$$

4. Using the previous item and the relation (11) of Proposition 2.3, we obtain

$$\begin{aligned}\sum_{j=0}^{n}(L_j+Le_j)&=L_{n+2}+Le_{n+2}-(n+3)\\&=L_{n+2}+2\left(\frac{L_{n+2}+L_{n+4}}{5}\right)-1-(n+3)\,,\end{aligned}$$

and a simple calculation completes the proof.

5. Finally, we use again the previous item 3, the relation (12) of Proposition 2.3, and (4) of Lemma 1.1, obtaining

$$\begin{aligned}\sum_{j=0}^{n}(L_j+Le_j)&=L_{n+2}+Le_{n+2}-(n+3)=L_{n+2}+\frac{L_n+L_{n+6}}{5}-(n+4)\\&=\frac{L_n+L_{n+6}+5L_{n+2}}{5}-(n+4)=2F_{n+3}+L_{n+2}-(n+4)\,.\end{aligned}$$

□

The next result is about the sum of the square of the first n terms of Leonardo numbers, and the expression is given by Fibonacci numbers.

Proposition 3.3. *For $n \geq 0$, the following identity holds*

$$\sum_{j=0}^{n} (Le_j)^2 = 4\,(F_{n+1}-1)\,(F_{n+2}-1) + (n+1)\,, \tag{15}$$

where Le_j is the jth Leonardo number and F_j is the jth Fibonacci number.

Proof. The result follows from the Propositions 2.1 i) and 4i) with $k = 1$ from [**9**], and the Proposition 2.2. In fact, we have

$$\begin{aligned}\sum_{j=0}^{n} (Le_j)^2 &= \sum_{j=0}^{n} (2F_{j+1}-1)^2 = \sum_{j=0}^{n}\Big(4\,(F_{j+1})^2 - 4F_{j+1} + 1\Big)\\ &= 4\sum_{j=0}^{n}(F_{j+1})^2 - 4\sum_{j=0}^{n}F_{j+1} + (n+1)\\ &= 4\,(F_{n+1}F_{n+2}) - 4\sum_{j=1}^{n+1}F_j + (n+1)\\ &= 4F_{n+1}F_{n+2} + 4\,(F_{n+1}+F_{n+2}-1) + (n+1)\\ &= 4\,(F_{n+1}-1)\,(F_{n+2}-1) + (n+1)\,,\end{aligned}$$

as required. □

According the previous result and by the use of the sum of the square of the first n terms of Lucas and Fibonacci numbers given in Proposition 4 with $k = 1$ [**9**] we easily prove the following result involving square terms of Fibonacci, Lucas, and Leonardo numbers. Note that in the result of Proposition 3.3, we also can use the expression using of Leonardo numbers and in this case the expression, is given by

$$\sum_{j=0}^{n} (Le_j)^2 = (Le_n - 1)\,(Le_{n+1}-1) + (n+1)\,.$$

Proposition 3.4. *For $n \geq 0$, the following identities are true:*

$$\sum_{j=0}^{n}\Big((Le_j)^2 + (F_j)^2\Big) = 4\,(F_{n+1}-1)\,(F_{n+2}-1) + (n+1) + F_nF_{n+1}, \tag{16}$$

$$\sum_{j=0}^{n}\Big((Le_j)^2 + (L_j)^2\Big) = 4\,(F_{n+1}-1)\,(F_{n+2}-1) + (n+1) + L_nL_{n+1} + 2, \tag{17}$$

where L_j is the jth Lucas number, F_j is the jth Fibonacci number, and Le_j is the jth Leonardo number.

The next result is related with the sum of the product of the terms of the Fibonacci and Lucas with Leonardo sequence.

Proposition 3.5. *For $n \geq 0$, the following identities are true:*

$$\sum_{j=0}^{n} Le_j F_{j+1} = F_{n+1} Le_{n+1} - F_{n+2} + 1, \tag{18}$$

$$\sum_{j=0}^{n} Le_j L_{j+2} = L_{n+3}\left(L_{n+2} - 1\right) - L_{n+2} - F_{2n+5} + 3, \tag{19}$$

where L_i is the ith Lucas number, F_i is the ith Fibonacci number and Le_i is the ith Leonardo number.

Proof. For the proof of (18,) we use the relation of Proposition 2.2 and the expressions of the sum of the squares of the terms, and the sums of first n terms of Fibonacci sequence in Propositions 2 and 4 with $k = 1$ from [**9**]. We have

$$\begin{aligned}
\sum_{j=0}^{n} Le_j F_{j+1} &= \sum_{j=0}^{n} \left(2F_{j+1} - 1\right) F_{j+1} = 2\sum_{j=0}^{n} \left(F_{j+1}\right)^2 - \sum_{j=0}^{n} F_{j+1} \\
&= 2F_{n+1}F_{n+2} - \left(F_n + F_{n+1} - 1 + F_{n+1}\right) \\
&= 2F_{n+1}F_{n+2} - \left(F_{n+2} - 1 + F_{n+1}\right) \\
&= F_{n+1}\left(2F_{n+2} - 1\right) - \left(F_{n+2} - 1\right) = F_{n+1}Le_{n+1} - F_{n+2} + 1,
\end{aligned}$$

as required.

The relation (19) is proved by the use of relation (13) of Proposition 2.3, and the results of the sum of the first n terms of Lucas numbers, the sum of the product of Lucas and Fibonacci numbers, and also the sum of the squares of the terms of Lucas numbers in Propositions 2, 4, and 5 with $k = 1$ [**9**]. Hence we have

$$\begin{aligned}
\sum_{j=0}^{n} Le_j L_{j+2} &= \sum_{j=0}^{n} \left(L_{j+2} - F_{j+2} - 1\right) L_{j+2} \\
&= \sum_{j=0}^{n} \left(L_{j+2}\right)^2 - \sum_{j=0}^{n} F_{j+2}L_{j+2} - \sum_{j=0}^{n} L_{j+2} \\
&= \left(L_{n+2}L_{n+3} + 2 - \left(L_0\right)^2 - \left(L_1\right)^2\right) - \left(F_{2n+5} - 1 - F_1L_1\right) \\
&\quad - \left(L_{n+3} + L_{n+2} - 1 - L_0 - L_1\right) \\
&= \left(L_{n+2}L_{n+3} - 3\right) - \left(F_{2n+5} - 2\right) - \left(L_{n+3} + L_{n+2} - 4\right) \\
&= L_{n+2}L_{n+3} - F_{2n+5} - L_{n+3} - L_{n+2} + 3,
\end{aligned}$$

and the results follows. □

4. Some identities

In this section, we deduce Catalan's, Cassini's, and d'Ocagne's identities for Leonardo numbers. For that purpose we start recalling Catalan's identity for Fibonacci numbers, that is, for $n > r,\ r \geq 0$,

$$F_{n-r}F_{n+r} - (F_n)^2 = (-1)^{n+1-r}(F_r)^2 . \tag{20}$$

The next result shows the expression of Catalan's identity for Leonardo sequence.

Proposition 4.1 (Catalan's identity)**.** *For $n > r,\ r \geq 1$,*

$$(Le_n)^2 - Le_{n-r}Le_{n+r} = Le_{n-r} + Le_{n+r} - 2Le_n - (-1)^{n-r}(Le_{r-1}+1)^2 , \tag{21}$$

where Le_i is the ith Leonardo number.

Proof. The proof follows from the relation (10) of Proposition 2.2 and Catalan's identity for Fibonacci numbers, after some calculations. In fact, we have

$$\begin{aligned}(Le_n)^2 - Le_{n-r}Le_{n+r} &= (2F_{n+1}-1)^2 - (2F_{n-r+1}-1)(2F_{n+r+1}-1)\\ &= 4\left((F_{n+1})^2 - F_{n+1-r}F_{n+1+r}\right) - 4F_{n+1}\\ &\quad + 2F_{n-r+1} + 2F_{n+r+1}\\ &= 4(-1)^{n+1-r}(F_r)^2 - 4F_{n+1} + 2F_{n-r+1} + 2F_{n+r+1}\\ &= 4(-1)^{n+1-r}(F_r)^2 - 2(2F_{n+1}-1) + Le_{n-r} + Le_{n+r}\\ &= 4(-1)^{n+1-r}(F_r)^2 - 2Le_n + Le_{n-r} + Le_{n+r}\\ &= Le_{n-r} + Le_{n+r} - 2Le_n + 4(-1)^{n+1-r}(F_r)^2\end{aligned}$$

and the result follows. □

Taking $r = 1$ in the previous identity, we obtain Cassini's identity for the Leonardo sequence. In fact, we have

$$(Le_n)^2 - Le_{n-1}Le_{n+1} = Le_{n-1} + Le_{n+1} - 2Le_n - (-1)^{n-1}(Le_0+1)^2 ,$$

and taking account that $Le_0 = 1$, we obtain the expression of Cassini's identity as expressed in the next result.

Proposition 4.2 (Cassini's identity)**.** *For $n \geq 2$,*

$$(Le_n)^2 - Le_{n-1}Le_{n+1} = Le_{n-1} - Le_{n-2} + 4(-1)^n . \tag{22}$$

The use of d'Ocagne's identity for Fibonacci numbers, that is, for $m > n$,

$$F_mF_{n+1} - F_{m+1}F_n = (-1)^n F_{m-n}, \tag{23}$$

Proposition 2.2, and also the relation (8) allow us to deduce d'Ocagne's identity for Leonardo numbers.

Proposition 4.3 (d'Ocagne's identity)**.** *If $m > n,\ n \geq 1$, then*

$$Le_mLe_{n+1} - Le_{m+1}Le_n = 2(-1)^{n+1}(Le_{m-n-1}+1) + Le_{m-1} - Le_{n-1}. \tag{24}$$

Proof. The result follows from the relation (8), Proposition 2.2, and d'Ocagne's identity (23) for Fibonacci numbers, after some calculations. In fact, we have,

$$\begin{aligned}
&Le_m Le_{n+1} - Le_{m+1} Le_n \\
&= (2F_{m+1} - 1)(2F_{n+2} - 1) - (2F_{m+2} - 1)(2F_{n+1} - 1) \\
&= 4(F_{m+1}F_{n+2} - F_{m+2}F_{n+1}) - 2(F_{m+1} + F_{n+2} - F_{m+2} - F_{n+1}) \\
&= 4\left((-1)^{n+1} F_{m+1-(n+1)}\right) + 2(F_{m+2} - F_{m+1}) + 2(F_{n+1} - F_{n+2}) \\
&= 4(-1)^{n+1} F_{m-n} + (Le_{m+1} - Le_m) + (Le_n - Le_{n+1}) \\
&= 2(-1)^{n+1}(Le_{m-n-1} + 1) + Le_{m-1} + 1 - Le_{n-1} - 1 \\
&= 2(-1)^{n+1}(Le_{m-n-1} + 1) + Le_{m-1} - Le_{n-1},
\end{aligned}$$

as required. □

5. Generating function

Next we give the generating functions for the Leonardo sequence. Consider the Leonardo sequence $\{Le_j\}_{j=0}^{\infty}$. By the definition of generating functions of a sequence, the generating associated function $g_{Le}(t)$ is defined by

$$g_{Le}(t) = \sum_{n=0}^{\infty} Le_n t^n. \tag{25}$$

We obtain the following result

Proposition 5.1. *For $1-2t+t^3 \neq 0$, the generating function for the Leonardo sequence is given by*

$$g_{Le}(t) = \frac{1-t+t^2}{1-2t+t^3}. \tag{26}$$

Proof. Taking account, the identity (25), the initial condition of the Leonardo sequence, and the relation (9), we have

$$\begin{aligned}
g_{Le}(t) &= Le_0 + Le_1 t + Le_2 t^2 + \sum_{n=3}^{\infty} Le_n t^n \\
&= 1 + t + 3t^2 + \sum_{n=3}^{\infty} Le_n t^n \\
&= 1 + t + 3t^2 + \sum_{n=3}^{\infty} (2Le_{n-1} - Le_{n-3})\, t^n \\
&= 1 + t + 3t^2 + 2t\sum_{n=3}^{\infty} Le_{n-1} t^{n-1} - t^3 \sum_{n=3}^{\infty} Le_{n-3} t^{n-3} \\
&= 1 + t + 3t^2 + 2t\Big(\sum_{n=0}^{\infty} Le_n t^n - Le_0 - Le_1 t\Big) - t^3 \sum_{n=0}^{\infty} Le_n t^n
\end{aligned}$$

$$= 1 + t + 3t^2 + 2t\Big(\sum_{n=0}^{\infty} Le_n t^n - 1 - t\Big) - t^3 \sum_{n=0}^{\infty} Le_n t^n$$
$$= 1 + t + 3t^2 - 2t - 2t^2 + 2t \sum_{n=0}^{\infty} Le_n t^n - t^3 \sum_{n=0}^{\infty} Le_n t^n$$
$$= 1 - t + t^2 + 2t \sum_{n=0}^{\infty} Le_n t^n - t^3 \sum_{n=0}^{\infty} Le_n t^n$$
$$= 1 - t + t^2 + 2t g_{Le}(t) - t^3 g_{Le}(t).$$

Therefore,

$$g_{Le}(t) - 2t g_{Le}(t) + t^3 g_{Le}(t) = 1 - t + t^2$$
$$\Longleftrightarrow$$
$$g_{Le}(t)\left(1 - 2t + t^3\right) = 1 - t + t^2,$$

and the result immediately follows. □

6. Conclusion and remarks

In this paper, the sequence of Leonardo numbers was introduced. Some properties involving this sequence, including the Binet formula, a generating function and some identities, were presented. Also several expressions involving sums and products with the terms of this sequence were established.

Acknowledgment. The authors would like to thank the referees for their pertinent comments and valuable suggestions, which significantly improve the final version of the manuscript.

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