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NEW BOUNDS FOR THE SPREAD OF A MATRIX USING THE RADIUS OF THE SMALLEST DISC THAT CONTAINS ALL EIGENVALUES

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Abstract. Let $\mathcal{D}$ denote the smallest disc containing all eigenvalues of the matrix A. Without knowing the eigenvalues of A, we can estimate the spread of A and the radius of $\mathcal{D}$. Some new bounds for the radius of $\mathcal{D}$ and the spread of A are given. These bounds involve the entries of A. Also sufficient conditions for equality are obtained for some inequalities. New proofs of some known results are presented, too.

1. Introduction

Throughout this paper, we assume that $n \geq 3$, $A = (a_{ij})$ is an $n \times n$ complex matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. *The spread* of the matrix A is defined as $\mathrm{Sp}(A) = \max_{i,j} |\lambda_i - \lambda_j|$, introduced by L. Mirsky [**10**].

Many authors have given several different bounds for the spread (see [**13**]). We write $\mathrm{Sp}_{\mathrm{Re}}(A) = \max_{i,j} |\operatorname{Re}(\lambda_i) - \operatorname{Re}(\lambda_j)|$ and $\mathrm{Sp}_{\mathrm{Im}}(A) = \max_{i,j} |\operatorname{Im}(\lambda_i) - \operatorname{Im}(\lambda_j)|$.

Let $R(A)$ and c denote the radius and center of the smallest disc $\mathcal{D}$ which contains all eigenvalues of A. Let $R_{\mathrm{Re}}(A)$, and $R_{\mathrm{Im}}(A)$ denote the radius of two smallest discs containing all the real parts and the imaginary parts of the eigenvalues of A, respectively.

If all eigenvalues of A are real, then $\mathrm{Sp}(A) = 2R(A)$, this is so if A is Hermitian, i.e., $A = A^*$ where A^* is the conjugate transpose of A. In the general case, $\sqrt{3}R(A) \leq \mathrm{Sp}(A)$, (see [**2**], [**3**]).

Let $m = \operatorname{tr} A/n$ where $\operatorname{tr} A$ is *the trace* of A. Let $\operatorname{tr}_2 A = \sum_{i<j} \lambda_i \lambda_j$ denote the sum of all principal 2-rowed minors of A. We denote the e_i the column vector whose i-th component is 1 while all the other components are 0.

The distance of A from scalar matrices is defined by $\Delta(A) = \inf_{z \in \mathbb{C}} \|A - zI\|$ where $\|\cdot\|$ is the spectral norm. For any matrix A, we have $R(A) \leq \Delta(A)$, and

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equality holds if A is normal, i.e., $AA^* = A^*A$, we refer the reader to [**2**] for more details.

For every normal matrix A, the following bound was shown by E. R. Barnes and A. J. Hoffman [**1**],

$$\max_{i,j}\left\{|a_{ii}-a_{jj}|^2+2\sum_{k\neq i}|a_{ki}|^2+2\sum_{k\neq j}|a_{kj}|^2\right\}\leq 4R^2(A). \tag{1}$$

An upper bound for $R(A)$ was given by R. Bhatia and R. Sharma [**2**], that says that for any matrix A,

$$R^2(A)\leq \max_{\|x\|=1}\left(\|Ax\|^2-|\langle x,Ax\rangle|^2\right). \tag{2}$$

The present work proposes some lower and upper bounds for $R(A)$, and also shows sufficient conditions for equality.

In [**10**], L. Mirsky gave an upper bound for the spread of an arbitrary $n\times n$ matrix A,

$$\operatorname{Sp}(A)\leq\left\{2\|A\|_F^2-\frac{2}{n}|\operatorname{tr}A|^2\right\}^{1/2}, \tag{3}$$

where $\|A\|_F$ denotes the Frobenius norm. We reobtain this result and prove it by three different ways.

In [**11**], L. Mirsky gave some lower bounds for the spread of a Hermitian matrix A, such as

$$2\max_{i\neq j}|a_{ij}|\leq \operatorname{Sp}(A)$$

and

$$\max_{i\neq j}\left\{(a_{ii}-a_{jj})^2+4|a_{ij}|^2\right\}^{1/2}\leq \operatorname{Sp}(A).$$

Since $\operatorname{Sp}(A)=2R(A)$ when A is Hermitian, then it follows that

$$\max_{i\neq j}|a_{ij}|\leq R(A)$$

and

$$\max_{i\neq j}\left\{\left(\frac{a_{ii}-a_{jj}}{2}\right)^2+|a_{ij}|^2\right\}^{1/2}\leq R(A).$$

In [**14**], R. A. Smith and L. Mirsky gave the following upper bound for $R(A)$, $2R^2(A)\leq\sum_{i=1}^n|\lambda_i|^2$. Since $R(A)$ is invariant under translation, then

$$2R^2(A)\leq\sum_{i=1}^n|\lambda_i-m|^2. \tag{4}$$

An upper bound for $\sum_{i=1}^n|\lambda_i|^2$ due to Schur [**9**], [**10**] states that

$$\sum_{i=1}^n|\lambda_i|^2\leq\|A\|_F^2. \tag{5}$$

Equality occurs in (5) if and only if A is normal.

Let $\operatorname{Sp}(A)=|\lambda_{i_0}-\lambda_{j_0}|$, where $i_0\neq j_0$ and $i_0,j_0\in\{1,\dots,n\}$, we say that:

1. The n eigenvalues satisfy *condition* $\mathcal{H}_1$ if and only if $(n-2)$ among them are equal to each other and to the arithmetic mean of the remaining two.
2. The n eigenvalues satisfy *condition* $\mathcal{H}_2$ if and only if n is even, i.e., $n = 2k$ and k among them are equal to each other and to λ_{i_0}, while all the remaining eigenvalues, $(n-k)$, are also equal to each other and to λ_{j_0}.

2. Bounds for $R(A)$ and $\mathrm{Sp}(A)$

Theorem 2.1. *Let A be an $n \times n$ matrix. Then*

$$R(A) \leq \frac{1}{\sqrt{2}}\left\{\|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}\right\}^{1/2}. \tag{6}$$

If A is Hermitian and its eigenvalues satisfy condition $\mathcal{H}_1$, then equality holds.

Proof. We have

$$\begin{aligned}\sum_{i=1}^{n} |\lambda_i - m|^2 &= \sum_{i=1}^{n} \left(|\lambda_i|^2 - m\overline{\lambda_i} - \overline{m}\lambda_i + |m|^2\right) \\ &= \sum_{i=1}^{n} |\lambda_i|^2 - \frac{|\operatorname{tr} A|^2}{n}\ leq \|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}.\end{aligned}$$

By applying (4), the statement (6) follows immediately.
For equality, we assume without loss of generality that $\lambda_2 = \cdots = \lambda_{n-1} = \frac{1}{2}(\lambda_1 + \lambda_n)$ and A is a Hermitian matrix, it follows that $\frac{1}{2}\left(\|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}\right) = \left(\frac{\lambda_n - \lambda_1}{2}\right)^2 = R^2(A)$. □

From (6), we conclude the following corollary.

Corollary 2.2. $2R^2(A) \leq \|A\|_F^2$.

Theorem 2.3. *Let A be an $n \times n$ normal matrix. Define*

$$M_1(A) = \left\{\|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}\right\}^{1/2}.$$

Then

$$\frac{1}{\sqrt{n}} M_1(A) \leq R(A) \leq \frac{1}{\sqrt{2}} M_1(A). \tag{7}$$

If A is Hermitian and its eigenvalues satisfy condition $\mathcal{H}_1$ [$\mathcal{H}_2$], then equality on the left [right] of (7) *occurs.*

Proof. The second inequality (the upper bound of $R(A)$) is proved in the previous theorem. To prove the first inequality, we have

$$\sum_{i=1}^{n} |\lambda_i - c|^2 \leq nR^2(A). \tag{8}$$

On the other hand,

$$\sum_{i=1}^{n} |\lambda_i - c|^2 = \sum_{i=1}^{n} \big(|\lambda_i|^2 - c\overline{\lambda_i} - \overline{c}\lambda_i + |c|^2\big)$$
$$= \sum_{i=1}^{n} |\lambda_i|^2 - \frac{|\operatorname{tr} A|^2}{n} + n\left|c - \frac{\operatorname{tr} A}{n}\right|^2.$$

It is clear that the choice $c = \operatorname{tr} A/n$ gives the smallest possible value for this last expression. Hence $\{\|A\|_F^2 - |\operatorname{tr} A|^2/n\}/n \leq R^2(A)$.

For the second equality, it is proved in the previous theorem. For the first equality, we assume without loss of generality that n is even, $\lambda_1 = \lambda_2 = \cdots = \lambda_k$, $\lambda_{k+1} = \cdots = \lambda_n$, and A is a Hermitian matrix. It follows that

$$\frac{1}{n}\Big(\|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}\Big) = \Big(\frac{\lambda_n - \lambda_1}{2}\Big)^2 = R^2(A). \qquad \square$$

Remark. We can deduce the lower bound from [**2**, Theorem 1] using inequality (27) and relationship between $\Delta(A)$ and $R(A)$ for normal matrices.

E. Jiang and X. Zhan [**8**] proposed a slightly simpler proof for the lower bound of (9). We give here a new simpler proof of (9) using the previous theorem.

Theorem 2.4 ([**1**]). *Let A be an $n \times n$ Hermitian matrix. Then*

$$\frac{2}{\sqrt{n}} M_1(A) \leq \operatorname{Sp}(A) \leq \frac{2}{\sqrt{2}} M_1(A). \tag{9}$$

If the eigenvalues of A satisfy condition $\mathcal{H}_1$ [$\mathcal{H}_2$], then equality holds on the left [right] of (9).

Proof. Since A is Hermitian, then A is normal. Using $\operatorname{Sp}(A) = 2R(A)$ and (7), the result (9) is obtained immediately. The proof of equality in (9) is the same as the previous proof of equality in (7) using $\operatorname{Sp}(A) = 2R(A)$. $\square$

When $n = 2$, both sides of (7) and (9) are equal, so

$$R(A) = \frac{1}{\sqrt{2}}\Big\{\|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}\Big\}^{1/2} = \frac{|\lambda_2 - \lambda_1|}{2}$$

and

$$\operatorname{Sp}(A) = \frac{2}{\sqrt{2}}\Big\{\|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}\Big\}^{1/2} = |\lambda_2 - \lambda_1|.$$

Remark. When n is odd, then A. Brauer, A. C. Mewborn [**4**], and Popoviciu [**12**] showed that the lower bound in (9) may be strengthened to

$$\sqrt{\frac{4n}{n^2-1}}\Big(\sum_{i=n}^{n} \lambda_i^2 - \frac{(\sum_{i=1}^{n} \lambda_i)^2}{n}\Big)^{1/2} \leq \operatorname{Sp}(A),$$

and equality holds if and only if $\lambda_1 = \cdots = \lambda_{k+1}$ and $\lambda_{k+2} = \cdots = \lambda_n$ with $n = 2k+1$, see ([**7**]).

Lemma 2.5 ([**6**]). *Let $z_1, z_2, \dots, z_n$ be complex numbers. Then*

$$\frac{1}{n}\sum_{1\le i<j\le n}|z_i-z_j|^2=\sum_{i=1}^{n}|z_i|^2-\frac{1}{n}\Big|\sum_{i=1}^{n}z_i\Big|^2.$$

The proof is left to the reader.

The upper bound in the following theorem was given by L. Mirsky in [**10**].

Theorem 2.6. *Let A be an $n\times n$ normal matrix. Then*

$$\sqrt{\frac{2}{n-1}}M_1(A)\le \operatorname{Sp}(A)\le \frac{2}{\sqrt{2}}M_1(A). \tag{10}$$

If the eigenvalues of A satisfy condition $\mathcal{H}_1$, then equality holds on the left of (10).

Proof. The second inequality (upper bound of $\operatorname{Sp}(A)$) follows from $\operatorname{Sp}(A)\le 2R(A)$ and (6).

To prove the first inequality, let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A. Taking $z_i=\lambda_i$ and $z_j=\lambda_j$ in the previous lemma, it follows that

$$\frac{1}{n}\sum_{1\le i<j\le n}|\lambda_i-\lambda_j|^2=\sum_{i=1}^{n}|\lambda_i|^2-\frac{|\operatorname{tr}A|^2}{n}.$$

On the other hand, we have

$$\sum_{1\le i<j\le n}|\lambda_i-\lambda_j|^2\le\frac{n(n-1)}{2}\operatorname{Sp}^2(A),$$

where $\frac{1}{2}n(n-1)$ is the number of $|\lambda_i-\lambda_j|^2$ when $1\le i<j\le n$. By (5) the assertion follows immediately.

Furthermore, for equality we assume without loss of generality that $\lambda_2=\cdots=\lambda_{n-1}=\frac{1}{2}(\lambda_1+\lambda_n)$ and A is normal. Then

$$2\left(\|A\|_F^2-|\operatorname{tr}A|^2/n\right)=|\lambda_n-\lambda_1|^2=\operatorname{Sp}^2(A). \qquad \square$$

Theorem 2.7. *Let A be an $n\times n$ matrix. Define $M_2(A)=\Big\{\Big(1-\frac{1}{n}\Big)(\operatorname{tr}A)^2-2\operatorname{tr}_2 A\Big\}^{1/2}$. If all eigenvalues of A are real, then*

$$\frac{1}{\sqrt{n}}M_2(A)\le R(A)\le\frac{1}{\sqrt{2}}M_2(A). \tag{11}$$

If the eigenvalues of A satisfy condition $\mathcal{H}_1$ [$\mathcal{H}_2$], then equality on the left [right] of (11) *occurs.*

Proof. We have

$$\begin{aligned}\sum_{i=1}^{n}(\lambda_i-m)^2&=\sum_{i=1}^{n}\left(\lambda_i^2-2m\lambda_i+m^2\right)\\&=\sum_{i=1}^{n}\lambda_i^2-\frac{(\operatorname{tr}A)^2}{n}=\Big(1-\frac{1}{n}\Big)(\operatorname{tr}A)^2-2\operatorname{tr}_2 A.\end{aligned}$$

By (4), and (8) the required result follows immediately.
The proof of equality is similar to that of (7). □

The upper bound in the following corollary was proved by L. Mirsky in [**10**].

Corollary 2.8. *Let A be an $n \times n$ matrix. If all eigenvalues of A are real, then*

$$\frac{2}{\sqrt{n}} M_2(A) \leq \mathrm{Sp}(A) \leq \frac{2}{\sqrt{2}} M_2(A). \tag{12}$$

If the eigenvalues of A satisfy condition $\mathcal{H}_1$ [$\mathcal{H}_2$], then equality on the left [right] of (12) *occurs.*

Proof. Using $\mathrm{Sp}(A) = 2R(A)$ and (11), the result (12) is obtained immediately.
The proof of equality is similar to that of (9). □

It may be noted that the upper bound of (12) was obtained for the first time by J. v. Sz. Nagy [**4**] on algebraic equations and later by L. Mirsky [**10**].

3. Bounds for $R_{\mathrm{Re}}(A), R_{\mathrm{Im}}(A), \mathrm{Sp}_{\mathrm{Re}}(A)$ and $\mathrm{Sp}_{\mathrm{Im}}(A)$

For any normal matrix A, the eigenvalues of $\frac{1}{2}(A + A^*)$ are $\mathrm{Re}(\lambda_1), \dots, \mathrm{Re}(\lambda_n)$. Hence $R_{\mathrm{Re}}(A) = R\{\frac{1}{2}(A + A^*)\}$ and $\mathrm{Sp}_{\mathrm{Re}}(A) = \mathrm{Sp}\{\frac{1}{2}(A + A^*)\}$. Also the eigenvalues of $\frac{1}{2}(A - A^*)/\mathrm{i}$ are $\mathrm{Im}(\lambda_1), \dots, \mathrm{Im}(\lambda_n)$. Hence $R_{\mathrm{Im}}(A) = R\{\frac{1}{2}(A - A^*)/\mathrm{i}\}$ and $\mathrm{Sp}_{\mathrm{Im}}(A) = \mathrm{Sp}\{\frac{1}{2}(A - A^*)/\mathrm{i}\}$.

We propose some bounds for $R_{\mathrm{Re}}(A)$, $\mathrm{Sp}_{\mathrm{Re}}(A)$, $R_{\mathrm{Im}}(A)$ and $\mathrm{Sp}_{\mathrm{Im}}(A)$.

Theorem 3.1. *Let A be an $n \times n$ normal matrix. Define $M_3(A) = \Big\{\|A\|_F^2 + \mathrm{Re}(\mathrm{tr}(A^2)) - \frac{2}{n}(\mathrm{Re}(\mathrm{tr}\, A))^2\Big\}^{1/2}$. Then*

$$\frac{1}{\sqrt{2n}} M_3(A) \leq R_{\mathrm{Re}}(A) \leq \frac{1}{2} M_3(A). \tag{13}$$

If A is Hermitian and the real parts of its eigenvalues satisfy condition $\mathcal{H}_1$ [$\mathcal{H}_2$], then equality on the left [right] of (13) *occurs.*

Proof. Using (6), it follows that

$$\begin{aligned} R_{\mathrm{Re}}(A) &\leq \frac{1}{\sqrt{2}} \Bigg\{ \Big\| \frac{A + A^*}{2} \Big\|_F^2 - \frac{\big|\mathrm{tr}\left(\frac{A+A^*}{2}\right)\big|^2}{n} \Bigg\}^{1/2} \\ &= \frac{1}{\sqrt{2}} \Bigg\{ \frac{1}{4}\,\mathrm{tr}(A + A^*)(A^* + A) - \frac{(\mathrm{Re}(\mathrm{tr}\, A))^2}{n} \Bigg\}^{1/2} = \frac{1}{2} M_3(A). \end{aligned}$$

Since the matrix $\frac{1}{2}(A + A^*)$ is Hermitian, then it is normal. Using the lower bound of (7), it follows that

$$R_{\mathrm{Re}}(A) \geq \frac{1}{\sqrt{n}} \Bigg\{ \Big\| \frac{A + A^*}{2} \Big\|_F^2 - \frac{\big|\mathrm{tr}\left(\frac{A+A^*}{2}\right)\big|^2}{n} \Bigg\}^{1/2} = \frac{1}{\sqrt{2n}} M_3(A).$$

Equality on the left and on the right in (13), is a consequence of equality given in (7). □

The upper bound in the following corollary was given by L. Mirsky [**10**].

Corollary 3.2. *Let A be an $n \times n$ normal matrix. Then*

$$\sqrt{\frac{2}{n}} M_3(A) \leq \mathrm{Sp}_{\mathrm{Re}}(A) \leq M_3(A). \tag{14}$$

If the real parts of the eigenvalues of A satisfy condition $\mathcal{H}_1$, then equality on the left of (14) *occurs and if A is Hermitian and the real parts of its eigenvalues satisfy condition $\mathcal{H}_2$, then equality on the right of* (14) *occurs.*

Proof. Since the matrix $\frac{1}{2}(A + A^*)$ is Hermitian then it is normal. Using $\mathrm{Sp}_{\mathrm{Re}}(A) = 2R_{\mathrm{Re}}(A)$ and (13), we obtain the desired result.

Equality on the left [right] in (14) is a consequence of equality on the left [right] in (9). □

Theorem 3.3. *Let A be an $n \times n$ normal matrix. Define* $M_4(A) = \Big\{\|A\|_F^2 - \mathrm{Re}(\mathrm{tr}(A^2)) - \frac{2}{n}(\mathrm{Im}(\mathrm{tr}\, A))^2\Big\}^{1/2}$. *Then*

$$\frac{1}{\sqrt{2n}} M_4(A) \leq R_{\mathrm{Im}}(A) \leq \frac{1}{2} M_4(A). \tag{15}$$

If A is skew-Hermitian and the imaginary parts of its eigenvalues satisfy condition $\mathcal{H}_1$ [$\mathcal{H}_2$], then equality on the left [right] of (15) *occurs.*

Proof. The proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let A be an $n \times n$ normal matrix. Then*

$$\sqrt{\frac{2}{n}} M_4(A) \leq \mathrm{Sp}_{\mathrm{Im}}(A) \leq M_4(A). \tag{16}$$

If the imaginary parts of the eigenvalues of A satisfy condition $\mathcal{H}_1$, then equality on the left of (16) *occurs and if A is skew-Hermitian and the imaginary parts of its eigenvalues satisfy condition $\mathcal{H}_2$, then equality on the right of* (16) *occurs.*

Proof. The proof is also similar to that of Corollary 3.2. □

Theorem 3.5. *Let $A = (a_{ij})$ be an $n \times n$ matrix. Then*

$$R(A) \leq \sqrt{R^2_{\mathrm{Re}}(A) + R^2_{\mathrm{Im}}(A)}. \tag{17}$$

Proof. Let the eigenvalues of $\frac{1}{2}(A + A^*)$ and $\frac{1}{2}(A - A^*)/i$ be $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$, respectively. So all eigenvalues of A lie in the interior or on the boundary of the rectangle $\mathcal{G}$ constructed by the lines $x = \alpha_1, x = \alpha_n$; $y = \beta_1, y = \beta_n$. We have $\frac{1}{2}\, d(\mathcal{G}) = \frac{1}{2}\sqrt{(\alpha_n - \alpha_1)^2 + (\beta_n - \beta_1)^2} = \sqrt{R^2_{\mathrm{Re}}(A) + R^2_{\mathrm{Im}}(A)}$, where $d(\mathcal{G})$ denotes the diameter of the rectangle $\mathcal{G}$. On the other hand, all eigenvalues of A lie in the interior or on the circle $\mathcal{C}$ with radius equals to $\frac{1}{2}\, d(\mathcal{G})$ and center lies where two diagonals of $\mathcal{G}$ intersect each other.

Since $R(A)$ is the radius of the smallest disc that contains all eigenvalues of A, then $R(A) \leq \frac{1}{2}\, d(\mathcal{G})$. □

Corollary 3.6. *Let $A = (a_{ij})$ be an $n \times n$ matrix. Then*

$$\mathrm{Sp}(A) \leq \sqrt{\mathrm{Sp}^2_{\mathrm{Re}}(A) + \mathrm{Sp}^2_{\mathrm{Im}}(A)}. \tag{18}$$

Proof. The inequality follows from $\mathrm{Sp}(A) \leq 2R(A)$, $\mathrm{Sp}_{\mathrm{Re}}(A) = 2R_{\mathrm{Re}}(A)$, $\mathrm{Sp}_{\mathrm{Im}}(A) = 2R_{\mathrm{Im}}(A)$, and (17). □

Theorem 3.7. *Let $A = (a_{ij})$ be a normal matrix. Then*

$$\max_{i \neq j} |a_{ij}| \leq R(A), \tag{19}$$

$$\max_{i \neq j} \frac{|a_{ij} + \overline{a_{ji}}|}{2} \leq R_{\mathrm{Re}}(A), \tag{20}$$

$$\max_{i \neq j} \frac{|a_{ij} - \overline{a_{ji}}|}{2} \leq R_{\mathrm{Im}}(A), \tag{21}$$

$$\max_{i \neq j} \frac{|a_{ii} - a_{jj}|}{2} \leq R(A). \tag{22}$$

Proof. The statements (19) and (22) follow from (1).
– By (19), it follows that $\max_{i \neq j} \frac{1}{2}|a_{ij} + \overline{a_{ji}}| \leq R\{\frac{1}{2}(A + A^*)\} = R_{\mathrm{Re}}(A)$ and $\max_{i \neq j} \frac{1}{2}|a_{ij} - \overline{a_{ji}}| \leq R\{\frac{1}{2}(A - A^*)/\mathrm{i}\} = R_{\mathrm{Im}}(A)$. □

4. Other proofs of L. Mirsky's theorem for Sp(A)

In [**5**], E. Deutsch gave a new simpler proof of the theorem due to L. Mirsky [**10**]. For our part, we propose three slightly simpler proofs of this theorem.

Theorem 4.1 ([**10**])**.** *Let A be an $n \times n$ complex matrix. Then*

$$Sp(A) \leq \left\{2\|A\|_F^2 - \frac{2}{n}|\operatorname{tr} A|^2\right\}^{1/2}.$$

Proof. 1) We have $\mathrm{Sp}(A) \leq 2R(A)$, by (6), it follows that

$$\mathrm{Sp}(A) \leq \left\{2\|A\|_F^2 - \frac{2}{n}|\operatorname{tr} A|^2\right\}^{1/2}.$$ □

Proof. 2) Using (18) and the upper bound in (16), (14), we obtain

$$\mathrm{Sp}(A) \leq \sqrt{M_4^2(A) + M_3^2(A)} \leq \sqrt{2\|A\|_F^2 - \frac{2}{n}|\operatorname{tr} A|^2}.$$ □

Proof. 3) We have $\mathrm{Sp}(A) \leq 2R(A)$ and $2R^2(A) \leq \|A\|_F^2$, thus $\mathrm{Sp}^2(A) \leq 2\|A\|_F^2$. The spread of A is invariant under translation, so we have

$$\mathrm{Sp}^2(A) = \mathrm{Sp}^2(A - m) \leq 2\|A - m\|_F^2.$$

On the other hand,

$$\|A-m\|_F^2 = \operatorname{tr}(A-m)(A^*-\overline{m}) = \operatorname{tr}\left(AA^* - mA^* - \overline{m}A + |m|^2\right)$$
$$= \|A\|_F^2 - \frac{2}{n}|\operatorname{tr} A|^2 + \frac{1}{n}|\operatorname{tr} A|^2 = \|A\|_F^2 - \frac{|\operatorname{tr} A|^2}{n}.$$

Then the desired result is obtained. □

5. The circumferential spread of matrices

Theorem 5.1. *Let $\Gamma(A)$ denote the circumference of the smallest disc $\mathcal{D}$ which contains all eigenvalues of the matrix A. Then*

$$\sup\left(\frac{\operatorname{Sp}(A)}{\Gamma(A)}\right) = \frac{1}{\pi} \tag{23}$$

and

$$\sup\left(\frac{\Gamma(A)}{\operatorname{Sp}(A)}\right) = \frac{2}{\sqrt{3}}\pi, \tag{24}$$

where the supremum is taken over all nonzero $n \times n$ matrices A.

Proof. Since $\Gamma(A) = 2\pi R(A)$, it is sufficient to prove for the first assertion that $\sup(\operatorname{Sp}(A)/R(A)) = 2$. We have $\operatorname{Sp}(A) \le 2R(A)$. On the other hand, taking $A = \operatorname{diag}(-1, 0, \dots, 0, 1)$, it follows that $R(A) = 1$ and $\operatorname{Sp}(A) = 2$. Hence $\sup \operatorname{Sp}(A)/R(A) = 2$.

The second assertion of the theorem is equivalent to $\sup(R(A)/\operatorname{Sp}(A)) = 1/\sqrt{3}$. We have $\sqrt{3}R(A) \le \operatorname{Sp}(A)$. On the other hand, the solutions of the equation $z^3 = 1$ are $z_0 = 1$, $z_1 = \mathrm{e}^{2\pi \mathrm{i}/3}$, and $z_2 = \mathrm{e}^{4\pi \mathrm{i}/3}$. We see that z_0, z_1, z_2 lie on the unit circle. Taking the matrix $A = \operatorname{diag}(z_0, z_1, z_2, 0, \dots, 0)$, it follows that $R(A) = 1$ and $\operatorname{Sp}(A) = \sqrt{3}$. Hence $\sup(R(A)/\operatorname{Sp}(A)) = 1/\sqrt{3}$, this completes the proof. □

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