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# DISCRETE LYAPUNOV THEORY FOR EVOLUTION FAMILIES

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Abstract. The aim of this paper is to obtain some necessary and sufficient conditions for the uniform exponential dichotomy of discrete evolution families in Hilbert spaces. We prove the discrete versions of some theorems from [**17**], [**18**] for $C_0$-semigroups, differential systems and we use them to obtain Lyapunov type results. Also, we get generalizations for abstract evolution families.

## 1. Introduction

An important role in the study of asymptotic behavior of evolution families is represented by the results of Lyapunov type.

The theorem of A. M. Lyapunov states that if $A$ is an $n \times n$ complex matrix, then $A$ has all its characteristic roots with real parts negative if and only if for any positive definite Hermitian matrix $H$, there exists a unique positive definite Hermitian matrix $W$ satisfying the equation

$$A^*W + WA = -H,$$

where $*$ denotes the conjugate transpose of a matrix (see [**1**] for details).

This result was extended by M. G. Krein and J. L. Daleckij in [**5**]. They proved that the semigroup $T(t) = \mathrm{e}^{tA}$ (with $A$ a bounded linear operator) is exponentially stable if and only if there exists a bounded linear operator $W$, $W \gg 0$ (more precisely, there exists $m > 0$ such that $\langle Wx, x\rangle \geq m||x||$ for all $x \in X$) as a solution for the autonomous Lyapunov equation

$$A^*W + WA = -I.$$

Also, R. Datko [**6**] showed that a $C_0$-semigroup $\{T_t\}_{t\geq 0}$ on a Hilbert space $X$ is exponentially stable if and only if there exists a bounded linear operator $W$, $W^* = W$, $W \ll 0$ such that

$$\langle Ax, Wx\rangle + \langle Wx, Ax\rangle \leq -\|x\|^2 \qquad \text{for all } x \in D(A),$$

where

$$D(A) = \{x \in X : \exists \lim_{t\to 0_+} \frac{T(t)x - x}{t} \text{ in } X\} \qquad \text{and} \qquad Ax = \lim_{t\to 0_+} \frac{T(t)x - x}{t}.$$

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Other important results were obtained by C. Chicone, Y. Latushkin [**2**], L. Pandolfi [**13**], A. Pazy [**14**] for the case when $A$ is an unbounded operator.

A generalization of exponential stability is the so-called dichotomic behavior. In the same paper from 1974 ([**5**]), the authors showed that the semigroup $T(t) = \mathrm{e}^{tA}$ is exponentially dichotomic if and only if the autonomous Lyapunov equation has a bounded linear self-adjoint solution $W$.

Other results by exponentially dichotomy for $C_0$-semigroups were obtained by K. J. Engel, R. Nagel in [**8**] and P. Preda, M. Megan in [**17**], more precisely they generalized the result of exponential stability, obtained by R. Datko [**6**], to uniform exponential dichotomy.

The case of linear differential systems

$$\dot{x}(t) = A(t)x(t), \tag{A}$$

was studied by W. A. Coppel who in 1978 obtained that in finite dimensional spaces, the differential system $(A)$ is exponentially dichotomic if and only if the non-autonomous Lyapunov inequality

$$\dot{W}(t) + A^*(t)W(t) + W(t)A(t) \leq -I \qquad \text{for all } t \geq 0$$

has a bounded self-adjoint solution (see [**4**, page 59, Proposition 1 and Proposition 2]).

In [**11**, Chapter 9], J. L. Massera and J. J. Schäffer proved that the existence of the Lyapunov function ensures a dichotomic behavior for the differential system $(A)$ if the subspace $X_2$ (which is a complement of the space of initial conditions of initial solutions for the system $(A)$, denoted $X_1$) is finite dimensional. This fact is mentioned in terminology [**11**] in the form of $X_1$ has a finite codimension.

In the literature relating to the asymptotic behavior of solutions of the differential system $(A)$, the hypotesis $A \in M_1(\mathcal{B}(X))$ is used frequently which means,

$$\sup_{t\geq 0} \int_t^{t+1} \|A(u)\| \mathrm{d}u < \infty$$

and according to the references [**5**], [**11**], this ensures the uniform exponential growth property of the evolution family $\Phi$ (generated by the differential system $(A)$), i.e., exist $M, \omega > 0$ such that

$$\|\Phi(t, t_0)\| \leq M\, \mathrm{e}^{\omega(t-t_0)} \qquad \text{for all } t \geq t_0 \geq 0.$$

The extension of Datko's result from stability (see [**7**]) to dichotomy was made by P.Preda and M. Megan [**18**] for differential systems and for abstract evolution families (i.e. not necessary provided by a differential system).

Concerning the discrete-time approach, we can mention the papers of C. V. Coffman, J. J. Schäffer [**3**], La Salle [**9**], M. Megan, B. Sasu, A. L. Sasu [**12**], M. Pinto[**15**], A. Pogan, P. Preda, C. Preda [**16**] and P. Preda, A. Pogan, C. Preda [**19**]. Also, K. M. Przyluski [**20**] and K. M. Przyluski, S. Rolewicz [**21**] showed applications related to the discrete theory of stability for linear infinite-dimensional continuous-time systems.

In this paper, our aim is to give new results for uniform exponential dichotomy, in the discrete case for $C_0$-semigroups, differential systems and abstract evolution families in Hilbert spaces. We prove firstly the discrete versions of some theorems from the continuous case from [**17**], [**18**], and then use them to obtain the results of Lyapunov type.

## 2. Preliminaries

Firstly, we establish the main notations and we recall some definitions used in the present paper.

Let $X$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{B}(X)$ be the Banach algebra of all linear and bounded operators acting on $X$. The norms on $X$ and $\mathcal{B}(X)$ are denoted by $\| \cdot \|$. Also, $\mathbb{R}_+ = [0, \infty), \mathbb{N}$ represents the set of nonnegative integers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

**Definition 2.1.** A family $\{T_t\}_{t \geq 0}$ of linear and bounded operators on $X$ is called a $C_0$-*semigroup* if the following conditions hold:

(i) $T(0) = I$ (where I is the identity on $X$);
(ii) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$;
(iii) $\lim\limits_{t \to 0_+} T(t)x = x$ for all $x \in X$.

We suppose that the subspace $X_1 = \{x \in X : T(\cdot)x \in L^\infty(X)\}$ is a closed subspace in X, where $L^\infty(X)$ represents the Banach space of $X$-valued functions $f$ almost defined on $\mathbb{R}_+$, $f$ is strongly measurable and essentially bounded. By $X_2$ we denote a complement of $X_1$ and a projector by $P_1$ ($P_1 \in \mathcal{B}(X)$, $P_1^2 = P_1$, $\operatorname{Ker} P_1 = X_2$). Also $P_2 = I - P_1$.

**Definition 2.2.** We say that $\{T_t\}_{t \geq 0}$ is *exponentially dichotomic* if there exist $N_1$, $N_2$, $\nu > 0$ such that:

(i) $\|T(t)x\| \leq N_1 \, \mathrm{e}^{-\nu t} \, \|x\|$ for all $x \in X_1$;
(ii) $\|T(t)x\| \geq N_2 \, \mathrm{e}^{\nu t} \, \|x\|$ for all $x \in X_2$.

**Definition 2.3.** A family of operators $\Phi \colon \{(n, n_0) \in \mathbb{N} \times \mathbb{N}, n \geq n_0\} \to \mathcal{B}(X)$ is called a *discrete evolution family* if the following properties hold:

(i) $\Phi(n, n) = I$ for all $n \in \mathbb{N}$;
(ii) $\Phi(n, m)\Phi(m, n_0) = \Phi(n, n_0)$ for all $n \geq m \geq n_0$, $n, m, n_0 \in \mathbb{N}$.

*Remark* 2.1. If in addition to the conditions (i), (ii) from Definition 2.3, there are $M$, $\omega > 0$ such that

$$\|\Phi(n, n_0)\| \leq M \, \mathrm{e}^{\omega(n - n_0)} \qquad \text{for all } n \geq n_0 \geq 0,$$

then $\{\Phi(n, n_0)\}_{n \geq n_0 \geq 0}$ is called a discrete evolution family with uniform exponential growth.

We assume that for every $n_0 \in \mathbb{N}$, the vector subspace

$$X_1(n_0) = \{x \in X : \Phi(\cdot, n_0)x \in l^\infty(X)\}$$

is closed in $X$, where

$$l^\infty(X) = \left\{x\colon \mathbb{N} \to X : \sup_{n\in\mathbb{N}} \|x_n\| < \infty\right\}.$$

By $X_2(n_0)$ we denote a complement of $X_1(n_0)$ and a projector by $P_1(n_0)$ (i.e. $P_1(n_0) \in \mathcal{B}(X), P_1^2(n_0) = P_1(n_0)$) such that $\operatorname{Ker} P_1(n_0) = X_2(n_0)$ and $P_2(n_0) = I - P_1(n_0)$. Also $P_i(0)$ is denoted by $P_i, i = 1, 2$.

**Definition 2.4.** A discrete evolution family $\{\Phi(n, n_0)\}_{n\geq n_0\geq 0}$ is *uniformly exponentially dichotomic* if there exist the constants $N_1, N_2, \nu > 0$ such that:

(i) $\|\Phi(n, n_0)x\| \leq N_1 \,\mathrm{e}^{-\nu(n-n_0)}\, \|x\|$ for all $x \in X_1(n_0)$, $n \geq n_0$;
(ii) $\|\Phi(n, n_0)x\| \geq N_2 \,\mathrm{e}^{\nu(n-n_0)}\, \|x\|$, for all $x \in X_2(n_0)$, $n \geq n_0$.

**Definition 2.5.** A function $P_1\colon \mathbb{N} \to \mathcal{B}(X)$ is called a dichotomy projector family if:

- $P_1(n)\Phi(n, n_0) = \Phi(n, n_0)P_1(n_0)$ for all $n \geq n_0$;
- $\Phi(n, n_0)\colon \operatorname{Ker} P_1(n_0) \to \operatorname{Ker} P_1(n)$ is an isomorphism for all $n \geq n_0$;
- the discrete evolution family $\{\Phi(n, n_0)\}_{n\geq n_0\geq 0}$ is uniformly exponentially dichotomic with $X_1(n_0) = \operatorname{Im} P_1(n_0)$.

Also, we present an auxiliary lemma as follows:

**Lemma 2.1.** *Let $\psi(n), \rho(n)$ be two positive functions, $n \in \mathbb{N}$.*

*If $\inf_{n\in\mathbb{N}} \rho(n) < 1$ and $\psi(n) \leq \rho(n - n_0)\psi(n_0)$ for all $n \geq n_0 \geq 0$, there exist $N, \nu > 0$ such that*

$$\psi(n) \leq N \,\mathrm{e}^{-\nu(n-n_0)}\, \psi(n_0) \qquad \text{for all } n \geq n_0 \geq 0.$$

*If $\sup_{n\in\mathbb{N}} \rho(n) > 1$, and $\psi(n) \geq \rho(n - n_0)\psi(n_0)$ for all $n \geq n_0 \geq 0$, there exist $N', \nu' > 0$ such that*

$$\psi(n) \geq N' \,\mathrm{e}^{\nu'(n-n_0)}\, \psi(n_0) \qquad \text{for all } n \geq n_0 \geq 0.$$

*Proof.* Is similar to that of [**10**, Lemma 5.3]. □

**Definition 2.6.** The differential system (A) $\dot{x}(t) = A(t)x(t)$ is said to be *uniformly exponentially dichotomic* if there exist $N, \nu > 0$ such that:

$$\|U(t)P_1U^{-1}(s)\| \leq N \,\mathrm{e}^{-\nu(t-s)} \qquad \text{for all } t \geq s \geq 0;$$
$$\|U(t)P_2U^{-1}(s)\| \leq N \,\mathrm{e}^{-\nu(s-t)} \qquad \text{for all } s \geq t \geq 0.$$

In the following we recall characterizations for the dichotomy of a $C_0$-semigroup and differential systems in Banach spaces, results that will be used in our discrete researches into the dichotomy of a $C_0$-semigroup and differential systems.

**Theorem 2.1.** ([**17**], Corrolary 3.2) *Let $T(t)$ be a $C_0$-semigroup of linear operators defined on a Banach space $X$. Then $T(t)$ is exponentially dichotomic if and only if there exist $m, c > 0$ and $p > 0$ such that:*

- $\displaystyle\int_t^\infty \|T(u - t)P_1x\|^p \mathrm{d}u \leq c^p \cdot \|P_1x\|^p,$

- $\int_0^t \|T(s)P_2x\|^p \mathrm{d}s \le c^p \cdot \|T(t)P_2x\|^p$,
- $\|T(1)P_2x\| \ge m\|P_2x\|$

*for all $t \ge 0$ and $x \in X$.*

**Theorem 2.2.** ([**18**], Corollary 3.2 and Corollary 3.3) *The differential system $(A)$ $\dot{x}(t) = A(t)x(t)$ is uniformly exponentially dichotomic if and only if there exist $N, p > 0$ such that:*

(i) $\|U(t)P_1U^{-1}(t)\| \le N$ *for all* $t \ge 0$;

(ii) $\Big(\int_t^\infty \|U(\tau)P_1U^{-1}(t)x\|^p d\tau\Big)^{1/p} + \Big(\int_0^t \|U(\tau)P_2U^{-1}(t)x\|^p d\tau\Big)^{1/p} \le N\|x\|$ *for all* $t \ge 0$, $x \in X$.

## 3. The discrete Lyapunov method for the dichotomy of $C_0$-semigroups

Firstly, we obtain the discrete version of the Theorem 2.1 as follows:

**Proposition 3.1.** *If there exists $L, p > 0$ such that:*

(i) $\sum_{k=0}^{\infty} \|T(k)x\|^p < \infty$ *for all* $x \in X_1$;

(ii) $\Big(\sum_{k=0}^{n-1} \|T(k)x\|^p\Big)^{\frac{1}{p}} \le L\|T(n)x\|$ *for all* $n \ge 1$, $x \in X_2$

*then $\{T_t\}_{t \ge 0}$ is exponentially dichotomic.*

*Proof.* Let $x \in X_1$, $t \ge 0$, $k = [t]$, (where $[\cdot]$ represents the largest integer less than or equal to $t$). We have that

$$\|T(t)x\| = \|T(t-k)T(k)x\| \le M\,\mathrm{e}^{\omega}\,\|T(k)x\|,$$

and then

$$\sum_{k=0}^{\infty} \int_k^{k+1} \|T(t)x\|^p \mathrm{d}t \le (M\,\mathrm{e}^{\omega})^p \sum_{k=0}^{\infty} \|T(k)x\|^p < \infty.$$

It follows that

$$\int_0^\infty \|T(t)x\|^p \mathrm{d}t < \infty \qquad \text{for all } x \in X_1. \tag{1}$$

Taking now $x \in X_2$, $t \ge 0$, $\tau \in [0, t]$, $n = [t]$, $k = [\tau]$, we get

$$\|T(\tau)x\| = \|T(\tau-k)T(k)x\| \le M\,\mathrm{e}^{\omega}\,\|T(k)x\|.$$

Proceeding similarly as above, we obtain

$$\sum_{k=0}^{n} \int_k^{k+1} \|T(\tau)x\|^p d\tau \le (M\,\mathrm{e}^{\omega})^p \sum_{k=0}^{n} \|T(k)x\|^p \le (M\,\mathrm{e}^{\omega}\,L)^p \|T(n+1)x\|^p,$$

which implies

$$\int_0^{n+1} \|T(\tau)x\|^p d\tau \leq (M\,e^{\omega}\,L)^p \|T(n+1-t)\|^p \|T(t)x\|^p.$$

Thus,

$$\Big(\int_0^t \|T(\tau)x\|^p d\tau\Big)^{\frac{1}{p}} \leq (M\,e^{\omega})^2 L \|T(t)x\| \quad \text{for all } t \geq 0,\ x \in X_2. \tag{2}$$

By (ii), it results that

$$\|T(1)x\| \geq \frac{1}{L}\|x\| \qquad \text{for all } x \in X_2. \tag{3}$$

From the relations (1), (2), (3) and Theorem 2.1, we obtain that $\{T_t\}_{t\geq 0}$ is exponentially dichotomic. □

**Proposition 3.2.** *If* $T(n)P_1 = P_1T(n)$ *for all* $n \in \mathbb{N}$, *where* $P_1X = X_1$, *then* $\{T_t\}_{t\geq 0}$ *is exponentially dichotomic if and only if there exist* $L$, $p > 0$ *such that*

(i) $\sum\limits_{k=0}^{\infty} \|T(k)x\|^p < \infty$ *for all* $x \in X_1$;

(ii) $\Big(\sum\limits_{k=0}^{n-1} \|T(k)x\|^p\Big)^{\frac{1}{p}} \leq L\|T(n)x\|$ *for all* $n \geq 1,\ x \in X_2$.

*Proof. Necessity.* We assume that $\{T_t\}_{t\geq 0}$ is exponentially dichotomic. By Definition 2.2, we have

$$\sum_{k=0}^{\infty} \|T(k)x\|^p \leq N_1^p \sum_{k=0}^{\infty} e^{-\nu pt}\,\|x\|^p = \frac{N_1^p}{1-e^{-\nu p}}\|x\|^p < \infty$$

for all $x \in X_1,\ p > 0$. Let now $x \in X_2,\ n \geq 1,\ k \in \{0, 1, \ldots, n-1\}$. Then

$$\|T(n)x\| = \|T(n-k)T(k)x\| \geq N_2\,e^{\nu(n-k)}\,\|T(k)x\|,$$

which implies

$$\sum_{k=0}^{n-1} \|T(k)x\|^p \leq \frac{1}{N_2^p}\sum_{k=0}^{n-1} e^{-\nu p(n-k)}\,\|T(n)x\|^p \leq \frac{1}{N_2^p(1-e^{-\nu p})}\|T(n)x\|^p.$$

We obtain

$$\Big(\sum_{k=0}^{n-1} \|T(k)x\|^p\Big)^{\frac{1}{p}} \leq \frac{1}{N_2(1-e^{-\nu p})^{\frac{1}{p}}}\|T(n)x\| \quad \text{for all } n \geq 1,\ x \in X_2.$$

*Sufficiency.* It follows from Proposition 3.1. □

**Theorem 3.1.** *Let* $\{T_t\}_{t\geq 0}$ *be a* $C_0$*-semigroup. If there exists* $W = W^* \in \mathcal{B}(X)$ *such that:*

(i) $T^*(n)WT(n) + \sum\limits_{k=0}^{n-1} T^*(k)T(k) \leq W$ *for all* $n \in \mathbb{N}^*$;

(ii) $\langle Wx, x\rangle \geq 0$ *for all* $x \in X_1$;

(iii) $\langle Wx, x\rangle \leq 0$ *for all* $x \in X_2$,

*then* $\{T_t\}_{t\geq 0}$ *is exponentially dichotomic.*

*Proof.* Let $x \in X_1$. From (i), we have

$$\langle WT(n)x, T(n)x\rangle + \sum_{k=0}^{n-1} \|T(k)x\|^2 \leq \langle Wx, x\rangle.$$

Thus

$$\sum_{k=0}^{n-1} \|T(k)x\|^2 \leq |\langle Wx, x\rangle| \leq \|W\| \cdot \|x\|^2 \qquad \text{for all } n \geq 1.$$

We obtain

$$\sum_{k=0}^{\infty} \|T(k)x\|^2 \leq \|W\| \cdot \|x\|^2 < \infty \qquad \text{for all } x \in X_1. \tag{4}$$

Setting now $x \in X_2$, $n \geq 1$ and proceeding similarly as above we obtain

$$\begin{aligned}\sum_{k=0}^{n-1} \|T(k)x\|^2 &\leq \langle Wx, x\rangle - \langle WT(n)x, T(n)x\rangle \\ &\leq |\langle WT(n)x, T(n)x\rangle| \leq \|W\| \cdot \|T(n)x\|^2,\end{aligned}$$

which implies

$$\Big(\sum_{k=0}^{n-1} \|T(k)x\|^2\Big)^{\frac{1}{2}} \leq \sqrt{\|W\|} \cdot \|T(n)x\| \qquad \text{for all } n \geq 1,\ x \in X_2. \tag{5}$$

From (4), (5) and Proposition 3.1, it follows that $\{T_t\}_{t\geq 0}$ is exponentially dichotomic. □

**Theorem 3.2.** *Let* $\{T_t\}_{t\geq 0}$ *be a* $C_0$*-semigroup with* $T(n)P_1 = P_1T(n)$ *for all* $n \in \mathbb{N}$*, and there exist* $n \in \mathbb{N}$ *such that* $T(n)$ *restricted to* $X_2$ *is surjective. If* $\{T_t\}_{t\geq 0}$ *is exponentially dichotomic, then there exists* $W = W^* \in \mathcal{B}(X)$ *such that:*

(i) $T^*(n)WT(n) + \sum_{k=0}^{n-1} T^*(k)T(k) \leq W$ *for all* $n \in \mathbb{N}^*$;

(ii) $\langle Wx, x\rangle \geq 0$ *for all* $x \in X_1$;

(iii) $\langle Wx, x\rangle \leq 0$ *for all* $x \in X_2$.

*Proof.* Let $W = 2\sum_{k=0}^{\infty} (T(k)P_1)^*T(k)P_1 - 2\sum_{k=1}^{\infty} (T^{-1}(k)P_2)^*T^{-1}(k)P_2$.
Using Definition 2.2, we obtain

$$\|W\| \leq 2\sum_{k=0}^{\infty} N_1^2 \,\mathrm{e}^{-2\nu k} + 2\sum_{k=1}^{\infty} \frac{1}{N_2^2}\,\mathrm{e}^{-2\nu k} \leq \frac{2}{1-\mathrm{e}^{-2\nu}}\left(N_1^2 + \frac{1}{N_2^2}\right),$$

so $W \in \mathcal{B}(X)$ and from the properties of the self-adjoint, we obtain $W = W^*$. Then

$$\begin{aligned}
&T^*(n)WT(n)\\
&= 2\sum_{k=0}^{\infty} T^*(n)(T(k)P_1)^*T(k)P_1T(n) - 2\sum_{k=1}^{\infty} T^*(n)(T^{-1}(k)P_2)^*T^{-1}(k)P_2T(n)\\
&= 2\sum_{k=0}^{\infty}(T(k+n)P_1)^*T(k+n)P_1 - 2\sum_{k=1}^{\infty}(T^{-1}(k)T(n)P_2)^*T^{-1}(k)T(n)P_2\\
&= 2\sum_{i=n}^{\infty}(T(i)P_1)^*T(i)P_1 - 2\sum_{k=1}^{n}(T^{-1}(k)T(k)T(n-k)P_2)^*T(n-k)P_2\\
&\quad - 2\sum_{k=n+1}^{\infty}((T(n)T(k-n))^{-1}T(n)P_2)^*T^{-1}(k-n)P_2\\
&= 2\sum_{i=n}^{\infty}(T(i)P_1)^*T(i)P_1 - 2\sum_{i=0}^{n-1}(T(i)P_2)^*T(i)P_2 - 2\sum_{i=1}^{\infty}(T^{-1}(i)P_2)^*T^{-1}(i)P_2\\
&= W - 2\sum_{i=0}^{n-1}(T(i)P_1)^*T(i)P_1 - 2\sum_{i=0}^{n-1}(T(i)P_2)^*T(i)P_2.
\end{aligned}$$

We get

$$T^*(n)WT(n) + 2\sum_{i=0}^{n-1}(T(i)P_1)^*T(i)P_1 + 2\sum_{i=0}^{n-1}(T(i)P_2)^*T(i)P_2 = W.$$

But

$$\begin{aligned}
&2\Big\langle \sum_{i=0}^{n-1}(T(i)P_1)^*T(i)P_1x, x\Big\rangle + 2\Big\langle \sum_{i=0}^{n-1}(T(i)P_2)^*T(i)P_2x, x\Big\rangle\\
&= 2\sum_{i=0}^{n-1}\|T(i)P_1x\|^2 + 2\sum_{i=0}^{n-1}\|T(i)P_2x\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle Wx, x\rangle &= \langle WT(n)x, T(n)x\rangle + 2\sum_{i=0}^{n-1}\big(\|T(i)P_1x\|^2 + \|T(i)P_2x\|^2\big)\\
&\geq \langle WT(n)x, T(n)x\rangle + \sum_{i=0}^{n-1}\|T(i)(P_1+P_2)x\|^2\\
&= \langle WT(n)x, T(n)x\rangle + \sum_{i=0}^{n-1}\langle T^*(i)T(i)x, x\rangle \qquad \text{for all } x \in X,\ n \geq 1.
\end{aligned}$$

It results that

$$T^*(n)WT(n) + \sum_{i=0}^{n-1} T^*(i)T^(i) \leq W \qquad \text{for all } n \in \mathbb{N}^*.$$

For $x \in X_1$, we obtain

$$\langle Wx, x\rangle = 2\sum_{k=0}^{\infty} \|T(k)x\|^2 \geq 0$$

and if $x \in X_2$, then

$$\langle Wx, x\rangle = -2\sum_{k=1}^{\infty} \|T(k)x\|^2 \leq 0.$$

□

## 4. The discrete Lyapunov method for the dichotomy of evolution families

**Proposition 4.1.** *Let $\{\Phi(n, n_0)\}_{n\geq n_0\geq 0}$ be a discrete evolution family such that there exist $L, m > 0$ and $p \geq 1$ such that:*

(i) $\left(\sum\limits_{k=n_0}^{\infty} \|\Phi(k, n_0)x\|^p\right)^{\frac{1}{p}} \leq L\|x\|$ *for all* $x \in X_1(n_0)$*,* $n_0 \in \mathbb{N}$*;*

(ii) $\left(\sum\limits_{k=n_0}^{n-1} \|\Phi(k, n_0)x\|^p\right)^{\frac{1}{p}} \leq L\|\Phi(n, n_0)x\|$ *for all* $x \in X_2(n_0)$*,* $n \geq n_0 + 1$*;*

(iii) $\|\Phi(n + 1, n)x\| \geq m\|x\|$ *for all* $x \in X_2(n)$*,* $n \in \mathbb{N}$*.*

*Then $\{\Phi(n, n_0)\}_{n\geq n_0\geq 0}$ is uniformly exponentially dichotomic.*

*Proof.* Let $x \in X_1(n_0), n \geq n_0$ and $k \in \{n_0, n_0 + 1, \ldots, n\}$. We have

$$\|\Phi(n, n_0)x\| = \|\Phi(n, k)\Phi(k, n_0)x\| \leq L\|\Phi(k, n_0)x\|,$$

and then

$$(n - n_0 + 1)\|\Phi(n, n_0)x\|^p \leq L^p \sum_{k=n_0}^{n} \|\Phi(k, n_0)x\|^p \leq L^{2p}\|x\|^p.$$

We deduce that

$$\|\Phi(n, n_0)x\| \leq \frac{L^2}{(n - n_0 + 1)^{\frac{1}{p}}}\|x\| \qquad \text{for all } n \geq n_0,\ x \in X_1(n_0).$$

Taking now $n \geq k \geq n_0$, $x \in X_1(n_0)$, we get

$$\|\Phi(n, n_0)x\| = \|\Phi(n, k)\Phi(k, n_0)x\| \leq \frac{L^2}{(n - k + 1)^{\frac{1}{p}}}\|\Phi(k, n_0)x\|$$

and by Lemma 2.1, it follows that there exist $N_1, \nu > 0$ such that

$$\|\Phi(n, n_0)x\| \leq N_1\, \mathrm{e}^{-\nu(n-k)}\, \|\Phi(k, n_0)x\| \qquad \text{for all } n \geq k \geq n_0,\ x \in X_1(n_0).$$

Thus,

$$\|\Phi(n, n_0)x\| \leq N_1\, \mathrm{e}^{-\nu(n-n_0)}\, \|x\| \qquad \text{for all } n \geq n_0,\ x \in X_1(n_0). \tag{6}$$

Let $n \geq n_0 + 1, x \in X_2(n_0)$. We denote

$$\varphi(n) = \sum_{k=n_0}^{n-1} \|\Phi(k, n_0)x\|^p$$

and by (ii), we obtain

$$\varphi(n) \leq L^p \|\Phi(n, n_0)x\|^p \qquad \text{for all } n \geq n_0 + 1.$$

Thus,

$$(n - n_0 - 1)\|x\|^p \leq \sum_{k=n_0+1}^{n-1} \varphi(k) \leq L^p \sum_{k=n_0+1}^{n-1} \|\Phi(k, n_0)x\|^p \leq L^{2p} \|\Phi(n, n_0)x\|^p,$$

which implies

$$\|\Phi(n, n_0)x\| \geq \frac{(n - n_0 - 1)^{\frac{1}{p}}}{L^2} \|x\| \qquad \text{for all } n \geq n_0 + 1, x \in X_2(n_0).$$

We deduce that $\|\Phi(n, n_0)x\| \geq \frac{(n-k-1)^{\frac{1}{p}}}{L^2} \|\Phi(k, n_0)x\|$ for all $n \geq k \geq n_0 + 1$, $x \in X_2(n_0)$.

Applying Lemma 2.1, we obtain that there exist $N_2, \nu > 0$ such that

$$\|\Phi(n, n_0)x\| \geq N_2\, \mathrm{e}^{\nu(n-n_0)}\, \|x\| \qquad \text{for all } x \in X_2(n_0),\ n \geq n_0. \tag{7}$$

From relations (6), (7) and Definition 2.4, we obtain that the discrete evolution family $\{\Phi(n, n_0)\}_{n \geq n_0 \geq 0}$ is uniformly exponentially dichotomic. □

**Theorem 4.1.** *If there exist $m > 0$ and $W\colon \mathbb{N} \to \mathcal{B}(X)$ bounded, $W(n) = W^*(n)$ for all $n \in \mathbb{N}$, such that:*

(i) $\Phi^*(n, n_0)W(n)\Phi(n, n_0) + \sum_{k=n_0}^{n-1} \Phi^*(k, n_0)\Phi(k, n_0) \leq W(n_0)$ *for all* $n \geq n_0 + 1$;

(ii) $\langle W(n)x, x\rangle \geq 0$ *for all* $x \in X_1(n),\ n \in \mathbb{N}$;

(iii) $\langle W(n)x, x\rangle \leq 0$ *for all* $x \in X_2(n),\ n \in \mathbb{N}$;

(iv) $\|\Phi(n + 1, n)x\| \geq m\|x\|$ *for all* $x \in X_2(n),\ n \in \mathbb{N}$.

*Then the discrete evolution family $\{\Phi(n, n_0)\}_{n \geq n_0 \geq 0}$ is uniformly exponentially dichotomic.*

*Proof.* Let $x \in X_1(n_0)$, $n \geq n_0 + 1$. According to the hypothesis, we have

$$\begin{aligned}\sum_{k=n_0}^{n-1} \|\Phi(k, n_0)x\|^2 &\leq \langle W(n_0)x, x\rangle - \langle W(n)\Phi(n, n_0)x, \Phi(n, n_0)x\rangle \\ &\leq |\langle W(n_0)x, x\rangle| \leq \|W(n_0)\| \cdot \|x\|^2 \leq L\|x\|^2,\end{aligned}$$

where $L = \sup_{n \in \mathbb{N}} \|W(n)\|$. It follows that

$$\Big(\sum_{k=n_0}^{\infty} \|\Phi(k, n_0)x\|^2\Big)^{\frac{1}{2}} \leq \sqrt{L}\|x\| \qquad \text{for all } n_0 \in \mathbb{N},\ x \in X_1(n_0). \tag{8}$$

Let $x \in X_2(n_0),\ n \geq n_0 + 1$. Then

$$\begin{aligned}\sum_{k=n_0}^{n-1} \|\Phi(k, n_0)x\|^2 &\leq \langle W(n_0)x, x\rangle - \langle W(n)\Phi(n, n_0)x, \Phi(n, n_0)x\rangle \\ &\leq |\langle W(n)\Phi(n, n_0)x, \Phi(n, n_0)x\rangle| \leq L\|\Phi(n, n_0)x\|^2,\end{aligned}$$

which implies

$$(9)\quad \Big(\sum_{k=n_0}^{n-1}\|\Phi(k,n_0)x\|^2\Big)^{\frac{1}{2}} \le \sqrt{L}\|\Phi(n,n_0)x\| \quad \text{for all } n\ge n_0+1,\ x\in X_2(n_0).$$

From (8), (9), (iv) and Proposition 4.1, it follows that the discrete evolution family $\{\Phi(n,n_0)\}_{n\geq n_0\geq 0}$ is uniformly exponentially dichotomic. □

**Theorem 4.2.** *If $P_1:\mathbb{N}\to\mathcal{B}(X)$ is a dichotomy projector family associated to a discrete evolution family $\{\Phi(n,n_0)\}_{n\geq n_0\geq 0}$, then there exist $m>0$ and $W:\mathbb{N}\to\mathcal{B}(X)$ bounded, $W(n)=W^*(n)$ for all $n\in\mathbb{N}$ such that:*

(i) $\Phi^*(n,n_0)W(n)\Phi(n,n_0)+\sum_{k=n_0}^{n-1}\Phi^*(k,n_0)\Phi(k,n_0)\le W(n_0)$ *for all* $n\geq n_0+1$;
(ii) $\langle W(n)x,x\rangle\geq 0$ *for all* $x\in X_1(n)$, $n\in\mathbb{N}$;
(iii) $\langle W(n)x,x\rangle\leq 0$ *for all* $x\in X_2(n)$, $n\in\mathbb{N}$;
(iv) $m\|x\|\leq\|\Phi(n+1,n)x\|$ *for all* $x\in X_2(n)$, $n\in\mathbb{N}$.

*Proof.* Let $W:\mathbb{N}\to\mathcal{B}(X)$,

$$\begin{aligned} W(n) = 2\sum_{k=n}^{\infty}&(\Phi(k,n)P_1(n))^*\Phi(k,n)P_1(n)\\ &- 2\sum_{k=0}^{n-1}(\Phi^{-1}(n,k)P_2(n))^*\Phi^{-1}(n,k)P_2(n)\end{aligned}$$

It follows easily that $W$ is bounded and $W(n)=W^*(n)$ for all $n\in\mathbb{N}$, Thus,

$$\begin{aligned}
&\Phi^*(n,n_0)W(n)\Phi(n,n_0)\\
&= 2\sum_{k=n}^{\infty}\Phi^*(n,n_0)(\Phi(k,n)P_1(n))^*\Phi(k,n)P_1(n)\Phi(n,n_0)\\
&\quad - 2\sum_{k=0}^{n-1}\Phi^*(n,n_0)(\Phi^{-1}(n,k)P_2(n))^*\Phi^{-1}(n,k)P_2(n)\Phi(n,n_0)\\
&= 2\sum_{k=n}^{\infty}(\Phi(k,n_0)P_1(n_0))^*\Phi(k,n_0)P_1(n_0)\\
&\quad - 2\sum_{k=0}^{n_0-1}(\Phi^{-1}(n,k)\Phi(n,n_0)P_2(n_0))^*\Phi^{-1}(n,k)\Phi(n,n_0)P_2(n_0)\\
&\quad - 2\sum_{k=n_0}^{n-1}(\Phi^{-1}(n,k)\Phi(n,n_0)P_2(n_0))^*\Phi^{-1}(n,k)\Phi(n,n_0)P_2(n_0)
\end{aligned}$$

$$= 2\sum_{k=n_0}^{\infty}(\Phi(k,n_0)P_1(n_0))^*\Phi(k,n_0)P_1(n_0) - 2\sum_{k=n_0}^{n-1}(\Phi(k,n_0)P_1(n_0))^*\Phi(k,n_0)P_1(n_0)$$

$$- 2\sum_{k=0}^{n_0-1}(\Phi^{-1}(n_0,k)P_2(n_0))^*\Phi^{-1}(n_0,k)P_2(n_0) - 2\sum_{k=n_0}^{n-1}(\Phi(k,n_0)P_2(n_0))^*\Phi(k,n_0)P_2(n_0)$$

$$= W(n_0) - 2\sum_{k=n_0}^{n-1}(\Phi(k,n_0)P_1(n_0))^*\Phi(k,n_0)P_1(n_0)$$

$$- 2\sum_{k=n_0}^{n-1}(\Phi(k,n_0)P_2(n_0))^*\Phi(k,n_0)P_2(n_0).$$

But

$$2\Big\langle \sum_{k=n_0}^{n-1}(\Phi(k,n_0)P_1(n_0))^*\Phi(k,n_0)P_1(n_0)x, x\Big\rangle$$

$$+ 2\Big\langle \sum_{k=n_0}^{n-1}(\Phi(k,n_0)P_2(n_0))^*\Phi(k,n_0)P_2(n_0)x, x\Big\rangle$$

$$= 2\sum_{k=n_0}^{n-1}\|\Phi(k,n_0)P_1(n_0)x\|^2 + 2\sum_{k=n_0}^{n-1}\|\Phi(k,n_0)P_2(n_0)x\|^2.$$

We obtain

$$\langle W(n_0)x, x\rangle = \langle W(n)\Phi(n,n_0)x, \Phi(n,n_0)x\rangle$$

$$+ 2\sum_{k=n_0}^{n-1}(\|\Phi(k,n_0)P_1(n_0)x\|^2 + \|\Phi(k,n_0)P_2(n_0)x\|^2)$$

$$\geq \langle W(n)\Phi(n,n_0)x, \Phi(n,n_0)x\rangle + \sum_{k=n_0}^{n-1}\|\Phi(k,n_0)x\|^2 \qquad \text{for all } x \in X.$$

It results that

$$\Phi^*(n,n_0)W(n)\Phi(n,n_0) + \sum_{k=n_0}^{n-1}\Phi^*(k,n_0)\Phi(k,n_0) \leq W(n_0) \quad \text{for all } n \geq n_0+1.$$

If $x \in X_1(n)$, then

$$\langle W(n)x, x\rangle = 2\sum_{k=n}^{\infty}\|\Phi(k,n)x\|^2 \geq 0$$

and for $x \in X_2(n)$, we have

$$\langle W(n)x, x\rangle = -2\sum_{k=0}^{n-1}\|\Phi^{-1}(n,k)x\|^2 \leq 0.$$

By Definition 2.4, we obtain

$$\|\Phi(n_0+1,n_0)x\| \geq N_2\,\mathrm{e}^{\nu}\,\|x\| \qquad \text{for all } n_0\in\mathbb{N},\ x\in X_2(n_0).$$

□

**Theorem 4.3.** *If there exist $m>0$ and $W\colon\mathbb{N}\to\mathcal{B}(X)$ bounded, $W(n)=W^*(n)$ for all $n\in\mathbb{N}$ such that:*

(i) $\Phi^*(n,n_0)W(n)\Phi(n,n_0)+\sum\limits_{k=n_0}^{n-1}\Phi^*(k,n_0)\Phi(k,n_0)\leq W(n_0)$ *for all* $n\geq n_0+1$*;*

(ii) $\langle W(n)x,x\rangle\geq 0$ *for all* $x\in X_1(n),\ n\in\mathbb{N}$*;*

(iii) $\langle W(n)x,x\rangle\leq -m\|x\|^2$ *for all* $x\in X_2(n),\ n\in\mathbb{N}$*.*

*Then the discrete evolution family $\{\Phi(n,n_0)\}_{n\geq n_0\geq 0}$ is uniformly exponentially dichotomic.*

*Proof.* Denoting $L=\sup\limits_{n\in\mathbb{N}}\|W(n)\|$, we obtain in a similar way as in Theorem 4.1 that

$$\Big(\sum_{k=n_0}^{\infty}\|\Phi(k,n_0)x\|^2\Big)^{\frac{1}{2}}\leq\sqrt{L}\|x\| \qquad \text{for all } n_0\in\mathbb{N},\ x\in X_1(n_0) \tag{10}$$

and

$$\Big(\sum_{k=n_0}^{n-1}\|\Phi(k,n_0)x\|^2\Big)^{\frac{1}{2}}\leq\sqrt{L}\|\Phi(n,n_0)x\| \quad \text{for all } n\geq n_0+1,\ x\in X_2(n_0). \tag{11}$$

Now we take $x\in X_2(n_0),\ n_0\in\mathbb{N}$. By $(i)$ and (iii), we obtain

$$\langle W(n_0+1,n_0)\Phi(n_0+1,n_0)x,\Phi(n_0+1,n_0)x\rangle\leq\langle W(n_0)x,x\rangle\leq -m\|x\|^2$$

and then

$$m\|x\|^2\leq-\langle W(n_0+1,n_0)\Phi(n_0+1,n_0)x,\Phi(n_0+1,n_0)x\rangle\leq L\|\Phi(n_0+1,n_0)x\|^2.$$

It follows that

$$\|\Phi(n_0+1,n_0)x\|\geq\sqrt{\frac{m}{L}}\|x\| \qquad \text{for all } x\in X_2(n_0),\ n_0\in\mathbb{N}. \tag{12}$$

From the relations (10), (11), (12) and Proposition 4.1, we deduce that the discrete evolution family $\{\Phi(n,n_0)\}_{n\geq n_0\geq 0}$ is uniformly exponentially dichotomic. □

## 5. The discrete Lyapunov method for the dichotomy of differential systems

In the following we consider $A\colon\mathbb{R}_+\to L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathcal{B}(X))$, where $L^1_{\mathrm{loc}}(\mathbb{R}_+,\mathcal{B}(X))$ is the space of strongly measurable and locally Bochner integrable functions with $\sup_{t\geq 0}\int_t^{t+1}\|A(\tau)\|\mathrm{d}\tau<\infty$. It is known that the differential equation

$$(A)\dot{x}(t)=A(t)x(t),$$

has the general solution given by

$$x(t)=\Phi(t,t_0)x_0,$$

where $\Phi(t,t_0) = U(t)U^{-1}(t_0)$ and $U$ is the solution of the operatorial Cauchy problem

$$\begin{cases} \dot{U}(t) = A(t)U(t) \\ U(0) = I, \end{cases}$$

$x_0 \in X$ (see [**5**], [**11**] for details). We obtain a similar result to that one in Theorem 2.2 for the discrete case, by replacing the Lebesgue measure with the counting measure.

**Proposition 5.1.** *If $A \in M_1(\mathcal{B}(X))$, then $(A)$ is uniformly exponentially dichotomic if and only if there exist $L, p > 0$ such that:*

(i) $\|U(n)P_1U^{-1}(n)\| \leq L$ *for all* $n \in \mathbb{N}$;

(ii) $\Big(\sum\limits_{k=n}^{\infty} \|U(k)P_1U^{-1}(n)x\|^p\Big)^{1/p} + \Big(\sum\limits_{k=0}^{n} \|U(k)P_2U^{-1}(n)x\|^p\Big)^{1/p} \leq L\|x\|$ *for all* $n \in \mathbb{N}$, $x \in X$.

*Proof.* The necessity is a simple verification of the inequalities $(i)$ and $(ii)$, using Definition 2.6.
For the sufficiency, we take $t \geq 0$, $n = [t]$ . Then

$$\|U(t)P_1U^{-1}(t)\| = \|U(t)U^{-1}(n)U(n)P_1U^{-1}(n)U(n)U^{-1}(t)\| \leq L(M\,\mathrm{e}^{\omega})^2.$$

Putting now $L_1 = L(M\,\mathrm{e}^{\omega})^2$, we obtain

$$\|U(t)P_1U^{-1}(t)\| \leq L_1 \qquad \text{for all } t \geq 0. \tag{13}$$

Let $t \geq 0$, $\tau \geq t+1$, $n = [t]$, $k = [\tau]$ and $x \in X$. We get $k \geq n+1$ and

$$\begin{aligned} \|U(\tau)P_1U^{-1}(t)x\| &= \|U(\tau)U^{-1}(k)U(k)P_1U^{-1}(n+1)U(n+1)U^{-1}(t)x\| \\ &\leq (M\,\mathrm{e}^{\omega})^2\|U(k)P_1U^{-1}(n+1)\| \cdot \|x\|. \end{aligned}$$

Therefore

$$\int_k^{k+1} \|U(\tau)P_1U^{-1}(t)x\|^p \mathrm{d}\tau \leq (M\,\mathrm{e}^{\omega})^{2p}\|U(k)P_1U^{-1}(n+1)\|^p \cdot \|x\|^p$$

and

$$\begin{aligned} \sum_{k=n+1}^{\infty} \int_k^{k+1} \|U(\tau)P_1U^{-1}(t)x\|^p \mathrm{d}\tau &\leq (Me^{\omega})^{2p} \sum_{k=n+1}^{\infty} \|U(k)P_1U^{-1}(n+1)\|^p\|x\|^p \\ &\leq L_1^p\|x\|^p. \end{aligned}$$

We obtain

$$\Big(\int_t^{\infty} \|U(\tau)P_1U^{-1}(t)x\|^p \mathrm{d}\tau\Big)^{1/p} \leq L_1\|x\| \qquad \text{for all } t \geq 0,\ x \in X. \tag{14}$$

Taking now $\tau \geq 0$, $t \geq \tau+1$, $n = [t]$, $k = [\tau]$ and $x \in X$, we have $n \geq k+1$ and

$$\begin{aligned} \|U(\tau)P_2U^{-1}(t)x\| &= \|U(\tau)U^{-1}(k+1)U(k+1)P_2U^{-1}(n)U(n)U^{-1}(t)x\| \\ &\leq (M\,\mathrm{e}^{\omega})^2\|U(k+1)P_2U^{-1}(n)\| \cdot \|x\|. \end{aligned}$$

Proceeding similarly as above, we obtain

$$\sum_{k=0}^{n-1}\int_{k}^{k+1}\|U(\tau)P_2U^{-1}(t)x\|^p\mathrm{d}\tau \le (M\,\mathrm{e}^{\omega})^{2p}\sum_{k=0}^{n-1}\|U(k+1)P_2U^{-1}(n)\|^p\cdot\|x\|^p$$
$$\le (M\,\mathrm{e}^{\omega})^{2p}\sum_{i=1}^{n}\|U(i)P_2U^{-1}(n)\|^p\|x\|^p \le L_1^p\|x\|^p,$$

that is equivalent to

$$\int_0^n \|U(\tau)P_2U^{-1}(t)x\|^p\mathrm{d}\tau \le L_1^p\|x\|^p.$$

But

$$\int_0^t \|U(\tau)P_2U^{-1}(t)x\|^p\mathrm{d}\tau = \int_0^n \|U(\tau)P_2U^{-1}(t)x\|^p\mathrm{d}\tau + \int_n^t \|U(\tau)P_2U^{-1}(t)x\|^p\mathrm{d}\tau$$
$$\le L_1^p\|x\|^p + \int_n^t \|U(\tau)U^{-1}(t)U(t)P_2U^{-1}(t)x\|^p\mathrm{d}\tau$$
$$\le (L_1^p + (M\,\mathrm{e}^{\omega})^p(L+1)^p)\|x\|^p, \qquad \text{for all } t\ge 1,\ x\in X.$$

If $t\in[0,1)$, then

$$\int_0^t \|U(\tau)P_2U^{-1}(t)x\|^p\mathrm{d}\tau \le (M\,\mathrm{e}^{\omega})^p(L+1)^p\|x\|^p \ \le L_2\|x\|^p,$$

where $L_2 = L_1^p + (Me^{\omega})^p(L+1)^p$. Finally, we have

$$\Big(\int_0^t \|U(\tau)P_2U^{-1}(t)x\|^p\mathrm{d}\tau\Big)^{1/p} \le L_2^{1/p}\|x\| \qquad \text{for all } t\ge 0, x\in X. \tag{15}$$

By (13), (14), (15) and Theorem 2.2, it follows that $(A)$ is uniformly exponentially dichotomic which completes the proof. $\square$

**Theorem 5.1.** *Let $A\in M_1(\mathcal{B}(X))$. The differential system $(A)$ is exponentially dichotomic if and only if there exist $L>0$ and $W\colon \mathbb{N}\to\mathcal{B}(X)$ bounded, $W(n) = W^*(n)$ for all $n\in\mathbb{N}$ with the following properties:*

(i) $\|U(n)P_1U^{-1}(n)\|\le L$ *for all* $n\in\mathbb{N}$*;*

(ii) $\Phi^*(n,n_0)W(n)\Phi(n,n_0)x + \sum_{k=n_0}^{n-1}\Phi^*(k,n_0)\Phi(k,n_0)x = W(n_0)x$ *for all* $n,n_0\in\mathbb{N}$, $n\ge n_0+1$, $x\in X$*;*

(iii) $\langle W(n)x,x\rangle\ge 0$ *for all* $x\in X_1(n)=P_1(n)X$ *and* $n\in\mathbb{N}$, *where* $P_1(n)=U(n)P_1U^{-1}(n)$*;*

(iv) $\langle W(n)x,x\rangle\le 0$ *for all* $x\in X_2(n)=P_2(n)X$*, and* $n\in\mathbb{N}$, *where* $P_2(n)=U(n)P_2U^{-1}(n)$*.*

*Proof. Necessity.* By Definition 2.6, we have that there exist $N, \nu > 0$ such that

$$\|U(n)P_1U^{-1}(m)\| \leq N\,\mathrm{e}^{-\nu(n-m)} \qquad \text{for all } n, m \in \mathbb{N},\ n \geq m.$$

Putting $m := n$, we obtain the first assertion.
We consider $W\colon \mathbb{N} \to \mathcal{B}(X)$, $n \geq n_0 + 1$,

$$W(n)x = \sum_{k=n}^{\infty} \Phi_1^*(k,n)\Phi_1(k,n)x - \sum_{k=0}^{n-1} \Phi_2^*(k,n)\Phi_2(k,n)x,$$

where $\Phi_i(n, n_0) = U(n)P_iU^{-1}(n_0)$, $i = \overline{1,2}$.

Using the Definition 2.6, we have

$$\|W(n)x\| \leq 2N^2\|x\| \sum_{i=0}^{\infty} \mathrm{e}^{-2\nu i} = \frac{2N^2}{1-\mathrm{e}^{-2\nu}}\|x\|,$$

and from the properties of the self-adjoint, we get $W(n) = W^*(n)$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned}
&\Phi^*(n,n_0)W(n)\Phi(n,n_0)x + \sum_{k=n_0}^{n-1} \Phi^*(k,n_0)\Phi(k,n_0)x \\
&= \sum_{k=n}^{\infty} \Phi^*(n,n_0)\Phi_1^*(k,n)\Phi_1(k,n)\Phi(n,n_0)x \\
&\quad - \sum_{k=0}^{n-1} \Phi^*(n,n_0)\Phi_2^*(k,n)\Phi_2(k,n)\Phi(n,n_0)x \\
&\quad + \sum_{k=n_0}^{n-1} \Phi^*(k,n_0)\Phi(k,n_0)x \\
&= \sum_{k=n}^{\infty}(U^{-1}(n_0))^*P_1U^*(k)U(k)P_1U^{-1}(n_0)x \\
&\quad - \sum_{k=0}^{n-1}(U^{-1}(n_0))^*P_2U^*(k)U(k)P_2U^{-1}(n_0)x \\
&\quad + \sum_{k=n_0}^{n-1} (U^{-1}(n_0))^*P_1U^*(k)U(k)P_1U^{-1}(n_0)x \\
&\quad + \sum_{k=n_0}^{n-1} (U^{-1}(n_0))^*P_2U^*(k)U(k)P_2U^{-1}(n_0)x \\
&= \sum_{k=n_0}^{\infty} (U(k)P_1U^{-1}(n_0))^*U(k)P_1U^{-1}(n_0)x \\
&\quad - \sum_{k=0}^{n_0-1} (U(k)P_2U^{-1}(n_0))^*U(k)P_2U^{-1}(n_0)x = W(n_0)x
\end{aligned}$$

for all $n, n_0 \in \mathbb{N}$, $n \geq n_0 + 1$, $x \in X$.

It is easily seen that the conditions (iii) and (iv) are satisfied.

*Sufficiency.* Let $x \in X_1(n_0), n \geq n_0 + 1$. From (ii) we have

$$\sum_{k=n_0}^{n-1} \|\Phi(k, n_0)x\|^2 = \langle W(n_0)x, x\rangle - \langle W(n)\Phi(n, n_0)x, \Phi(n, n_0)x\rangle$$
$$\leq |\langle W(n_0)x, x\rangle| \leq \|W(n_0)\| \cdot \|x\|^2 \leq \sup_{n\in\mathbb{N}} \|W(n)\| \cdot \|x\|^2.$$

But $x \in X_1(n_0)$ implies that $x = U(n_0)P_1U^{-1}(n_0)y$, $y \in X$ and $\Phi(k, n_0)x = U(k)P_1U^{-1}(n_0)y$, $y \in X$. We obtain

$$\sum_{k=n_0}^{n-1} \|U(k)P_1U^{-1}(n_0)y\|^2 \leq L_1 \cdot L^2\|y\|^2,$$

where $L_1 = \sup_{n\in\mathbb{N}} \|W(n)\|$ and

$$\text{(16)} \quad \Big(\sum_{k=n_0+1}^{\infty} \|U(k)P_1U^{-1}(n_0 + 1)y\|^2\Big)^{\frac{1}{2}} \leq \sqrt{L_1} \cdot L\|y\| \quad \text{for all } n_0 \in \mathbb{N},\ y \in X.$$

Let $x \in X_2(n_0)$, $n \geq n_0 + 1$. From the hypothesis we have

$$\sum_{k=n_0}^{n-1} \|\Phi(k, n_0)x\|^2 = \langle W(n_0)x, x\rangle - \langle W(n)\Phi(n, n_0)x, \Phi(n, n_0)x\rangle$$
$$\leq |\langle W(n)\Phi(n, n_0)x, \Phi(n, n_0)x\rangle| \leq L_1 \cdot \|\Phi(n, n_0)x\|^2.$$

Putting $x = P_2(n_0)U(n_0)U^{-1}(n)y$, $y \in X$, we get $x \in X_2(n_0)$ and

$$\Phi(k, n_0)x = U(k)P_2U^{-1}(n)y.$$

Therefore,

$$\sum_{k=n_0}^{n-1} \|U(k)P_2U^{-1}(n)y\|^2 \leq L_1 \cdot \|P_2(n)\|^2 \cdot \|y\|^2 \leq L_1 \cdot (L + 1)^2 \cdot \|y\|^2$$

and

$$\sum_{k=n_0}^{n} \|U(k)P_2U^{-1}(n)y\|^2 \leq L_2 \cdot \|y\|^2 \qquad \text{for all } n \geq n_0 + 1,\ y \in X,$$

where $L_2 = L_1(L + 1)^3$.
Putting now $n := n_0 + 1$ and $n_0 := 0$ in the last inequality, we have

$$\text{(17)} \quad \Big(\sum_{k=0}^{n_0+1} \|U(k)P_2U^{-1}(n_0 + 1)y\|^2\Big)^{\frac{1}{2}} \leq \sqrt{L_2} \cdot \|y\| \quad \text{for all } n_0 \in \mathbb{N},\ y \in X.$$

By (16), (17) and Proposition 5.1, we have $(A)$ is uniformly exponentially dichotomic. $\square$

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