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A NOTE ON k-PELL, k-PELL-LUCAS AND MODIFIED k-PELL NUMBERS WITH ARITHMETIC INDEXES

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Abstract. In this paper, we define the k-Pell, k-Pell-Lucas and Modified k-Pell numbers with arithmetic indexes, study some properties of these sequences of numbers and give their generating functions.

1. Introduction

In this paper, we consider new sequences of numbers with arithmetic indexes, and we give some properties and the generating functions for these types of sequences. One of the sequences of positive integers (also defined recursively) that has been studied over several years is the well-known Fibonacci (and Lucas) sequence. Many works of researchers are dedicated to the Fibonacci sequence, such as the works of Hoggatt [**15**], Vorobiov [**25**], and Koshy [**21**], among others. The studies of several authors are devoted to other sequences satisfying a second-order recurrence relation such as the sequences $\{P_n\}_n (n \geq 0)$, $\{Q_n\}_n$ $(n \geq 0)$ and $\{q_n\}_n$ $(n \geq 0)$ of Pell, Pell-Lucas, and Modified Pell numbers, respectively. Recall that the second-order recurrence relation and the initial conditions for the Pell numbers P_n, Pell-Lucas numbers Q_n, and the Modified Pell numbers q_n, are given by

$$P_{n+1} = 2P_n + P_{n-1}, \qquad P_0 = 0,\ P_1 = 1, \qquad (n \geq 1), \tag{1}$$

$$Q_{n+1} = 2Q_n + Q_{n-1}, \qquad Q_0 = Q_1 = 2, \qquad (n \geq 1), \tag{2}$$

and

$$q_{n+1} = 2q_n + q_{n-1}, \qquad q_0 = q_1 = 1, \qquad (n \geq 1), \tag{3}$$

respectively. More recently, generalizations for any positive real number k of certain sequences of positive integers have been studied by many researchers. For example, the study of the k-Pell sequence, the k-Pell-Lucas sequence, and the Modified k-Pell sequence has appeared in [**4**], [**5**], [**7**], [**8**], [**9**] and [**10**], among

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others. Studies of other generalizations can be found in [**1**], [**2**], [**3**], [**6**], [**11**], [**12**], [**17**], [**18**], [**19**], and [**20**].

In this paper, we work with sequences related with the k-Pell, the k-Pell-Lucas and the Modified k-Pell sequence, and such generalizations are defined by a recurrence relation of second order given for any positive real number k, by,

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \qquad n \geq 2, \tag{4}$$

with initial terms $P_{k,0} = 0$ and $P_{k,1} = 1$, for the k-Pell sequence,

$$Q_{k,n} = 2Q_{k,n-1} + kQ_{k,n-2}, \qquad n \geq 2, \tag{5}$$

with initial terms $Q_{k,0} = Q_{k,1} = 2$, for the k-Pell-Lucas sequence, and

$$q_{k,n} = 2q_{k,n-1} + kq_{k,n-2}, \qquad n \geq 2, \tag{6}$$

with initial terms $q_{k,0} = q_{k,1} = 1$ for the Modified k-Pell sequence. Note that in the particular case where $k = 1$, (4)-(5)-(6) reduces to (1)-(2)-(3), respectively.

The Binet style formula for these sequences are given by

$$P_{k,n} = \frac{(s_1)^n - (s_2)^n}{s_1 - s_2}, \qquad Q_{k,n} = (s_1)^n + (s_2)^n, \qquad q_{k,n} = \frac{(s_1)^n + (s_2)^n}{2}, \tag{7}$$

respectively, where $s_1 = 1 + \sqrt{1+k}$ and $s_2 = 1 - \sqrt{1+k}$ are the roots of the characteristic equation $s^2 - 2s - k = 0$ associated with the above recurrence relations (4), (5) and (6). Note that $s_1 + s_2 = 2$, $s_1 s_2 = -k$ and $s_1 - s_2 = 2\sqrt{1+k}$.

The works of Falcon [**13**], and Falcon and Plaza [**14**] are dedicated to the study of the k-Lucas and the k-Fibonacci numbers of arithmetic indexes, respectively. Similar studies with other sequences can be found in the work of Jhala, Rathore and Sisodiya [**16**], where the authors stated some properties of the k-Jacobsthal numbers with arithmetic indexes. Also, Uygun [**23**] studied the k-Jacobsthal Lucas numbers of arithmetic indexes and Singh, Bhadouria and Sikhwal [**22**] determined sum properties for the k-Lucas numbers with arithmetic indexes.

Motivated essentially by these works, we aim at introducing and analogously investigating versions of *arithmetic indexes* of each one of these three classes of numbers: k-Pell, k-Pell-Lucas, and Modified k-Pell numbers. We begin recalling some properties involving these sequences of numbers and then, for each type, we introduce and present some properties and the generating function.

2. On the k-Pell numbers of indexes $an + r$

We focus here on the subsequences of k-Pell numbers with indexes in an arithmetic sequence, say $an + r$ for fixed integers a, r with $0 \leq r \leq a - 1$. In this section, we derive some formulae for the sums of these indexes of numbers. First, we state some results involving some properties of this sequence.

Lemma 2.1. *For any positive real number k and any natural number n, we have*

$$(s_1)^n + (s_2)^n = P_{k,n+1} + kP_{k,n-1}.$$

Proof. Applying the respective Binet formula and taking into account that $s_1 s_2 = -k$,

$$\begin{aligned}
P_{k,n+1} + kP_{k,n-1} &= \frac{1}{s_1 - s_2}\left((s_1)^{n+1} - (s_2)^{n+1} + k(s_1)^{n-1} - k(s_2)^{n-1}\right)\\
&= \frac{1}{s_1 - s_2}\left((s_1)^{n+1} - (s_2)^{n+1} - (s_1 s_2)(s_1)^{n-1} + (s_1 s_2)(s_2)^{n-1}\right)\\
&= \frac{1}{s_1 - s_2}\left((s_1)^n(s_1 - s_2) + (s_2)^n(s_1 - s_2)\right)\\
&= (s_1)^n + (s_2)^n,
\end{aligned}$$

as required. □

Proposition 2.2. *For any positive real number k, any natural number n, and for fixed integers a, r with $0 \le r \le a-1$, we obtain*

$$P_{k,a(n+2)+r} = \left(P_{k,a+1} + kP_{k,a-1}\right) P_{k,a(n+1)+r} - (-k)^a P_{k,an+r}.$$

Proof. Taking into account Lemma 2.1, the respective Binet formula and the fact that $s_1 s_2 = -k$, we obtain

$$\begin{aligned}
&\left(P_{k,a+1} + kP_{k,a-1}\right) P_{k,a(n+1)+r}\\
&= \left((s_1)^a + (s_2)^a\right)\left(\frac{(s_1)^{a(n+1)+r} - (s_2)^{a(n+1)+r}}{s_1 - s_2}\right)\\
&= \frac{1}{s_1 - s_2}\left((s_1)^{a(n+2)+r} - (s_2)^{a(n+2)+r} + (-k)^a\left((s_1)^{an+r} - (s_2)^{an+r}\right)\right)\\
&= P_{k,a(n+2)+r} + (-k)^a P_{k,an+r},
\end{aligned}$$

as required. □

Now, we discuss the sum formula for the k-Pell numbers with arithmetic indexes $an + r$, where a and r are fixed integers such that $0 \le r \le a-1$.

Theorem 2.3. *For any positive real number k, any natural number n, and fixed integers a, r with $0 \le r \le a-1$, the sum of k-Pell numbers of kind $an+r$ is*

$$\sum_{i=0}^{n} P_{k,ai+r} = \frac{(-k)^a P_{k,an+r} - P_{k,a(n+1)+r} + P_{k,r} + (-k)^r P_{k,a-r}}{(-k)^a - \left(P_{k,a+1} + kP_{k,a-1}\right) + 1}.$$

Proof. Applying Binet's formula for the k-Pell numbers, the fact that $s_1 s_2 = -k$, and Lemma 2.1, we have

$$\begin{aligned}
\sum_{i=0}^{n} P_{k,ai+r} &= \sum_{i=0}^{n} \left(\frac{(s_1)^{ai+r} - (s_2)^{ai+r}}{s_1 - s_2} \right) \\
&= \frac{1}{s_1 - s_2} \left(\sum_{i=0}^{n} (s_1)^{ai+r} - \sum_{i=0}^{n} (s_2)^{ai+r} \right) \\
&= \frac{1}{s_1 - s_2} \left(\frac{(s_1)^{an+r+a} - (s_1)^r}{(s_1)^a - 1} - \frac{(s_2)^{an+r+a} - (s_2)^r}{(s_2)^a - 1} \right) \\
&= \frac{(-k)^a P_{k,an+r} - P_{k,a(n+1)+r} + P_{k,r} + (-k)^r P_{k,a-r}}{(-k)^a - (P_{k,a+1} + kP_{k,a-1}) + 1}. \qquad \square
\end{aligned}$$

The next two results give us the sum of certain k-Pell numbers with arithmetic indexes.

Corollary 2.4. *If $a = 2p+1$, then* Theorem 2.3 *gives*

$$\begin{aligned}
&\sum_{i=0}^{n} P_{k,(2p+1)i+r} \\
&= \frac{(-k)^{2p+1} P_{k,(2p+1)n+r} - P_{k,(2p+1)(n+1)+r} + P_{k,r} + (-k)^r P_{k,(2p+1)-r}}{(-k)^{2p+1} - (P_{k,2p+2} + kP_{k,2p}) + 1}.
\end{aligned}$$

In particular, if $p = 0$, then $a = 1$ and $r = 0$, and we obtain

$$\begin{aligned}
\sum_{i=0}^{n} P_{k,i} &= \frac{(-k)P_{k,n} - P_{k,n+1} + P_{k,0} + (-k)^0 P_{k,1}}{(-k) - (P_{k,2} + kP_{k,0}) + 1} \\
&= \frac{(-k)P_{k,n} - P_{k,n+1} + 1}{-k - 2 + 1} = \frac{kP_{k,n} + P_{k,n+1} - 1}{k+1},
\end{aligned}$$

the formula stated in [**10**, Proposition 1].

If $p = 1$, then $a = 3$, and if $r = 0$ or $r = 1$ or $r = 2$, we obtain

$$\begin{aligned}
\sum_{i=0}^{n} P_{k,3i} &= \frac{-k^3 P_{k,3n} - P_{k,3n+3} + 4 + k}{-k^3 - 6k - 7}, \\
\sum_{i=0}^{n} P_{k,3i+1} &= \frac{-k^3 P_{k,3n+1} - P_{k,3n+4} + 1 - 2k}{-k^3 - 6k - 7}, \\
\sum_{i=0}^{n} P_{k,3i+2} &= \frac{-k^3 P_{k,3n+2} - P_{k,3n+5} + 2 + k^2}{-k^3 - 6k - 7},
\end{aligned}$$

respectively.

Corollary 2.5. *If* $a = 2p$, *then* Theorem 2.3 *gives*

$$\begin{aligned}&\sum_{i=0}^{n} P_{k,2pi+r}\\ &= \frac{(-k)^{2p} P_{k,2pn+r} - P_{k,2p(n+1)+r} + P_{k,r} + (-k)^r P_{k,2p-r}}{(-k)^{2p} - (P_{k,2p+1} + kP_{k,2p-1}) + 1}.\end{aligned}$$

In particular, if $p = 1$, then $a = 2$, and we obtain

$$\begin{aligned}\sum_{i=0}^{n} P_{k,2i+r} &= \frac{k^2 P_{k,2n+r} - P_{k,2(n+1)+r} + P_{k,r} + (-k)^r P_{k,2-r}}{k^2 - (P_{k,3} + kP_{k,1}) + 1}\\ &= \frac{k^2 P_{k,2n+r} - P_{k,2(n+1)+r} + P_{k,r} + (-k)^r P_{k,2-r}}{k^2 - (k + 4 + k) + 1}\\ &= \frac{k^2 P_{k,2n+r} - P_{k,2(n+1)+r} + P_{k,r} + (-k)^r P_{k,2-r}}{k^2 - 2k - 3}.\end{aligned}$$

If $r = 0$ or $r = 1$, then the previous equation gives

$$\sum_{i=0}^{n} P_{k,2i} = \frac{k^2 P_{k,2n} - P_{k,2n+2} + 2}{k^2 - 2k - 3}$$

and

$$\sum_{i=0}^{n} P_{k,2i+1} = \frac{k^2 P_{k,2n+1} - P_{k,2n+3} + 1 - k}{k^2 - 2k - 3},$$

respectively, formula equivalent to the formula stated in the [**24**, Proposition 3], but obtained in a different way.

Now, we consider the alternating sequence $\{(-1)^n P_{k,an+r}\}$. It is easy to find the sum formula for this sequence.

Theorem 2.6. *For any positive real number* k*, any natural number* n*, and fixed integers* a*,* r *with* $0 \le r \le a-1$*, the sum of the first* $n+1$ *terms of the alternating sequence of the* k*-Pell numbers with arithmetic indexes is given by*

$$\sum_{i=0}^{n} (-1)^i P_{k,ai+r} = \frac{P_{k,r} + (-1)^n P_{k,a(n+1)+r} - (-k)^r P_{k,a-r} + (-k)^a (-1)^n P_{k,an+r}}{1 + P_{k,a+1} + kP_{k,a-1} + (-k)^a}.$$

Proof. Applying the Binet formula for the k-Pell numbers once more, the fact that $s_1 s_2 = -k$ and $(s_1)^{-1} = -s_2 k^{-1}$, $(s_2)^{-1} = -s_1 k^{-1}$, and Lemma 2.1, we have

$$\begin{aligned}\sum_{i=0}^{n} (-1)^i P_{k,ai+r} &= \frac{1}{s_1 - s_2} \sum_{i=0}^{n} (-1)^i \left(s_1^{ai+r} - s_2^{ai+r}\right)\\ &= \frac{1}{s_1 - s_2}\left(\frac{(s_1)^r\left(1 + (-1)^n s_1^{a(n+1)}\right)}{1 + (s_1)^a} - \frac{(s_2)^r\left(1 + (-1)^n (s_2)^{a(n+1)}\right)}{1 + (s_2)^a}\right)\\ &= \frac{P_{k,r} + (-1)^n P_{k,a(n+1)+r} - (-k)^r P_{k,a-r} + (-k)^a (-1)^n P_{k,an+r}}{1 + P_{k,a+1} + kP_{k,a-1} + (-k)^a}.\end{aligned}$$ □

The next two results give the sum of certain terms of the alternating sequence.

Corollary 2.7. *If $a = 2p + 1$, then* Theorem 2.6 *gives*

$$\sum_{i=0}^{n}(-1)^i P_{k,(2p+1)i+r}$$
$$= \frac{P_{k,r}+(-1)^n P_{k,(2p+1)(n+1)+r}-(-k)^r P_{k,2p+1-r}+(-k)^{2p+1}(-1)^n P_{k,(2p+1)n+r}}{1+P_{k,2p+2}+kP_{k,2p}+(-k)^{2p+1}}.$$

In particular, if $p = 0$, then $a = 1$ and $r = 0$, and we obtain

$$\sum_{i=0}^{n}(-1)^i P_{k,i} = \frac{(-1)^n\left(P_{k,n+1}-kP_{k,n}\right)-1}{3-k}.$$

If $p = 1$, then $a = 3$ and if $r = 0$ or $r = 1$ or $r = 2$, we obtain

$$\sum_{i=0}^{n}(-1)^i P_{k,3i} = \frac{(-1)^n\left(P_{k,3(n+1)}-k^3P_{k,3n}\right)-k-4}{-k^3+6k+9},$$

$$\sum_{i=0}^{n}(-1)^i P_{k,3i+1} = \frac{(-1)^n\left(P_{k,3n+4}-k^3P_{k,3n+1}\right)+2k+1}{-k^3+6k+9},$$

and

$$\sum_{i=0}^{n}(-1)iP_{k,3i+2} = \frac{(-1)^n\left(P_{k,3n+5}-k^3P_{k,3n+2}\right)-k^2+2}{-k^3+6k+9},$$

respectively.

Corollary 2.8. *If $a = 2p$, then* Theorem 2.6 *gives*

$$\sum_{i=0}^{n}(-1)^i P_{k,2pi+r}$$
$$= \frac{P_{k,r}+(-1)^n P_{k,2p(n+1)+r}-(-k)^r P_{k,2p-r}+(-k)^{2p}(-1)^n P_{k,2pn+r}}{1+P_{k,2p+1}+kP_{k,2p-1}+(-k)^{2p}}.$$

In particular, if $p = 1$, then $a = 2$, and we obtain

$$\sum_{i=0}^{n}(-1)^i P_{k,2i+r}$$
$$= \frac{P_{k,r}+(-1)^n P_{k,2(n+1)+r}-(-k)^r P_{k,2-r}+k^2(-1)^n P_{k,2n+r}}{k^2+2k+5}.$$

If $r = 0$ or $r = 1$, then the previous equation gives

$$\sum_{i=0}^{n}(-1)^i P_{k,2i} = \frac{(-1)^n\left(P_{k,2(n+1)}+k^2P_{k,2n}\right)-2}{k^2+2k+5},$$

and

$$\sum_{i=0}^{n}(-1)^i P_{k,2i+1} = \frac{(-1)^n\left(P_{k,2n+3}+k^2P_{k,2n+1}\right)+k+1}{k^2+2k+5}$$

respectively.

For any positive real number k and fixed integers a, r with $0 \leq r \leq a-1$, let denote the generating function of the sequence $\{P_{k,an+r}\}$ by $f_{a,r}(k,x)$. Then we get

$$\begin{aligned}
f_{a,r}(k,x) &= \sum_{n=0}^{\infty} P_{k,an+r} x^n \\
&= P_{k,r} + P_{k,a+r} x + \sum_{n=2}^{\infty} P_{k,an+r} x^n \\
&= P_{k,r} + P_{k,a+r} x + \sum_{n=2}^{\infty} \left(2P_{k,a(n-1)+r} + kP_{k,a(n-2)+r}\right) x^n \\
&= P_{k,r} + P_{k,a+r} x + 2x \sum_{n=2}^{\infty} P_{k,a(n-1)+r} x^{n-1} + kx^2 \sum_{n=2}^{\infty} P_{k,a(n-2)+r} x^{n-2}.
\end{aligned}$$

Consider $j = n-2$ and $l = n-1$, then

$$\begin{aligned}
f_{a,r}(k,x) &= P_{k,r} + P_{k,a+r} x + 2x \Big(\sum_{l=0}^{\infty} P_{k,al+r} x^l - P_{k,r} \Big) + kx^2 \sum_{j=0}^{\infty} P_{k,aj+r} x^j \\
&= P_{k,r} + P_{k,a+r} x - 2x P_{k,r} + 2x \sum_{n=0}^{\infty} P_{k,an+r} x^n + kx^2 \sum_{n=0}^{\infty} P_{k,an+r} x^n,
\end{aligned}$$

which is equivalent to

$$\sum_{n=0}^{\infty} P_{k,an+r} x^n \left(1 - 2x - kx^2\right) = P_{k,r} - x\left(2P_{k,r} - P_{k,a+r}\right)$$

and then the following result is proved.

Proposition 2.9. *For any positive real number k, any natural number n, and fixed integers a, r with $0 \leq r \leq a-1$, the generating function of the sequence $\{P_{k,an+r}\}$ is*

$$f_{a,r}(k,x) = \frac{P_{k,r} - x\left(2P_{k,r} - P_{k,a+r}\right)}{1 - 2x - kx^2}.$$

In particular, if $a = 1$, we have $r = 0$, then we get the generating function of k-Pell numbers

$$f_{1,0}(k,x) = \frac{x}{1 - 2x - kx^2}.$$

If $a = 2$ and $r = 0$ or $r = 1$, we obtain

$$f_{2,0}(k,x) = \frac{2x}{1 - 2x - kx^2} \qquad \text{and} \qquad f_{2,1}(k,x) = \frac{1 + 2x + kx}{1 - 2x - kx^2},$$

respectively.

3. On the k-Pell-Lucas and Modified k-Pell numbers of indexes $an+r$

In this section, we introduce the k-Pell-Lucas and Modified k-Pell numbers of indexes $an+r$ for fixed integers a, r with $0 \leq r \leq a-1$, and present some properties involving these sequences, including the sum formulae and the generating functions. Since all the results, which include these sequences, can be proved by a similar way used in the previous section, we decide to state these results without any proof.

Note that it is easy to see that

$$Q_{k,n} = 2q_{k,n}. \tag{8}$$

This fact is used in the next results for the case of the Modified k-Pell numbers of indexes $an+r$. The correspondent result of Proposition 2.2 for the k-Pell-Lucas and Modified k-Pell numbers of indexes $an+r$ is the following

Proposition 3.1. *For any positive real number k, any natural number n, and for fixed integers a, r with $0 \leq r \leq a-1$, we obtain*

1. $Q_{k,a(n+2)+r} = Q_{k,a}Q_{k,a(n+1)+r} - (-k)^a Q_{k,an+r}$,
2. $q_{k,a(n+2)+r} = 2q_{k,a}q_{k,a(n+1)+r} - (-k)^a q_{k,an+r}$.

Now, we present the sum formulae for the k-Pell-Lucas, and Modified k-Pell numbers with arithmetic indexes $an+r$, where a and r are fixed integers such that $0 \leq r \leq a-1$.

Theorem 3.2. *For any positive real number k, any natural number n and for fixed integers a, r with $0 \leq r \leq a-1$, the sum of k-Pell-Lucas and Modified k-Pell numbers of indexes $an+r$ is*

1. $\sum_{i=0}^{n} Q_{k,ai+r} = \dfrac{(-k)^a Q_{k,an+r} - Q_{k,a(n+1)+r} + Q_{k,r} - (-k)^r Q_{k,a-r}}{(-k)^a + Q_{k,a} + 1}$,
2. $\sum_{i=0}^{n} q_{k,ai+r} = \dfrac{(-k)^a q_{k,an+r} - q_{k,a(n+1)+r} + q_{k,r} - (-k)^r q_{k,a-r}}{(-k)^a + 2q_{k,a} + 1}$.

The next results give, the sum of certain k-Pell-Lucas and Modified k-Pell numbers respectively, with arithmetic indexes.

Corollary 3.3. *If $a = 2p+1$, then* Theorem 3.2 *gives*

1. $\sum_{i=0}^{n} Q_{k,(2p+1)i+r}$
$$= \frac{(-k)^{2p+1} Q_{k,(2p+1)n+r} - Q_{k,(2p+1)(n+1)+r} + Q_{k,r} - (-k)^r Q_{k,(2p+1)-r}}{(-k)^{2p+1} + Q_{k,2p+1} + 1},$$
2. $\sum_{i=0}^{n} q_{k,(2p+1)i+r}$
$$= \frac{(-k)^{2p+1} q_{k,(2p+1)n+r} - q_{k,(2p+1)(n+1)+r} + q_{k,r} - (-k)^r q_{k,(2p+1)-r}}{(-k)^{2p+1} + 2q_{k,2p+1} + 1}.$$

Corollary 3.4. *If $a = 2p$, then the equation stated in* Theorem 3.2 *gives*

1. $\sum_{i=0}^{n} Q_{k,2pi+r} = \dfrac{(-k)^{2p}Q_{k,2pn+r} - Q_{k,2p(n+1)+r} + Q_{k,r} - (-k)^r Q_{k,2p-r}}{(-k)^{2p} + Q_{k,2p} + 1}$,
2. $\sum_{i=0}^{n} q_{k,2pi+r} = \dfrac{(-k)^{2p}q_{k,2pn+r} - q_{k,2p(n+1)+r} + q_{k,r} - (-k)^r q_{k,2p-r}}{(-k)^{2p} + 2q_{k,2p} + 1}$.

Now, for any positive real number k and fixed integers $a,\ r$ with $0 \le r \le a-1$, we consider the alternating sequences $\{(-1)^n Q_{k,an+r}\}$ and $\{(-1)^n q_{k,an+r}\}$. The sum formulae for these sequences are stated in the next result.

Theorem 3.5. *For any positive real number k, any natural number n, and fixed integers $a,\ r$ with $0 \le r \le a-1$, the sum of the first $n+1$ terms of the alternating sequence of the k-Pell-Lucas and Modified k-Pell numbers, respectively, with arithmetic indexes are given by*

1. $\sum_{i=0}^{n}(-1)^i Q_{k,ai+r}$
$$= \frac{Q_{k,r} + (-1)^n Q_{k,a(n+1)+r} + (-k)^r Q_{k,a-r} + (-k)^a(-1)^n Q_{k,an+r}}{1 + Q_{k,a} + (-k)^a},$$
2. $\sum_{i=0}^{n}(-1)^i q_{k,ai+r}$
$$= \frac{q_{k,r} + (-1)^n q_{k,a(n+1)+r} + (-k)^r q_{k,a-r} + (-k)^a(-1)^n q_{k,an+r}}{1 + 2q_{k,a} + (-k)^a}.$$

Finally, for any positive real number k and fixed integers $a,\ r$ with $0 \le r \le a-1$, let denote the generating function of the sequence $\{Q_{k,an+r}\}$ by $g_{a,r}(k,x)$, and for the sequence $\{q_{k,an+r}\}$ by $h_{a,r}(k,x)$. Note that $2h_{a,r}(k,x) = g_{a,r}(k,x)$ and then a similar result of Proposition 2.9 concerning these sequences is

Proposition 3.6. *For any positive real number k, any natural number n, and fixed integers $a,\ r$ with $0 \le r \le a-1$, the generating function of the sequences $\{Q_{k,an+r}\}$ and $\{q_{k,an+r}\}$, are respectively*

1. $g_{a,r}(k,x) = \dfrac{Q_{k,r} - x\left(2Q_{k,r} - Q_{k,a+r}\right)}{1 - 2x - kx^2}$,
2. $h_{a,r}(k,x) = \dfrac{q_{k,r} - x\left(2q_{k,r} - q_{k,a+r}\right)}{1 - 2x - kx^2}$.

4. Conclusions

Sequences of numbers have been studied over several years with emphasis on studies of the well known Fibonacci sequence (and then the Lucas sequence) related to the golden ratio and of the Pell sequence that is related to the silver ratio. In this paper, we also contribute to the study of subsequences of k-Pell, k-Pell-Lucas, and Modified k-Pell numbers with indexes in an arithmetic sequence. We have deduced some formulae for the sums of such numbers and presented the generating functions.

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