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SOME NEW RESULTS ON EQUITABLE COLORING PARAMETERS OF GRAPHS

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Abstract. An equitable coloring of a graph G is a proper vertex coloring $\mathbb{C}$ of G such that the cardinalities of any two color classes in G with respect to $\mathbb{C}$ differ by at most one. Coloring the vertices of a graph G subject to given conditions can be considered as a random experiment. In this context, a discrete random variable X can be defined as the color of a vertex chosen at random, with respect to the given type of coloring of G and a probability mass function for this random variable can be defined accordingly. In this paper, we discuss two statistical parameters of the powers of certain graph classes with respect to their equitable colorings.

1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [**2, 8, 25**] and for the terminology of graph coloring, we refer to [**3, 9, 10**]. For the terminology in Statistics, see [**16, 17**]. Unless mentioned otherwise, all graphs considered in this paper are simple, finite, connected, and undirected.

Graph coloring is an assignment of colors or labels, or weights to the vertices, edges and faces of a graph under consideration. Unless stated otherwise, the graph coloring, is meant to be an assignment of colors to the vertices of a graph subject to certain conditions. A *proper coloring* of a graph G is a coloring with respect to which vertices of G are colored in such a way that no two adjacent vertices G have the same color. The *chromatic number* of graphs is the minimum number of colors required in a proper coloring of the given graph. The set of vertices of a graph G having a particular color c_i is called the color class of c_i in G and the cardinality of this color class is called the *strength* or the *weight* of the color c_i and denoted by $\theta(c_i)$. The general coloring sums with respect to different classes were studied in [**11, 18**].

Coloring of the vertices of a given graph G can be considered as a random experiment. For a proper k-coloring with color set $\mathbb{C} = \{c_1, c_2, c_3, \ldots, c_k\}$ of G, we can define a random variable ($r.v.$) X which denotes the color (or precisely, the

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subscript i of the color c_i) of any arbitrary vertex in G. As the sum of all weights of colors of G is the order of G, the real valued function $f(i)$ defined by

$$f(i) = \begin{cases} \dfrac{\theta(c_i)}{|V(G)|}, & i = 1, 2, 3, \dots, k, \\ 0, & \text{elsewhere.} \end{cases}$$

is the *probability mass function (p.m.f.)* of the random variable (*r.v.*) X (see [**19**]), where $\theta(c_i)$ is strength of the color c_i.

If the context is clear, we can also say that $f(i)$ is the probability mass function of the graph G with respect to the given coloring $\mathbb{C}$. Further results on certain related coloring parameters of certain graph classes were investigated in [**20, 22, 23**].

An *equitable coloring* of a graph G is a proper coloring $\mathbb{C}$ of G which assigns colors to the vertices of G such that the numbers of vertices in any two color classes differ by at most one (see [**4, 5, 12, 14**] for some related studies on equitable coloring of graphs). The *equitable chromatic number* of a graph G is the smallest number k such that G has an equitable coloring with k colors. Throughout this paper, we denote the equitable chromatic number of a graph by k.

Some studies on equitable coloring parameters of certain graph classes were conducted in [**6, 15, 21**]. Motivated by the studies on different types graph colorings, coloring parameters and their applications, we discuss the concepts of arithmetic mean and variance, two statistical parameters, to equitable coloring of certain graph classes in this paper. Throughout the paper, we follow the convention that $0 \le \theta(c_i) - \theta(c_j) \le 1$ when $i < j$.

The chromatic means and variances corresponding to an equitable coloring of a graph G are defined in [**21**] as follows.

Definition 1.1 ([**21**])**.** Let $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_k\}$ be a color set corresponding to an equitable k-coloring of a given graph G and X be the random variable which denotes the number of vertices having a particular color in $\mathbb{C}$, with the *p.m.f.* $f(i)$. Then

(i) the *equitable coloring mean* of a coloring $\mathbb{C}$ of a given graph G, denoted by $\mu_{\chi_e}(G)$, is defined to be $\mu_{\chi_e}(G) = \sum\limits_{i=1}^{k} i\, f(i)$,

(ii) *equitable coloring variance* of a coloring $\mathbb{C}$ of a given graph G, denoted by $\sigma^2_{\chi_e}(G)$, is defined to be $\sigma^2_{\chi_e}(G) = \sum\limits_{i=1}^{n} i^2\, f(i) - \Big(\sum\limits_{i=1}^{k} i\, f(i)\Big)^2$.

If the context is clear, the above-defined parameters can be called the equitable coloring mean and equitable coloring variance of the graph G with respect to the coloring $\mathbb{C}$. Some interesting studies on the equitable coloring parameters of certain graph classes can be seen in [**6, 15, 21**]. Motivated by these studies, in this paper, we discuss the equitable chromatic parameters for certain wheel related graph classes.

2. New results

Recall that the *k-th power* of a graph G, denoted by G^k, is the graph obtained by adding edges between the vertices which are at a distance at most k (see [**24**]). It is proved in [**24**] that the power graph G^d is a complete graph, where d is the diameter of the graph G. The chromatic number of graph powers was studied in [**1**].

First, recall the following result proved in [**21**].

Proposition 2.1 ([**21**]). *For a complete graph K_n, we have $\mu_{\chi_e}(K_n) = \frac{n+1}{2}$ and $\sigma^2_{\chi_e}(K_n) = \frac{n^2-1}{12}$.*

We first discuss the equitable coloring parameters of various powers of a path P_n.

Theorem 2.2. *For the r-th power of a path P_n, we have*

$$\mu_{\chi_e}(P_n^r) = \frac{n^3-(k-1)n^2-kn+k(k+1)(r+1)}{2n(r+1)}.$$

Proof. Note that for $r \le n$, corresponding to every P_{r+1} in P_n, P_n^r contains a complete graph K_{r+1} (see Figure 1). Hence, $r+1$ distinct colors are required for coloring the vertices of P_n^r.

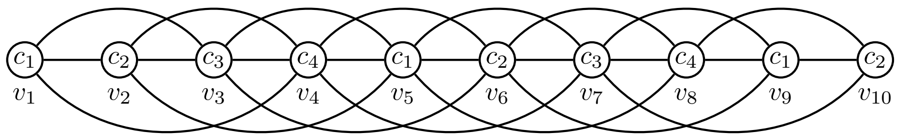


Figure 1. Equitable coloring of the graph P_{10}^3.

From Figure 1, it can be observed that $\theta(c_i) = \lfloor \frac{n}{r+1} \rfloor + 1 = \frac{n-k+r+1}{r+1}$ for $1 \le i \le k$, and $\theta(c_i) = \lfloor \frac{n}{r+1} \rfloor = \frac{n-k}{r+1}$ for $k+1 \le i \le n$, where $n \equiv k (\text{mod } r+1)$, $1 \le k \le r$. Therefore, the corresponding *probability mass function* is given by

$$f(i) = P(X=i) = \begin{cases} \dfrac{n-k+r+1}{n(r+1)}, & 1 \le i \le k, \\ \dfrac{n-k}{n(r+1)}, & k+1 \le i \le n. \end{cases}$$

Therefore,

$$\begin{aligned}
\mu_{\chi_e} &= \Big(\sum_{i=1}^{k} i\Big)\cdot\Big(\frac{n-k+r+1}{n(r+1)}\Big) + \Big(\sum_{i=k+1}^{n} i\Big)\cdot\Big(\frac{n-k}{n(r+1)}\Big)\\
&= \Big(\frac{k(k+1)}{2}\Big)\cdot\Big(\frac{n-k+r+1}{n(r+1)}\Big) + \Big(\frac{n-k}{2}\Big)(n+k+1)\cdot\Big(\frac{n-k}{n(r+1)}\Big)\\
&= \frac{k^2r+k^2-kn+kr+k-kn^2+n^3+n^2}{2n(r+1)}\\
&= \frac{n^3-(k-1)n^2-kn+k(k+1)(r+1)}{2n(r+1)}.
\end{aligned}$$

Computation of variance, in this case, is complex, and hence we leave the calculation unattempted. □

Note that the diameter d of the cycle C_n is $\lfloor \frac{n}{2} \rfloor$. If $r = \lfloor \frac{n}{2} \rfloor$, then $C_n^{\lfloor \frac{n}{2} \rfloor}$ is a complete graph. Therefore, the p.m.f. of the corresponding coloring is given by

$$f(i) = \begin{cases} \frac{1}{n} & \text{if } 1 \le i \le n,\\ 0, & \text{elsewhere.}\end{cases}$$

Hence, in view of Proposition 2.1, we have $\mu_{\chi_e} = \frac{n+1}{2}$ and $\sigma^2_{\chi_e} = \frac{n^2-1}{12}$ for all such graphs.

The following theorem determines the chromatic parameters of the powers of a cycle C_n for the remaining cases.

Theorem 2.3. *For the r-th power of a cycle C_n, where $1 < r < \lfloor \frac{n}{2} \rfloor$, we have*

$$\mu_{\chi_e}(C_n^r) = \begin{cases} \frac{r+2}{2} & \text{if } n \equiv 0 \,(\text{mod } r+1),\\ \frac{n+2}{4} & \text{if } n \equiv k \,(\text{mod } r+1) \text{ and } n \text{ is even, } k \neq 0,\\ \frac{(n+1)^2}{4n} & \text{if } n \equiv k\,(\text{mod } r+1) \text{ and } n \text{ is odd, } k \neq 0.\end{cases}$$

and

$$\sigma^2_{\chi_e}(C_n^r) = \begin{cases} \frac{r^2-10r-12}{12} & \text{if } n \equiv 0\,(\text{mod } r+1).\\ \frac{n^2-4}{12} & \text{if } n \equiv k\,(\text{mod } r+1) \text{ and } n \text{ is even, } k \neq 0.\\ \frac{(n+1)(n^3-n^2+3n-3)}{48n^2} & \text{if } n \equiv k\,(\text{mod } r+1) \text{ and } n \text{ is odd, } k \neq 0.\end{cases}$$

Proof. Given that $1 < r < \lfloor \frac{n}{2} \rfloor$. Then, we have to consider the following cases:

Case-1: Let $n \equiv 0\,(\text{mod } r+1)$. Then, we can color the vertices of $C_n^{\lfloor \frac{n}{2} \rfloor}$ in an equitable manner using $r+1$ colors, each of strength $\theta(c_i) = \frac{n}{r+1}$, where $1 \le i \le r+1$ (see Figure 2, for example).

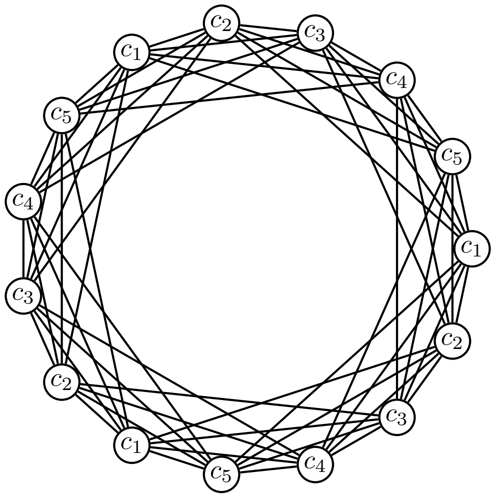


Figure 2. Equitable coloring of C_{15}^4

Then the corresponding p.m.f. is given by

$$f(i) = \begin{cases} \frac{1}{r+1} & \text{if } i = 1, 2, 3, \dots, r, r+1, \\ 0, & \text{elsewhere.} \end{cases}$$

Then

$$\mu_{\chi_e} = \sum_{i=1}^{r+1} i \frac{1}{r+1} = \frac{(r+1)(r+2)}{2} \cdot \frac{1}{r+1} = \frac{r+2}{2}$$

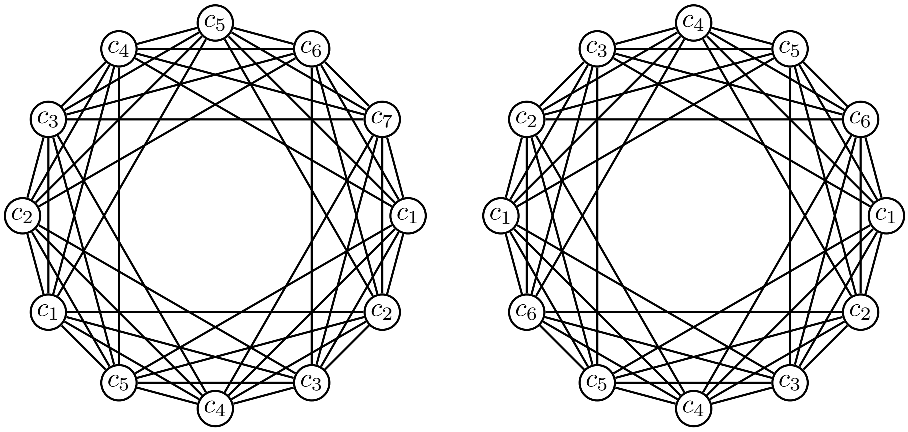
and

$$\sigma_{\chi_e}^2 = \Big(\sum_{i=1}^{r+1} i^2 \frac{1}{r+1}\Big) - \Big(\frac{r+2}{2}\Big)^2 = \frac{r(r+1)(2r+1)}{6} \cdot \frac{1}{r+1} - \Big(\frac{r+2}{2}\Big)^2$$
$$= \frac{r(2r+1)}{6} - \Big(\frac{r+2}{2}\Big)^2 = \frac{r^2 - 10r - 12}{12}.$$

<u>*Case-2:*</u> Let $k \neq 0$, $n \equiv k \pmod{r+1}$, and n is even. Here, the minimum number of colors is required for coloring $C_n^{\lfloor \frac{n}{2} \rfloor}$. But, if we color $C_n^{\lfloor \frac{n}{2} \rfloor}$ using $r+1$ colors, all its final k vertices must have different colors, all different from the above $r+1$ colors (see Figure 3a for illustration).

Hence, with respect to an equitable coloring of $C_n^{\lfloor \frac{n}{2} \rfloor}$, every color class can have at most 2 elements. Therefore, the required equitable coloring must have $\frac{n}{2}$ distinct colors (see Figure 3b), and hence the corresponding pm.f. is given by

$$f(i) = \begin{cases} \frac{2}{n} & \text{if } i = 1, 2, 3, \frac{n}{2}, \\ 0, & \text{elsewhere.} \end{cases}$$



(a) A minimal coloring of C_{12}^4 **(b)** An equitable coloring of C_{12}^4

Figure 3

Therefore, we have

$$\mu_{\chi_e} = \sum_{i=1}^{\frac{n}{2}} i\frac{2}{n} = \frac{n(n+2)}{8} \cdot \frac{2}{n} = \frac{n+2}{4}$$

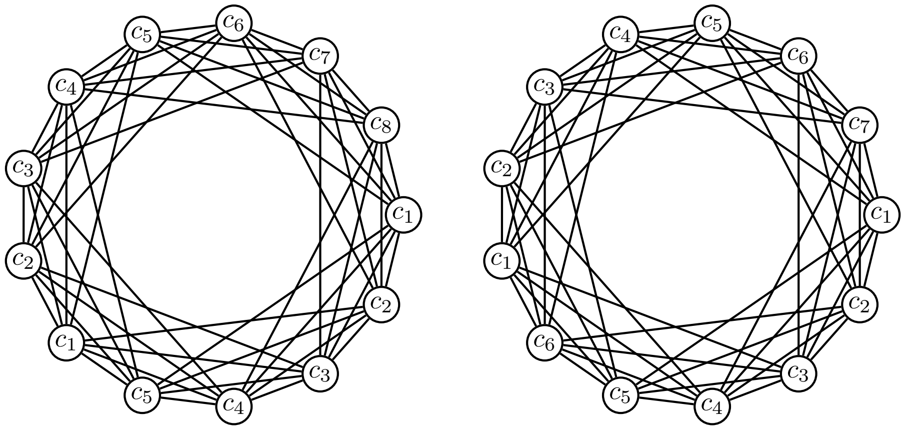
and

$$\begin{aligned}\sigma^2_{\chi_e} &= \Big(\sum_{i=1}^{\frac{n}{2}} i^2\frac{2}{n}\Big) - \Big(\frac{n+2}{4}\Big)^2 = \frac{n(n+1)(n+2)}{24} \cdot \frac{2}{n} - \Big(\frac{n+2}{4}\Big)^2 \\ &= \frac{n^2+3n+2}{12} - \frac{n^2+4n+4}{16} = \frac{n^2-4}{12}.\end{aligned}$$

<u>*Case-3:*</u> Let $k \neq 0$, $n \equiv k \,(\mathrm{mod}\ r+1)$, and n is odd. Here, as mentioned in the previous case, if we color $C_n^{\lfloor \frac{n}{2} \rfloor}$ using $r+1$ colors, all its final k vertices must have different colors, all different from the above $r+1$ colors (see Figure 4a for illustration).

Hence, in this case, with respect to an equitable coloring of $C_n^{\lfloor \frac{n}{2} \rfloor}$, every color class can have at most 2 elements. Therefore, the required equitable coloring must have $\frac{n+1}{2}$ distinct colors (see Figure 3b) such that the color classes of the first $\frac{n-1}{2}$ colors have exactly 2 elements and the color class of the last color $c_{\frac{n+1}{2}}$ is a singleton set. Hence the corresponding pm.f. is given by

$$f(i) = \begin{cases} \frac{2}{n} & \text{if } i = 1, 2, 3, \frac{n-1}{2}, \\ \frac{1}{n} & \text{if } i = \frac{n+1}{2}, \\ 0, & \text{elsewhere.} \end{cases}$$



(a) A minimal coloring of C_{12}^4 **(b)** An equitable coloring of C_{13}^4

Figure 4

Therefore, we have

$$\mu_{\chi_e} = \Big(\sum_{i=1}^{\frac{n-1}{2}} i\frac{2}{n}\Big) + \frac{n+1}{2}\cdot\frac{1}{n} = \frac{(n-1)(n+1)}{8}\cdot\frac{2}{n} + \frac{n+1}{2n} = \frac{(n+1)^2}{4n}$$

and

$$\begin{aligned}\sigma^2_{\chi_e} &= \Big[\Big(\sum_{i=1}^{\frac{n-1}{2}} i^2\frac{2}{n}\Big) + \frac{(n+1)^2}{4}\cdot\frac{1}{n}\Big] - \Big(\frac{(n+1)^2}{4n}\Big)^2 \\ &= \frac{n(n-1)(n+1)}{24}\cdot\frac{2}{n} + \frac{(n+1)^2}{4}\frac{1}{n} - \Big(\frac{(n+1)^2}{4n}\Big)^2 \\ &= \frac{n^2-1}{12} + \frac{(n+1)^2}{4n} - \frac{n^2+4n+4}{16n^2} = \frac{(n+1)(n^3-n^2+3n-3)}{48n^2}.\end{aligned}$$

This completes the proof. □

3. Equitable chromatic parameters of the powers of certain cycle related graphs

The equitable chromatic parameters of certain cycle related graphs were determined in [**6**]. Among them, wheel graphs, double wheel graphs, djembe graphs, flower graphs, and blossom graphs (see [**6, 7**] for definitions of these cycle related graph classes) are those graphs whose squares are complete graphs. Hence, their square graphs have the p.m.f.

$$f(i) = \begin{cases} \frac{1}{|V|} & \text{if } i = 1,2,3,\dots,|V|, \\ 0, & \text{elsewhere,}\end{cases}$$

the equitable chromatic mean $\mu_{\chi_e} = \frac{1+|V|}{2}$, and the equitable chromatic variance $\sigma^2_{\chi_e} = \frac{|V|^2-1}{12}$ (Refer Proposition 2.1). We consider some other cycle related (wheel related) graphs in our following discussion.

A *helm graph* H_n is a graph obtained by attaching a pendant edge to every rim vertex of a wheel graph W_n. Next, we consider the case of the powers helm graph H_n to determine the equitable chromatic parameters. Equitable chromatic parameters of helm graphs were found out in [**6**]. We also note that the diameter of a helm graph is 4, and hence H_n^4 is a complete graph on $2n+1$ vertices. Therefore, by the above remark, $\mu_{\chi_e}(H_n^4) = n+1$ and $\sigma^2_{\chi_e}(H_n^4) = \frac{n(n+1)}{3}$. Now, we need to consider the cube and fourth power of H_n. Therefore, we have tne following theorem.

Theorem 3.1. *The chromatic parameters of H_n^r is given by*

$$\mu_{\chi_e}(H_n^r) = \begin{cases} n+1 & \text{if } r=2, \\ \frac{5n^2+10n+4}{4(2n+1)} & \text{if } r=3 \text{ and } n \text{ is even}, \\ \frac{5n^2+11n+5}{4(2n+1)} & \text{if } r=3 \text{ and } n \text{ is odd}. \end{cases}$$

and

$$\sigma^2_{\chi_e}(H_n^r) = \begin{cases} \frac{(n+1)^2(2n^2+7n+6)}{3(2n+1)} & \text{if } r=2, \\ \frac{37n^4+200n^3+374n^2+286n+72}{48(2n+1)^2} & \text{if } r=3 \text{ and } n \text{ is even}, \\ \frac{37n^4+158n^3+263n^2+190n+45}{48(2n+1)^2} & \text{if } r=3 \text{ and } n \text{ is odd}. \end{cases}$$

Proof. Let u be the central vertex, $V = \{v_1, v_2, v_3, \ldots, v_n\}$ the set of rim vertices, and $U = \{u_1, u_2, u_3, \ldots, u_n\}$ the pendant vertices of the helm graph H_n. Then, we have the following cases:

Case-1: When $r = 2$, it can be noted that the induced graph $\langle U \cup \{u\}\rangle$ is a clique on $n+1$ vertices in H_n^2. Hence, $n+1$ distinct colors are required to color the vertices in the induced graph $\langle U \cup \{u\}\rangle$. Now, it can be noted that for every vertex $v \in V$, there exists at least one vertex in U which is not adjacent to v. Therefore, v and this vertex can have the same color. Therefore, the above-mentioned $n+1$ colors are required in an equitable coloring of H_n^2 (see Figure 5 for illustration).

Hence, the first n colors have strength 2 and the color c_{n+1} has strength 1. Therefore, the corresponding p.m.f. is given by

$$f(i) = \begin{cases} \frac{2}{2n+1} & \text{if } i = 1, 2, 3, \ldots n, \\ \frac{1}{2n+1} & \text{if } i = n+1, \\ 0, & \text{elsewhere}. \end{cases}$$

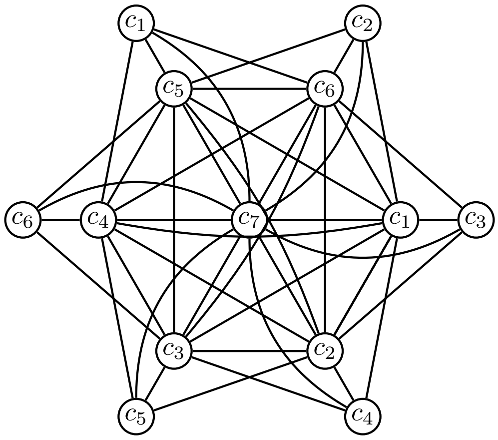


Figure 5. A minimal coloring of H_6^2

Therefore, we have

$$\begin{aligned}\mu_{\chi_e} &= \left[\sum_{i=1}^{n} i\frac{2}{2n+1} + (n+1)\cdot\frac{1}{2n+1}\right]\\ &= \frac{2n(n+1)}{2n+1} + \frac{n+1}{2n+1}, = n+1\\ \sigma^2_{\chi_e} &= \left[\sum_{i=1}^{n} i^2\frac{2}{2n+1} + (n+1)^2\cdot\frac{1}{2n+1}\right] - (n+1))^2\\ &= \frac{n(n+1)(2n+1)}{6}\cdot\frac{2}{2n+1} + \frac{(n+1)^2}{2n+1} - (n+1)^2\\ &= \frac{(n+1)^2}{2n+1}\left[1 - \frac{2n-3}{3(n+1)(2n+1)}\right] = \frac{(n+1)^2(2n^2+7n+6)}{3(2n+1)}.\end{aligned}$$

Case-2: When $r = 3$, as mentioned above, the induced subgraph graph $\langle U \cup \{u\}\rangle$ of H_n^3 is a clique on $n+1$ vertices in H_n^2. Also, note that every vertex $v \in V$ is adjacent to all vertices of U, and hence no vertices in V can have the same color of a vertex in $U \cup \{u\}$. Therefore, we have to consider the following subcases:

Subcase-2.1: If n is even, then $\frac{n}{2} + n + 1 = \frac{3n}{2} + 1$ colors are required in an equitable coloring of H_n^2 (see Figure 6 for illustration).

Note that the first $\frac{n}{2}$ colors have strength 2 and the remaining $n+1$ colors have strength 1. Therefore, the corresponding p.m.f. is given by

$$f(i) = \begin{cases} \frac{2}{2n+1} & \text{if } i = 1, 2, 3, \ldots, \frac{n}{2},\\ \frac{1}{2n+1} & \text{if } i = \frac{n}{2}+1, \frac{n}{2}+2, \frac{3n}{2}+1,\\ 0, & \text{elsewhere.}\end{cases}$$

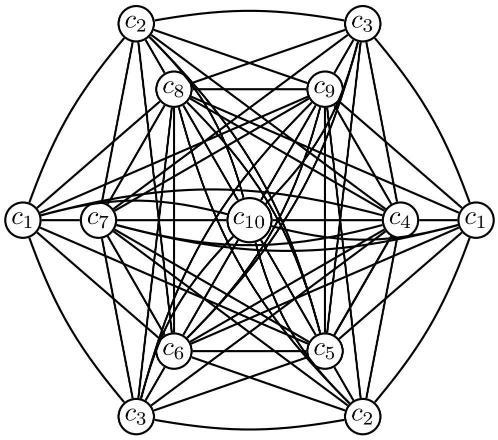


Figure 6. An equitable coloring of H_6^3

Then, we have

$$\mu_{\chi_e} = \sum_{i=1}^{\frac{n}{2}} \frac{2i}{2n+1} + \sum_{i=\frac{n}{2}+1}^{\frac{3n}{2}+1} \frac{i}{2n+1}$$
$$= \frac{n+2}{4} + \frac{(n+1)^2}{n} = \frac{5n^2+10n+4}{4(2n+1)},$$

$$\sigma^2_{\chi_e} = \sum_{i=1}^{\frac{n}{2}} \frac{2i^2}{2n+1} + \sum_{i=\frac{n}{2}+1}^{\frac{3n}{2}+1} \frac{i^2}{2n+1} - \left(\frac{5n^2+10n+4}{4(2n+1)}\right)^2$$
$$= \frac{n(n+1)(n+2)}{24} \cdot \frac{2}{2n+1} + \frac{26n^3+105n^2+139n+60}{24} \cdot \frac{1}{2n+1} - \left(\frac{5n^2+10n+4}{4(2n+1)}\right)^2$$
$$= \frac{28n^3+111n^2+143n+60}{24(2n+1)} - \left(\frac{5n^2+10n+4}{4(2n+1)}\right)^2$$
$$= \frac{37n^4+200n^3+374n^2+286n+72}{48(2n+1)^2}.$$

Subcase-2.2: If n is odd, then $\frac{n+1}{2}+n+1 = \frac{3(n+1)}{2}$ colors are required in an equitable coloring of H_n^2.

Note that the first $\frac{n-1}{2}$ colors have strength 2 and the remaining $n+2$ colors have strength 1. Therefore, the corresponding p.m.f. is given by

$$f(i) = \begin{cases} \frac{2}{2n+1} & \text{if } i = 1,2,3,\ldots,\frac{n-1}{2}, \\ \frac{1}{2n+1} & \text{if } i = \frac{n+1}{2}, \frac{n+1}{2}+1, \frac{3(n+1)}{2}, \\ 0, & \text{elsewhere.} \end{cases}$$

Then, we have

$$\begin{aligned}
\mu_{\chi_e} &= \sum_{i=1}^{\frac{n-1}{2}} \frac{2i}{2n+1} + \sum_{i=\frac{n+1}{2}}^{\frac{3(n+1)}{2}} \frac{i}{2n+1} \\
&= \frac{(n-1)(n+1)}{4(2n+1)} + \frac{(n+2)(n+1)}{2n+1} \\
&= \frac{5n^2+11n+5}{4(2n+1)}, \\
\sigma^2_{\chi_e} &= \left[\sum_{i=1}^{\frac{n-1}{2}} \frac{2i^2}{2n+1} + \sum_{i=\frac{n+1}{2}}^{\frac{3(n+1)}{2}} \frac{i^2}{2n+1}\right] - \left(\frac{5n^2+11n+5}{4(2n+1)}\right)^2 \\
&= \left[\frac{n(n^2-1)}{24(2n-1)} \cdot \frac{2}{2n+1} + \frac{(n+1)(13n^2+41n+30)}{12} \cdot \frac{1}{n}\right] \\
&\quad - \left(\frac{5n^2+11n+5}{4(2n+1)}\right)^2 \\
&= \frac{(n+1)(7n^2+20n+15)}{6(2n+1)} - \left(\frac{5n^2+11n+5}{4(2n+1)}\right)^2 \\
&= \frac{37n^4+158n^3+263n^2+190n+45}{48(2n+1)^2}.
\end{aligned}$$

This completes the proof. □

A closed helm graph CH_n is a graph obtained from the helm graph H_n by joining a pendant vertex v_i to the pendant vertex v_{i+1}, where $1 \leq i \leq n$ and $v_{n+i} = v_i$. Exactly in the same manner explained above, we can verify that the equitable chromatic parameters of closed helm graphs are the same as those ones of the corresponding helm graphs.

A *sunflower graph* SF_n is a graph obtained by replacing each edge of the rim of a wheel graph W_n by a triangle such that two triangles share a common vertex if and only if the corresponding edges in Wn are adjacent in W_n. The equitable chromatic parameters of sunflower graphs were determined in [**6**]. Now, we note that the diameter of SF_n is also 4. Hence, $\mu_{\chi_e}(SF_n^4) = n+1$ and $\sigma^2_{\chi_e}(SF_n^4) = \frac{n(n+1)}{3}$. The following theorem explains the equitable coloring parameters of the remaining powers of sunflower graph SF_n.

Theorem 3.2. *The chromatic parameters of SF_n^r are given by*

$$\mu_{\chi_e}(SF_n^r) = \begin{cases} n+1 & \text{if } r=2, \\ \frac{5n^2+10n+4}{4(2n+1)} & \text{if } r=3 \text{ and } n \text{ is even}, \\ \frac{5n^2+11n+5}{4(2n+1)} & \text{if } r=3 \text{ and } n \text{ is odd}, \end{cases}$$

and

$$\sigma^2_{\chi_e}(SF^r_n) = \begin{cases} \frac{(n+1)^2(2n^2+7n+6)}{3(2n+1)} & \text{if } r=2, \\ \frac{37n^4+200n^3+374n^2+286n+72}{48(2n+1)^2} & \text{if } r=3 \text{ and } n \text{ is even;} \\ \frac{37n^4+158n^3+263n^2+190n+45}{48(2n+1)^2} & \text{if } r=3 \text{ and } n \text{ is odd.} \end{cases}$$

Proof. The proof follows exactly as explained in the proof of Theorem 3.1. □

4. Equitable coloring parameters of Mycielskian of paths and cycles

Let G be a graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$. The *Mycielski graph* or the *Mycielskian* of a graph G, denoted by $\mu(G)$, is the graph with vertex set $V(\mu(G)) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, w\}$ such that $v_iv_j \in E(\mu(G)) \iff v_iv_j \in E(G)$, $v_iu_j \in E(\mu(G)) \iff v_iv_j \in E(G)$ and $u_iw \in E(\mu(G))$ for all $i = 1, \ldots, n$ (see [**13**]). Here, the vertex w may be called the *root vertex* of the Mycielskian graph $\mu(G)$. For ease of notation, we write $\breve{G}$ instead of $\mu(G)$ to denote the Mycielskian of G.

The equitable coloring parameters of Mycielski graphs of paths and cycles were determined in [**15**]. Following those results, the theorem given below discusses the equitable coloring parameters of Mycielskian of paths and cycles.

Theorem 4.1. *If G is the path P_n or the cycle C_n, then, we have*

$$\mu_{\chi_e}(\breve{G}^r) = \begin{cases} n+1 & \text{if } r=2, \\ \frac{5n^2+10n+4}{4(2n+1)} & \text{if } r=3 \text{ and } n \text{ is even,} \\ \frac{5n^2+11n+5}{4(2n+1)} & \text{if } r=3 \text{ and } n \text{ is odd,} \\ n+1 & \text{if } r=4. \end{cases}$$

and

$$\sigma^2_{\chi_e}(\breve{G}^r) = \begin{cases} \frac{(n+1)^2(2n^2+7n+6)}{3(2n+1)} & \text{if } r=2, \\ \frac{37n^4+200n^3+374n^2+286n+72}{48(2n+1)^2} & \text{if } r=3 \text{ and } n \text{ is even,} \\ \frac{37n^4+158n^3+263n^2+190n+45}{48(2n+1)^2} & \text{if } r=3 \text{ and } n \text{ is odd,} \\ \frac{n(n+1)}{3} & \text{if } r=4. \end{cases}$$

Proof. Let w be the root vertex of $\mu(G)$, V and W be the vertex sets as defined in the definition of Mycielskian of graphs given above. Then, we have the following cases:

Case-1: All vertices in U are at a distance 2 in $\mu(G)$, and hence if $r = 2$, then every vertex in U is adjacent to each other and to w, making the induced subgraph $\langle U \cup \{w\}$ a clique of $\breve{G}^2$. Also, every vertex of V is adjacent to w, but not adjacent to some vertices of U (and also non-adjacent to some vertices of V).

Hence, maximum two vertices can be seen in a color class of $\breve{G}^2$. Therefore, we need $n+1$ colors in an equitable coloring of $\breve{G}^2$.

Case-2: If $r = 3$, then in addition to the fact that $\langle U \cup \{w\} \rangle$ is a complete subgraph of $\breve{G}^3$, every vertex in V is adjacent to all vertices in $U \cup \{w\}$ and non-adjacent among some vertices in V. Hence, maximum two vertices can be also seen a color class of $\breve{G}^3$. Therefore, if n is even, then $\frac{n}{2}$ colors are required to color the vertices in V in addition to the $n+1$ colors required to color the vertices in $U \cup \{w\}$. Similarly, if n is odd, then $\frac{n+1}{2}$ colors are required to color the vertices in V in addition to the $n+1$ colors required to color the vertices in $U \cup \{w\}$.

Case-3: If $r = 4$, then the graph $\breve{G}^4$ is a complete graph on $2n+1$ vertices. Note that all the above-mentioned cases clearly match the corresponding cases mentioned in the computation of the equitable coloring parameters of the powers of helm graphs in Theorem 3.1, and hence the result follows exactly as mentioned in the proof of Theorem 3.1. □

The diameter of the Mycielskian of the complete graph $\breve{K}_n$ is 2, and hence Theorem 3.1 can be applied to the square of the Mycielskian of complete graphs. For all graphs with $\Delta(G) < n-1$, the diameter of their Mycielskians is 4, and hence Theorem 4.1 can be extended to them. For those graphs with $\Delta(G) = n-1$, the situation is slightly different. The number of vertices with maximum degree, structural characteristics, etc., play a vital role in this context. These aspects open promising research areas for further studies.

5. Conclusion

In this paper, we determined two important statistical parameters, related to equitable coloring of the powers of certain fundamental graph classes. We can use these statistical parameters defined for graph coloring problems for modelling problems in many areas like project management, communication networks, optimisation problems, etc. The concepts of equitable chromatic parameters can be utilised in certain particular real-life and industrial problems including routing, resource allocation, resource smoothing, inventory management, service, and distribution systems, etc.

The problems of χ_e-chromatic mean, and variance of several other graph classes still offer much for further studies. Investigating the sum, mean and variance corresponding to different types of edge colorings, map colorings, total colorings, etc., of graphs also offer much for future studies. The studies can be extended to determine the skewness and kurtosis, and other similar measures of graphs with respect to different graph colorings, and hence determine the most effective or economical graph coloring method in the context concerned.

We can associate many other statistical parameters to graph coloring and other notions like covering, matching, etc. All these facts highlight a wide scope for future studies in this area.

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