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ON THE MINIMAL DOUBLY RESOLVING SETS OF HARARY GRAPH

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Abstract. Let $G = (V, E)$ be a simple connected and undirected graph, where V and E represent the vertex and edge set, respectively. The vertices x and y doubly resolve the vertices u and v if the following condition is satisfied

$$d(u, x) - d(u, y) \neq d(v, x) - d(v, y).$$

A subset D of vertex set V of G is said to be doubly resolving set of G if for every pair x', y' of distinct vertices of G, there exist two vertices x, y in D which doubly resolve the vertices x', y'. A minimal doubly resolving set is a doubly resolving set which has minimum cardinality. The cardinality of minimal doubly resolving set is denoted by $\psi(G)$. Let $\beta(G)$ denotes the metric dimension of graph G which is the cardinality of minimal resolving set, then we have $\beta(G) \leq \psi(G)$ since every doubly resolving set is a resolving set, too.

Borchert and Gosselin et al. solved the problem of finding metric dimension for Harary graph $H_{4,n}$, $n \geq 8$. In this paper, we find the minimal doubly resolving set, and hence the cardinality $\psi(H_{4,n})$ for Harary graph $H_{4,n}$, $n \geq 8$.

1. Introduction and preliminary results

Cáceres [**5**] introduced the notion of doubly resolving set of graph G and defined it in the following way. Let $G = (V, E)$ be a simple, connected and undirected graph with vertex set V and edge set E, respectively. Let $d(x, y)$ denote the distance between the vertices x and y which is the shortest length of the path among the lengths of all paths between x and y. Consider a graph G of order at least 2. We say that two vertices u and v doubly resolve the vertices u' and v' of G if

$$d(u', u) - d(u', v) \neq d(v', u) - d(v', v).$$

A subset D of vertex set V of G is called doubly resolving set if for every pair u', v' of distinct vertices of G, there exist two vertices u and v in D which doubly resolve the pair u', v'. Let $D = \{x_1, x_2, \ldots, x_m\}$ be an ordered subset of vertex set V of G. For a vertex v of G, the m-tuple $r(v|D) = (d(v, x_1), d(v, x_2), \ldots, d(v, x_m))$

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is called a vector of metric coordinates of v with respect to D. Using the above definitions, the following important lemma can be easily deduced.

Lemma 1.1. *An ordered subset $D = \{x_1, x_2, \ldots, x_m\}$ of vertex set V of graph G is doubly resolving set of G if for every pair of distinct vertices $u, v \in V$, we have*

$$r(u|D) - r(v|D) \neq \underbrace{(\mu, \mu, \ldots, \mu)}_{m\ times}$$

for any integer μ, i.e., the difference between all the coordinates of the tuples $r(u|D)$ and $r(v|D)$ can not be the same integer.

Proof. Suppose that $D = \{x_1, x_2, \ldots, x_m\}$ of V is doubly resolving set of G and there exist two distinct vertices $u, v \in V$ such that

$$r(u|D) - r(v|D) = \underbrace{(\mu, \mu, \ldots, \mu)}_{m \text{ times}}$$

for some integer μ, then we have

$$d(u, x_i) - d(v, x_i) = d(u, x_j) - d(v, x_j) = \mu$$

for every $i \neq j$, that is,

$$d(u, x_i) - d(u, x_j) = d(v, x_i) - d(v, x_j)$$

for every $i \neq j$. This shows that by definition, no pair of vertices from the set D doubly resolves the vertices u and v, which is a contradiction to the fact that D is a doubly resolving set. Hence our supposition is wrong and the Lemma 1.1 holds. □

A doubly resolving set which the minimum cardinality among all doubly resolving sets is called minimal doubly resolving set of graph G. The cardinality of minimal doubly resolving set is denoted by $\psi(G)$. Note that if u, v doubly resolve the vertices u', v', then either $d(u', u) - d(u', v) \neq 0$ or $d(v', u) - d(v', v) \neq 0$ which follows that either u' or v' resolve the vertices u and v. Therefore, a doubly resolving set is also a resolving set and we have $\beta(G) \leq \psi(G)$, thus the minimal doubly resolving sets are used to obtain an upper bound on the metric dimension of a graph. Another application of minimal doubly resolving set includes locating the source of diffusion [**6**]. The problem of finding minimal doubly resolving set is NP-hard [**8**]. The problem of finding minimal doubly resolving set is considered for many families of graphs, for instance, minimal doubly resolving sets of prism, convex polytope and Hamming graphs were studied in [**4**], [**9**], and [**10**], respectively. The minimal doubly resolving sets of necklace and circulant graphs were found in [**1**] and [**2**].

Harary graph $H_{r,n}$ [**7, 11**] is an r-regular graph of order n with $V(H_{r,n}) = \{v_1, v_2, \ldots, v_n\}$ arranged cyclically. If r is an even integer, i.e., $r = 2k \leq n - 1$, then for non-negative integer $k \leq \frac{(n-1)}{2}$, the graph $H_{r,n}$ can be constructed by joining each vertex v_i to the k vertices that immediately follow v_i and k vertices that immediately precede v_i. In other words, join v_i to $v_{i+1}, v_{i+2}, \ldots, v_{i+k}$ and to $v_{i-1}, v_{i-2}, \ldots, v_{i-k}$ with indices taken modulo n. The graph $H_{4,12}$ is shown in Figure 1 below.

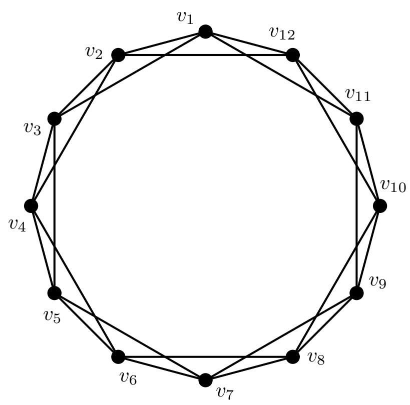

Figure 1. The Harary graph $H_{4,12}$.

If r is odd, then $n = 2m$ is even. Also $r = 2k+1 \leq n-1$ for some non negative integer $k \leq \frac{(n-2)}{2}$. In this case, $H_{r,n}$ can be constructed by joining each vertex v_i to $2k$ vertices as described above as well as to v_{i+m} with index taken modulo n, i.e., joining each vertex v_i to the k vertices that immediately follow v_i and k vertices that immediately precede v_i and the unique vertex "opposite" v_i. The graph $H_{5,12}$ is shown in Figure 2 below.

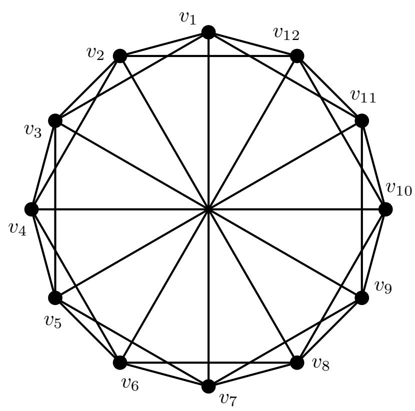

Figure 2. The Harary graph $H_{5,12}$.

In [**3**], Borchert et al. proved that

$$\beta(H_{4,n}) = \begin{cases} 3 & \text{if } n \not\equiv 1 (\text{mod } 4), \\ 4, & \text{otherwise.} \end{cases}$$

We determine the minimal doubly resolving set of $H_{4,n}$, $n \geq 8$.

2. The minimal doubly resolving sets for Harary graph $H_{4,n}$, $n \equiv 0, 2, 3 (\mathrm{mod}\ 4)$, $n \geq 8$

We have

$$\psi(H_{4,n}) \geq \beta(H_{4,n}) = \begin{cases} 3 & \text{if } n \not\equiv 1 (\mathrm{mod}\ 4), \\ 4, & \text{otherwise.} \end{cases}$$

(see [**3**]).

Let us define the set $S_i(v_1) = \{w \in V(H_{4,n}) : d(v_1, w) = i\}$, which is the set of vertices in $V(H_{4,n})$ at distance i from v_1. The following Table 1 displays the sets $S_i(v_1)$ for Harary graph $H_{4,n}$, where $n \geq 8$.

Table 1. $S_i(v_1)$ for $H_{4,n}$, $n \geq 8$.

n	i	$S_i(v_1)$
$4k$ $(k \geq 2)$	0	$\{v_1\}$
	$1 \leq i \leq k-1$	$\{v_{2i}, v_{2i+1}, v_{n+1-2i}, v_{n+2-2i}\}$
	k	$\{v_{2k}, v_{2k+1}, v_{2k+2}\}$
	$i \geq k+1$	$\emptyset$
$4k+1$ $(k \geq 2)$	0	$\{v_1\}$
	$1 \leq i \leq k$	$\{v_{2i}, v_{2i+1}, v_{n+1-2i}, v_{n+2-2i}\}$
	$i \geq k+1$	$\emptyset$
$4k+2$ $(k \geq 2)$	0	$\{v_1\}$
	$1 \leq i \leq k$	$\{v_{2i}, v_{2i+1}, v_{n+1-2i}, v_{n+2-2i}\}$
	$k+1$	$\{v_{2k+2}\}$
	$i \geq k+2$	$\emptyset$
$4k+3$ $(k \geq 2)$	0	$\{v_1\}$
	$1 \leq i \leq k$	$\{v_{2i}, v_{2i+1}, v_{n+1-2i}, v_{n+2-2i}\}$
	$k+1$	$\{v_{2k+2}, v_{2k+3}\}$
	$i \geq k+2$	$\emptyset$

Theorem 2.1. $\psi(H_{4,n}) = 3$ *for* $n \equiv 0, 2, 3 (\mathrm{mod}\ 4)$, $n \geq 8$.

Proof. We need to show that $\psi(H_{4,n}) \leq 3$ for $n \equiv 0, 2, 3 (\mathrm{mod}\ 4)$, $n \geq 8$. So it suffices to find a doubly resolving set of cardinality 3 in each case. Let us first consider the case when $n \equiv 0 (\mathrm{mod}\ 4)$, i.e., $n = 4k$ for $k \geq 2$. Using the sets $S_i(v_1)$ from Table 1, the following Table 2 displays the vectors of metric coordinates of every vertex of $H_{4,n}$ with respect to the set $D^* = \{v_1, v_2, v_{2k+1}\}$. Similarly, using Table 1, the following tables display the vectors of metric coordinates of vertices of $H_{4,n}$ for $n \equiv 2 (\mathrm{mod}\ 4)$ and $n \equiv 3 (\mathrm{mod}\ 4)$ with respect to the set $D^* = \{v_1, v_3, v_{2k+3}\}$ and $D^* = \{v_1, v_2, v_{2k+2}\}$, respectively.

From Tables 2, 3, 4, it can be verified directly that if two vertices x, y belong to $S_i(v_1)$ for some i, then the difference between first coordinates of x and y is

Table 2. Vectors of metric coordinates for $H_{4,n}$, $n = 4k$, $k \geq 2$.

i	$S_i(v_1)$	metric coordinates w.r.t $D^* = \{v_1, v_2, v_{2k+1}\}$
0	v_1	$(0, 1, k)$
$1 \leq i \leq k-1$	v_{2i}	$(i, i-1, k+1-i)$
	v_{2i+1}	$(i, i, k-i)$
	v_{n+1-2i}	$(i, i+1, k-i)$
	v_{n+2-2i}	$(i, i, k+1-i)$
k	v_{2k}	$(k, k-1, 1)$
	v_{2k+1}	$(k, k, 0)$
	v_{2k+2}	$(k, k, 1)$

Table 3. Vectors of metric coordinates for $H_{4,n}$, $n = 4k+2$, $k \geq 2$.

i	$S_i(v_1)$	metric coordinates w.r.t $D^* = \{v_1, v_3, v_{2k+3}\}$
0	v_1	$(0, 1, k)$
1	v_2	$(1, 1, k+1)$
	v_3	$(1, 0, k)$
	v_{4k+1}	$(1, 2, k-1)$
	v_{4k+2}	$(1, 2, k)$
$2 \leq i \leq k-1$	v_{2i}	$(i, i-1, k+2-i)$
	v_{2i+1}	$(i, i-1, k+1-i)$
	v_{n+1-2i}	$(i, i+1, k-i)$
	v_{n+2-2i}	$(i, i+1, k+1-i)$
k	v_{2k}	$(k, k-1, 2)$
	v_{2k+1}	$(k, k-1, 1)$
	v_{2k+3}	$(k, k, 0)$
	v_{2k+4}	$(k, k+1, 1)$
k+1	v_{2k+2}	$(k+1, k, 1)$

Table 4. Vectors of metric coordinates for $H_{4,n}$, $n = 4k+3$, $k \geq 2$.

i	$S_i(v_1)$	metric coordinates w.r.t $D^* = \{v_1, v_2, v_{2k+2}\}$
0	v_1	$(0, 1, k+1)$
$1 \leq i \leq k$	v_{2i}	$(i, i-1, k+1-i)$
	v_{2i+1}	$(i, i, k+1-i)$
	v_{n+1-2i}	$(i, i+1, k+1-i)$
	v_{n+2-2i}	$(i, i, k+2-i)$
k+1	v_{2k+2}	$(k+1, k, 0)$
	v_{2k+3}	$(k+1, k+1, 1)$

zero but the difference between all coordinates is not zero at the same time. For example, consider the representation of vertices v_{2i} and v_{2i+1} in Table 4, where

$1 \le i \le k$, we have

$$\begin{aligned} r(v_{2i}|D^*) - r(v_{2i+1}|D^*) &= (i, i-1, k+1-i) - (i, i, k+1-i) \\ &= (0, -1, 0). \end{aligned}$$

Also if $x \in S_i(v_1)$ and $y \in S_j(v_1)$ for $i \neq j$, then the difference between first coordinates of x and y is $i - j$ but the difference between all coordinates is not $i - j$ at the same time. For example, consider the representation of vertices v_{2i} and v_{2j} in Table 4, where $1 \le i, j \le k$ and $i \neq j$, we have

$$\begin{aligned} r(v_{2i}|D^*) - r(v_{2j}|D^*) &= (i, i-1, k+1-i) - (j, j-1, k+1-j) \\ &= (i-j, i-j, j-i). \end{aligned}$$

Thus using Lemma 1.1, $D^* = \{v_1, v_2, v_{2k+1}\}$, $D^* = \{v_1, v_3, v_{2k+3}\}$, and $D^* = \{v_1, v_2, v_{2k+2}\}$ are doubly resolving sets (indeed minimal doubly resolving sets) of $H_{4,n}$ for $n \equiv 0, 2, 3 (\mathrm{mod}\ 4)$, respectively, and hence Theorem 2.1 holds. □

Minimal doubly resolving sets of Harary graph $H_{4,n}$, $n \equiv 1 (\mathrm{mod}\ 4)$, $n \ge 9$ are found in the following theorem.

Theorem 2.2. $\psi(H_{4,n}) = 4$ *for* $n \equiv 1 (\mathrm{mod}\ 4)$, $n \ge 9$.

Proof. The proof is similar to the proof of Theorem 2.1. Consider the graph $H_{4,n}$ for $n \equiv 1 (\mathrm{mod}\ 4)$, $n \ge 9$, i.e., $n = 4k + 1$, where $k \ge 2$. We have $\psi(H_{4,n}) \ge \beta(H_{4,n}) = 4$. The following Table 5 displays the vectors of metric coordinates of vertices of $H_{4,n}$, $n \equiv 1 (\mathrm{mod}\ 4)$, with respect to the set $D^* = \{v_1, v_2, v_{2k}, v_{2k+1}\} \subset V(H_{4,n})$.

Table 5. Vectors of metric coordinates for $H_{4,n}$, $n = 4k + 1$, $k \ge 2$.

i	$S_i(v_1)$	metric coordinates w.r.t $D^* = \{v_1, v_2, v_{2k}, v_{2k+1}\}$
0	v_1	$(0, 1, k, k)$
1	v_2	$(1, 0, k-1, k)$
	v_3	$(1, 1, k-1, k-1)$
	v_{4k}	$(1, 2, k, k)$
	v_{4k+1}	$(1, 1, k, k)$
$2 \le i \le k-1$	v_{2i}	$(i, i-1, k-i, k+1-i)$
	v_{2i+1}	$(i, i, k-i, k-i)$
	v_{n+1-2i}	$(i, i+1, k+1-i, k+1-i)$
	v_{n+2-2i}	$(i, i, k+2-i, k+1-i)$
k	v_{2k}	$(k, k-1, 0, 1)$
	v_{2k+1}	$(k, k, 1, 0)$
	v_{2k+2}	$(k, k, 1, 1)$
	v_{2k+3}	$(k, k, 2, 1)$

Using this Table and Lemma 1.1, the set D^* is minimal doubly resolving set of the Harary graph $H_{4,n}$, and hence Theorem 2.2 holds. □

Conclusion. As the problem of finding minimal doubly resolving sets of graphs is NP-hard, therefore, it is being investigated for different families of graphs. In this paper, we find the minimal doubly resolving set for Harary graph $H_{4,n}$, $n \geq 8$. We have proved that

$$\psi(H_{4,n}) = \beta(H_{4,n}) = \begin{cases} 3 & \text{if } n \not\equiv 1 (\text{mod } 4), \\ 4, & \text{otherwise.} \end{cases}$$

In future, we are interested to find $\psi(H_{r,n})$ for all other possible values of r.

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