Acta Math. Univ. Comenianae
Vol. LXXXIX, 2 (2020), pp. 299–307

SOME RESULTS ON f-HARMONIC MAPS AND f-BIHARMONIC SUBMANIFOLDS

E. REMLI and A. M. CHERIF

Abstract. In this paper, we study the existence of f-harmonic maps into Riemannian manifolds admitting a homothetic vector fields. Also we present some properties for the f-biharmonicity of submanifolds of $\mathbb{R}^n$, where f is a smooth positive function on $\mathbb{R}^n$.

1. Preliminaries and Notations

Let (M,g) be a Riemannian manifold. By R^M, we denote the Riemannian curvature tensor of (M,g). Then R^M is defined by

$$R^M(X,Y)Z = \nabla^M_X \nabla^M_Y Z - \nabla^M_Y \nabla^M_X Z - \nabla^M_{[X,Y]} Z, \tag{1}$$

where ∇^M is the Levi-Civita connection with respect to g, and $X, Y, Z \in \Gamma(TM)$. The divergence of $(0,p)$-tensor α on M is defined by

$$(\operatorname{div}^M \alpha)(X_1, \ldots, X_{p-1}) = (\nabla^M_{e_i} \alpha)(e_i, X_1, \ldots, X_{p-1}), \tag{2}$$

where $X_1, \ldots, X_{p-1} \in \Gamma(TM)$ and $\{e_i\}$ is an orthonormal frame. Given a smooth function λ on M, the gradient of λ is defined by

$$g(\operatorname{grad}^M \lambda, X) = X(\lambda), \tag{3}$$

the Hessian of λ is defined by

$$(\operatorname{Hess}^M \lambda)(X,Y) = g(\nabla^M_X \operatorname{grad} \lambda, Y), \tag{4}$$

where $X, Y \in \Gamma(TM)$ (for more details, see, for example, [**9**]).

Let $\varphi\colon (M,g) \to (N,h)$ be a smooth map between two Riemannian manifolds, $\tau(\varphi)$ the tension field of φ (see, [**1, 2, 6**]) and f be a smooth positive function on $M \times N$, the f-tension field of φ is given by

$$\tau_f(\varphi) = f_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi) - e(\varphi)(\operatorname{grad}^N f) \circ \varphi, \tag{5}$$

Received October 11, 2018; revised April 24, 2020.
2010 *Mathematics Subject Classification.* Primary 53C43, 58E20, 53A30.
Key words and phrases. Homothetic vector fields; f-harmonic maps, f-biharmonic submanifolds.
The authors are supported by National Agency Scientific Research of Algeria and Laboratory of Geometry, Analysis, Controle and Applications, Algeria.

where f_φ is a smooth positive function on M, defined by

$$f_\varphi(x) = f(x, \varphi(x)) \qquad \text{for all } x \in M, \tag{6}$$

$e(\varphi)$ is the energy density of φ (see [**1**]), and grad^M (resp. grad^N) denotes the gradient operator with respect to g (resp., h). Then φ is called f-harmonic if the f-tension field vanishes, i.e., $\tau_f(\varphi) = 0$. We define the index form for f-harmonic maps by

$$I_f^\varphi(v, w) = \int_M h(J_f^\varphi(v), w) v^M \tag{7}$$

for all $v, w \in \Gamma(\varphi^{-1}TN)$, where

$$\begin{aligned} J_f^\varphi(v) = & -f_\varphi \operatorname{trace}_g R^N(v, d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi v \\ & + e(\varphi)(\nabla_v^N \operatorname{grad}^N f) \circ \varphi - \mathrm{d}\varphi(\operatorname{grad}^M v(f)) \\ & - v(f)\tau(\varphi) + \langle \nabla^\varphi v, d\varphi \rangle (\operatorname{grad}^N f) \circ \varphi, \end{aligned} \tag{8}$$

R^N is the curvature tensor of (N, h), ∇^N is the Levi-Civita connection of (N, h), ∇^φ denotes the pull-back connection on $\varphi^{-1}TN$, and v^M is the volume form of (M, g) (see [**1**],[**9**]). If $\tau_{2,f}(\varphi) \equiv J_f^\varphi(\tau_f(\varphi))$ is null on M, then φ is called an f-biharmonic map (for more details on the concept of f-harmonic and f-biharmonic maps, see [**4, 5, 10**]).

A vector field ξ on a Riemannian manifold (M, g) is called a homothetic if $L_\xi g = 2kg$ for some constant $k \in \mathbb{R}$, where $L_\xi g$ is the Lie derivative of the metric g with respect to ξ, that is,

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 2kg(X, Y) \qquad \text{for all } X, Y \in \Gamma(TM). \tag{9}$$

The constant k is then called the homothetic constant. If ξ is homothetic and $k \neq 0$, then ξ is called proper homothetic, while $k = 0$ is Killing (see [**1**],[**8**],[**13**]). Note that if a complete Riemannian manifold of dimension ≥ 2 admits a proper homothetic vector field, then the manifold is isometric to the Euclidean space (see [**7**] [**13**]).

2. Homothetic vector fields and f-harmonic maps

Theorem 2.1. *Let (M, g) be a compact orientable Riemannian manifold without boundary, (N, h) a Riemannian manifold admitting a homothetic vector field ξ with a homothetic constant k, and let f be a smooth positive function on $M \times N$ such that $2kf + \xi(f) \neq 0$ at any point. Then, any f-harmonic map φ from (M, g) to (N, h) is constant.*

Proof. We set

$$\omega(X) = h(\xi \circ \varphi, f_\varphi d\varphi(X)), \qquad \text{for all } X \in \Gamma(TM). \tag{10}$$

Let $\{e_i\}$ be a normal orthonormal frame at $x \in M$, we have

$$\begin{aligned} \operatorname{div}^M \omega &= e_i\big[h(\xi \circ \varphi, f_\varphi d\varphi(e_i))\big] \\ &= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, \nabla^\varphi_{e_i} f_\varphi d\varphi(e_i)) \\ &= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, f_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi)). \end{aligned} \tag{11}$$

By equation (11) and the f-harmonicity of φ, we get

$$\begin{aligned} \operatorname{div}^M \omega &= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, e(\varphi)(\operatorname{grad}^N f) \circ \varphi) \\ &= f_\varphi h(\nabla^N_{d\varphi(e_i)} \xi, d\varphi(e_i)) + h(\xi \circ \varphi, e(\varphi)(\operatorname{grad}^N f) \circ \varphi). \end{aligned}$$

Since ξ is a homothetic vector field with a homothetic constant k, we find that

$$\begin{aligned} \operatorname{div}^M \omega &= f_\varphi k h(d\varphi(e_i), d\varphi(e_i)) + e(\varphi) h(\xi \circ \varphi, (\operatorname{grad}^N f) \circ \varphi) \\ &= k f_\varphi |d\varphi|^2 + \frac{1}{2}|d\varphi|^2 h(\xi \circ \varphi, (\operatorname{grad}^N f) \circ \varphi) \\ &= \frac{|d\varphi|^2}{2}\big[2k f_\varphi + h(\xi \circ \varphi, (\operatorname{grad}^N f) \circ \varphi)\big] = \frac{|d\varphi|^2}{2}\big[2k f_\varphi + \xi(f) \circ \varphi\big]. \end{aligned}$$

Theorem 2.1 follows from the last equation and the divergence theorem [**1**] with $2kf + \xi(f) \neq 0$. □

From Theorem 2.1, we get the following results.

Corollary 2.2. [**3**] *Let (M, g) be a compact orientable Riemannian manifold without boundary and (N, h) be a Riemannian manifold admitting a homothetic vector field ξ with a homothetic constant $k \neq 0$. Then, any harmonic map φ from (M, g) to (N, h) is constant.*

If $f(x, y) = f_1(x)$ for all $(x, y) \in M \times N$, where f_1 is a smooth positive function on M, we have the following.

Corollary 2.3. *Let (M, g) be a compact orientable Riemannian manifold without boundary, (N, h) a Riemannian manifold admitting a proper homothetic vector field, and let f_1 be a smooth positive function on M. Then, any f_1-harmonic map φ from (M, g) to (N, h) is constant.*

In the case of non-compact Riemannian manifold, we obtain the following result.

Theorem 2.4. *Let (M, g) be a complete non-compact orientable Riemannian manifold, (N, h) a Riemannian manifold admitting a homothetic vector field ξ with a homothetic constant k, and let f be a smooth positive function on $M \times N$ such that $2(k-\mu)f + \xi(f) \neq 0$ (at any point) for some constant $\mu > 0$. If $\varphi\colon (M, g) \to (N, h)$ is an f-harmonic map satisfying*

$$\int_M f_\varphi |\xi \circ \varphi|^2 v^g < \infty,$$

then φ is constant.

Proof. Let ρ be a smooth function with compact support on M, we set

$$\omega(X) = h(\xi \circ \varphi, \rho^2 f_\varphi d\varphi(X)) \qquad \text{for all } X \in \Gamma(TM),$$

and let $\{e_i\}$ be a normal orthonormal frame at $x \in M$, we have

$$\begin{aligned} \operatorname{div}^M \omega &= e_i[h(\xi \circ \varphi, \rho^2 f_\varphi d\varphi(e_i))] \\ &= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), \rho^2 f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, \nabla^\varphi_{e_i} \rho^2 (f_\varphi d\varphi(e_i))) \\ &= h(\nabla^\varphi_{e_i}(\xi \circ \varphi), \rho^2 f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, e_i(\rho^2) f_\varphi d\varphi(e_i)) \\ &\quad + h(\xi \circ \varphi, \rho^2 \nabla^\varphi_{e_i} f_\varphi d\varphi(e_i)), \end{aligned}$$

so that

$$\begin{aligned} \operatorname{div}^M \omega = h(\nabla^\varphi_{e_i}(\xi \circ \varphi), \rho^2 f_\varphi d\varphi(e_i)) + h(\xi \circ \varphi, 2\rho e_i(\rho) f_\varphi d\varphi(e_i)) \\ + h(\xi \circ \varphi, \rho^2 [f_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi)]). \end{aligned} \tag{12}$$

By equation (12) and f-harmonicity condition of φ, we get

$$\begin{aligned} \operatorname{div}^M \omega = \rho^2 f_\varphi h(\nabla^N_{d\varphi(e_i)} \xi, d\varphi(e_i)) + 2\rho e_i(\rho) f_\varphi h(\xi \circ \varphi, d\varphi(e_i)) \\ + \rho^2 h(\xi \circ \varphi, e(\varphi)(\operatorname{grad}^N f) \circ \varphi). \end{aligned}$$

Since ξ is a homothetic vector field with a homothetic constant k, we find that

$$\operatorname{div}^M \omega = k\rho^2 f_\varphi h(d\varphi(e_i), d\varphi(e_i)) + 2\rho e_i(\rho) f_\varphi h(\xi \circ \varphi, d\varphi(e_i)) + \frac{1}{2}|d\varphi|^2 \rho^2 \xi(f) \circ \varphi,$$

that is,

$$\operatorname{div}^M \omega = k\rho^2 f_\varphi |d\varphi|^2 + 2\rho e_i(\rho) f_\varphi h(\xi \circ \varphi, d\varphi(e_i)) + \frac{1}{2}|d\varphi|^2 \rho^2 \xi(f) \circ \varphi. \tag{13}$$

By the Young's inequality, we have

$$-2\rho e_i(\rho) h(\xi \circ \varphi, d\varphi(e_i)) \leq \varepsilon \rho^2 |d\varphi|^2 + \frac{1}{\varepsilon} e_i(\rho)^2 |\xi \circ \varphi|^2$$

for all $\varepsilon > 0$, multiplying the last inequality by f_φ, we find that

$$-2 f_\varphi \rho e_i(\rho) h(\xi \circ \varphi, d\varphi(e_i)) \leq \varepsilon f_\varphi \rho^2 |d\varphi|^2 + \frac{1}{\varepsilon} f_\varphi e_i(\rho)^2 |\xi \circ \varphi|^2. \tag{14}$$

From (13), (14), we deduce the inequality

$$k\rho^2 f_\varphi |d\varphi|^2 - \operatorname{div}^M \omega + \frac{1}{2}|d\varphi|^2 \rho^2 \xi(f) \circ \varphi \leq \varepsilon f_\varphi \rho^2 |d\varphi|^2 + \frac{1}{\varepsilon} f_\varphi e_i(\rho)^2 |\xi \circ \varphi|^2, \tag{15}$$

we set $\varepsilon = \mu$, by (15), we have

$$(k - \mu)\rho^2 f_\varphi |d\varphi|^2 - \operatorname{div}^M \omega + \frac{1}{2}|d\varphi|^2 \rho^2 \xi(f) \circ \varphi \leq \frac{1}{\mu} f_\varphi e_i(\rho)^2 |\xi \circ \varphi|^2. \tag{16}$$

By the divergence theorem and (16), we have

$$\frac{1}{2} \int_M \rho^2 |d\varphi|^2 \big[2(k - \mu) f_\varphi + \xi(f) \circ \varphi\big] v^g \leq \frac{1}{\mu} \int_M f_\varphi e_i(\rho)^2 |\xi \circ \varphi|^2 v^g. \tag{17}$$

Now, consider the cut-off smooth function $\rho = \rho_R$ such that, $\rho \leq 1$ on M, $\rho = 1$ on the ball $B(\rho, R)$, $\rho = 0$ on $M \setminus B(\rho, 2R)$, and $|\operatorname{grad}^M \rho| \leq \frac{2}{R}$ (see [**12**]). From (17), we get

$$\frac{1}{2}\int_M \rho^2 |d\varphi|^2 \big[2(k-\mu)f_\varphi + \xi(f)\circ\varphi\big] v^g \leq \frac{4}{\mu R^2}\int_M f_\varphi |\xi\circ\varphi|^2 v^g. \tag{18}$$

Since $\int_M f_\varphi |\xi\circ\varphi|^2 v^g < \infty$, when $R \to \infty$, we obtain

$$\int_M |d\varphi|^2 \big[2(k-\mu)f_\varphi + \xi(f)\circ\varphi\big] v^g = 0. \tag{19}$$

Consequently, $|d\varphi| = 0$, that is, φ is constant, because $2(k-\mu)f + \xi(f) \neq 0$ at any point. □

From Theorem 2.4, we deduce

Corollary 2.5. [**3**] *Let (M, g) be a complete non-compact orientable Riemannian manifold and (N, h) be a Riemannian manifold admitting a proper homothetic vector field ξ. If $\varphi\colon (M, g) \to (N, h)$ is a harmonic map satisfying $\int_M |\xi\circ\varphi|^2 v^g < \infty$, then φ is constant.*

3. f-BIHARMONIC MAPS AND SUBMANIFOLDS

Let M be a submanifold of $\mathbb{R}^n$ of dimension m, $\mathbf{i}\colon M \hookrightarrow \mathbb{R}^n$ the canonical inclusion, $f \in C^\infty(\mathbb{R}^n)$ a smooth positive function such that $f\circ\mathbf{i} = 1$, and let $\{e_i\}$ be an orthonormal frame with respect to induced Riemannian metric on M by the inner product $<,>$ on $\mathbb{R}^n$. By ∇ (resp., ∇^M), we denote the Levi-Civita connection of $\mathbb{R}^n$ (resp., of M), by grad (resp., grad^M) the gradient operator in $\mathbb{R}^n$ (resp., in M), by B the second fundamental form of the submanifold M, by A the shape operator, by H the mean curvature vector field of M, and by $\nabla^\perp$ the normal connection of M (see, for example [**1**]). Under the notation above we have the following results.

Theorem 3.1. *The map $\mathbf{i}$ is f-biharmonic if and only if*

$$\begin{aligned}\frac{m}{2}\operatorname{grad}^M |H|^2 - 2A_{\nabla^\perp_{e_i}H}(e_i) - m(\nabla^\perp_{e_i}H)(f)e_i + A_{\nabla^\perp_{e_i}\operatorname{grad} f}(e_i)&\\ +\frac{m-2}{2}\operatorname{grad}^M H(f) - \frac{m-4}{8}\operatorname{grad}^M|\operatorname{grad} f|^2 &= 0,\end{aligned}$$

$$\begin{aligned}-B(e_i, A_H(e_i)) - \Delta^\perp H + \frac{1}{2}B(e_i, A_{\operatorname{grad} f}(e_i)) + \frac{1}{2}\Delta^\perp \operatorname{grad} f&\\ +\frac{m}{2}(\nabla_H \operatorname{grad} f)^\perp - \frac{m}{4}(\nabla_{\operatorname{grad} f}\operatorname{grad} f)^\perp - mH(f)H + \frac{m}{2}|\operatorname{grad} f|^2 H&\\ -m|H|^2\operatorname{grad} f + \frac{m}{2}H(f)\operatorname{grad} f &= 0.\end{aligned}$$

We need the following lemmas to prove Theorem 3.1.

Lemma 3.2 ([**14**])**.** *Let $\Delta^\perp$ the Laplacian in the normal bundle of M, then*

$$\operatorname{trace}\nabla^2 H = -\frac{m}{2}\operatorname{grad}^M(|H|^2) + 2A_{\nabla^\perp_{e_i}H}(e_i) + B(e_i, A_H(e_i)) + \Delta^\perp H.$$

Lemma 3.3. *On taking the trace of $\nabla^2 \operatorname{grad} f$, we obtain*

$$\operatorname{trace}\nabla^2 \operatorname{grad} f = -m(\nabla^\perp_{e_i}H)(f)e_i + 2A_{\nabla^\perp_{e_i}\operatorname{grad} f}(e_i) + B(e_i, A_{\operatorname{grad} f}(e_i)) + \Delta^\perp \operatorname{grad} f.$$

Proof. First, note that $\operatorname{grad} f$ is normal to M because f is constant on M. We suppose that $\nabla^M_{e_i}e_j = 0$ at $x \in M$ for all $i,j = 1,\dots,m$. Then calculating at x

$$\begin{aligned}\nabla_{e_i}\nabla_{e_i}\operatorname{grad} f &= \nabla_{e_i}\big(A_{\operatorname{grad} f}(e_i) + (\nabla_{e_i}\operatorname{grad} f)^\perp\big)\\ &= \nabla^M_{e_i}A_{\operatorname{grad} f}(e_i) + B(e_i, A_{\operatorname{grad} f}(e_i))\\ &\quad + A_{(\nabla_{e_i}\operatorname{grad} f)^\perp}(e_i) + \big(\nabla_{e_i}(\nabla_{e_i}\operatorname{grad} f)^\perp\big)^\perp.\end{aligned} \tag{20}$$

Since $\langle A_{\operatorname{grad} f}(X), Y\rangle = -\langle B(X,Y), \operatorname{grad} f\rangle$, for all $X, Y \in \Gamma(TM)$, we get the following

$$\begin{aligned}\nabla^M_{e_i}A_{\operatorname{grad} f}(e_i) &= \big\langle \nabla^M_{e_i}A_{\operatorname{grad} f}(e_i), e_j\big\rangle\, e_j = e_i\big(\big\langle A_{\operatorname{grad} f}(e_i), e_j\big\rangle\big)\, e_j\\ &= -e_i\big(\big\langle B(e_i,e_j), \operatorname{grad} f\big\rangle\big)\, e_j = -e_i\big(\big\langle \nabla_{e_j}e_i, \operatorname{grad} f\big\rangle\big)\, e_j,\end{aligned}$$

and since $\nabla_X\nabla_Y Z = \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z$, for all $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned}\nabla^M_{e_i}A_{\operatorname{grad} f}(e_i) &= -\big\langle \nabla_{e_i}\nabla_{e_j}e_i, \operatorname{grad} f\big\rangle\, e_j - \big\langle \nabla_{e_j}e_i, \nabla_{e_i}\operatorname{grad} f\big\rangle\, e_j\\ &= -\big\langle \nabla_{e_j}\nabla_{e_i}e_i, \operatorname{grad} f\big\rangle\, e_j - \big\langle B(e_i,e_j), (\nabla_{e_i}\operatorname{grad} f)^\perp\big\rangle\, e_j.\end{aligned}$$

Here, the Riemannian curvature tensor of $\mathbb{R}^n$ is null, so that

$$\begin{aligned}\nabla^M_{e_i}A_{\operatorname{grad} f}(e_i) = &- e_j\big(\big\langle \nabla_{e_i}e_i, \operatorname{grad} f\big\rangle\big)\, e_j + \big\langle \nabla_{e_i}e_i, \nabla_{e_j}\operatorname{grad} f\big\rangle\, e_j\\ &+ \big\langle A_{(\nabla_{e_i}\operatorname{grad} f)^\perp}(e_i), e_j\big\rangle\, e_j\\ = &- m e_j\big(\big\langle H, \operatorname{grad} f\big\rangle\big)\, e_j + m\big\langle H, \nabla_{e_j}\operatorname{grad} f\big\rangle\, e_j\\ &+ A_{(\nabla_{e_i}\operatorname{grad} f)^\perp}(e_i)\\ = &- m\big\langle \nabla_{e_j}H, \operatorname{grad} f\big\rangle\, e_j + A_{(\nabla_{e_i}\operatorname{grad} f)^\perp}(e_i).\end{aligned} \tag{21}$$

By (20) and (21), the lemma is as follows. □

Proof of Theorem 3.1. Note that the f-tension field of $\mathbf{i}$ is given by

$$\tau_f(\mathbf{i}) = \tau(\mathbf{i}) - e(\mathbf{i})(\operatorname{grad} f)\circ\mathbf{i} = mH - \frac{m}{2}\operatorname{grad} f$$

such that $\nabla^M_{e_i}e_j = 0$ at $x \in M$ for all $i,j = 1,\dots,m$. Then calculating at x,

$$\nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i}\tau_f(\mathbf{i}) = m\nabla_{e_i}\nabla_{e_i}H - \frac{m}{2}\nabla_{e_i}\nabla_{e_i}\operatorname{grad} f$$

and by Lemmas 3.2 and 3.3, we have

$$
\begin{aligned}
-\nabla^{\mathbf{i}}_{e_i}\nabla^{\mathbf{i}}_{e_i}\tau_f(\mathbf{i}) = {}& \frac{m^2}{2}\operatorname{grad}^M(|H|^2) - 2mA_{\nabla^{\perp}_{e_i}H}(e_i)\\
& - mB(e_i, A_H(e_i)) - m\Delta^{\perp}H - \frac{m^2}{2}(\nabla^{\perp}_{e_i}H)(f)e_i\\
& + mA_{\nabla^{\perp}_{e_i}\operatorname{grad} f}(e_i) + \frac{m}{2}B(e_i, A_{\operatorname{grad} f}(e_i)) + \frac{m}{2}\Delta^{\perp}\operatorname{grad} f.
\end{aligned}
\tag{22}
$$

In the same way, we have the following formulas

$$
\begin{aligned}
e(\mathbf{i})(\nabla_{\tau_f(\mathbf{i})}\operatorname{grad} f)\circ\mathbf{i} &= \frac{m^2}{2}\nabla_H\operatorname{grad} f - \frac{m^2}{4}\nabla_{\operatorname{grad} f}\operatorname{grad} f\\
&= \frac{m^2}{2}(\nabla_H\operatorname{grad} f)^{\perp} - \frac{m^2}{4}(\nabla_{\operatorname{grad} f}\operatorname{grad} f)^{\perp}\\
&\quad + \frac{m^2}{2}\langle\nabla_{e_i}\operatorname{grad} f, H\rangle e_i - \frac{m^2}{4}\langle\nabla_{e_i}\operatorname{grad} f, \operatorname{grad} f\rangle e_i\\
&= \frac{m^2}{2}(\nabla_H\operatorname{grad} f)^{\perp} - \frac{m^2}{4}(\nabla_{\operatorname{grad} f}\operatorname{grad} f)^{\perp}\\
&\quad + \frac{m^2}{2}\operatorname{grad}^M H(f) - \frac{m^2}{2}(\nabla^{\perp}_{e_i}H)(f)e_i\\
&\quad - \frac{m^2}{8}\operatorname{grad}^M|\operatorname{grad} f|^2,
\end{aligned}
\tag{23}
$$

$$
-d\mathbf{i}(\operatorname{grad}^M\tau_f(\mathbf{i})(f)) = -m\operatorname{grad}^M H(f) + \frac{m}{2}\operatorname{grad}^M|\operatorname{grad} f|^2,
\tag{24}
$$

$$
-\tau_f(\mathbf{i})(f)\tau(\mathbf{i}) = -m^2H(f)H + \frac{m^2}{2}|\operatorname{grad} f|^2H,
\tag{25}
$$

$$
\begin{aligned}
\langle\nabla^{\mathbf{i}}\tau_f(\mathbf{i}), d\mathbf{i}\rangle(\operatorname{grad} f)\circ\mathbf{i} &= \big[m\langle\nabla_{e_i}H, e_i\rangle - \frac{m}{2}\langle\nabla_{e_i}\operatorname{grad} f, e_i\rangle\big]\operatorname{grad} f\\
&= \big[-m\langle H, B(e_i,e_i)\rangle + \frac{m}{2}\langle\operatorname{grad} f, B(e_i,e_i)\rangle\big]\operatorname{grad} f\\
&= \big[-m^2|H|^2 + \frac{m^2}{2}H(f)\big]\operatorname{grad} f.
\end{aligned}
\tag{26}
$$

By definition (8) and equations (22–26), the theorem is as follows. □

Example 3.4. Let $\varepsilon\in\mathbb{R}$, the plane $M = \{(x,y,z)\in\mathbb{R}^3 | z=\varepsilon\}$ is proper f-biharmonic, i.e., the canonical inclusion $\mathbf{i}\colon M\hookrightarrow\mathbb{R}^3$ is an f-biharmonic non-f-harmonic map for $f(x,y,z) = F(z-\varepsilon)$, where F is a smooth positive function such that $F(0)=1$, $F'(0)\neq 0$, and $F''(0)=0$. For example, we consider the function

$$F(t) = \frac{1}{2} + \frac{1}{2}\left[t^2 - \exp(t)\right]^2.$$

Indeed, the function f satisfies the following formulas

$$\operatorname{grad} f = F'(z-\varepsilon)\partial_z, \qquad |\operatorname{grad} f|^2 = F'(0)^2 \quad \text{on } M,$$

$$\nabla_Z\operatorname{grad} f = F''(z-\varepsilon)\langle Z, \partial_z\rangle\partial_z,$$

for all $Z \in \Gamma(T\mathbb{R}^3)$, and for $X \in \Gamma(TM)$ we have

$$\nabla_X \operatorname{grad} f = 0,$$

Note that a unit normal vector field U on M is evidently parallel in $\mathbb{R}^3$ (constant Euclidean coordinates), hence $A_U X = \nabla_X U = 0$ for all tangent vectors X to M. Thus the shape operator is identically zero, so that $B = 0$ and $H = 0$. According to Theorem 3.1, the map $\mathbf{i}$ is f-biharmonic if and only if $F''(0)F'(0) = 0$.

Using the similar technique of Example 3.4, we have

Example 3.5. The sphere $\mathbb{S}^m$ of $\mathbb{R}^{m+1}$ is proper f-biharmonic for

$$f(y) = F\Big(\frac{|y|^2}{2}\Big) \quad \text{for all } y \in \mathbb{R}^n, \text{ where } F(t) = \frac{1}{5}\exp\Big(\frac{5}{2} - 5t\Big) - \frac{2}{5}t + 1.$$

Here, $H = -P$, where P is the position vector field on $\mathbb{R}^{m+1}$,

$$|H| = 1, \quad \nabla^{\perp}_X H = 0, \quad A_H X = -X, \quad B(X,Y) = -\langle X, Y\rangle P,$$

$$\operatorname{grad} f = F'\big(\tfrac{|y|^2}{2}\big)P, \quad H(f) = -F'\big(\tfrac{1}{2}\big), \quad A_{\operatorname{grad} f}X = F'\big(\tfrac{1}{2}\big)X$$

$$\nabla_Z \operatorname{grad} f = \langle Z, P\rangle F''\big(\tfrac{|y|^2}{2}\big)P + F'\big(\tfrac{|y|^2}{2}\big)Z,$$

where $X, Y \in \Gamma(T\mathbb{S}^m)$ and $Z \in \Gamma(T\mathbb{R}^{m+1})$. According to Theorem 3.1, the map $\mathbf{i}$ is f-biharmonic if and only if

$$\frac{1}{2}F''\Big(\frac{1}{2}\Big) + 3F'\Big(\frac{1}{2}\Big) + \frac{5}{4}F'\Big(\frac{1}{2}\Big)^2 + \frac{1}{4}F'\Big(\frac{1}{2}\Big)F''\Big(\frac{1}{2}\Big) + 1 = 0.$$

References

1. Baird P. and Wood J. C., *Harmonic Morphisms between Riemannian Manifolds*, Clarendon Press Oxford, 2003.
2. Caddeo R., Montaldo S. and Oniciuc C., *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math. **12** (2001), 867–876.
3. Cherif A. M., *Some results on harmonic and bi-harmonic maps*, Int. J. Geom. Methods Mod. Phys. **14**(7) (2017).
4. Course N., *f-harmonic maps which map the boundary of the domain to one point in the target*, New York J. Math. **13** (2007), 423–435.
5. Djaa M., Cherif A. M., Zagga K. and Ouakkas S., *On the generalized of harmonic and bi-harmonic maps*, Int. Electron. J. Geom. **5**(1) (2012), 90–100.
6. Eells J. and Sampson J. H., *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.
7. Kobayashi S., *A theorem on the affine transformation group of a Riemannian manifold*, Nagoya Math. J. **9** (1955), 39–41.
8. Kühnel W. and Rademacher H., *Conformal transformations of pseudo-Riemannian manifolds*, Differential Geom. Appl. **7** (1997), 237–250.
9. O'Neil B., *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
10. Ouakkas S., Nasri R. and Djaa M., *On the f-harmonic and f-biharmonic maps*, JP Journal of Geometry and Topology V. **10**(1) (2010), 11–27.
11. Xin Y. L., *Geometry of Harmonic Maps*, Progress in Nonlinear Differential Equations and Their Aplications, Birkhuser Boston, 1996.
12. Yau S. T., *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

13. Yano K. and Nagano T., *The de Rham decomposition, isometries and affine transformations in Riemannian space*, Japan. J. Math. **29** (1959), 173–184.

14. Zegga K., Cherif A. M. and Djaa M., *On the f-biharmonic maps and submanifolds*, Kyungpook Math. J. **55** (2015), 157–168.

E. Remli, University Mustapha Stambouli Mascara, Faculty of Exact Sciences, Department of Mathematics, 29000, Algeria,
e-mail: `ambarka.ramli@univ-mascara.dz`

A. M. Cherif, University Mustapha Stambouli Mascara, Faculty of Exact Sciences, Department of Mathematics, 29000, Algeria,
e-mail: `a.mohammedcherif@univ-mascara.dz`