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KOROVKIN TYPE APPROXIMATION THEOREM ON AN INFINITE INTERVAL IN $A^{\mathcal{I}}$-STATISTICAL SENSE

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Abstract. In this paper, we consider the notion of $A^{\mathcal{I}}$-statistical convergence for real sequences and establish a Korovkin type approximation theorem for positive linear operators on $UC_*[0,\infty)$, the Banach space of all real valued uniform continuous functions on $[0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists finitely for any $f \in UC_*[0,\infty)$. We then construct an example which shows that our new result is stronger than its classical version. In the section 3, we extend the Korovkin type approximation theorem for positive linear operators on $UC_*([0,\infty)\times[0,\infty))$.

1. Introduction and background

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers. Approximation theory has important applications in the theory of polynomial approximation in various areas of functional analysis. For a sequence $\{L_n\}_{n\in\mathbb{N}}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [**12**] first established the necessary and sufficient conditions for the uniform convergence of $\{L_n(f)\}_{n\in\mathbb{N}}$ to a function f by using the test functions $e_1 = 1$, $e_2 = x$, $e_3 = x^2$ [**1**]. The study of the Korovkin type approximation theory has a long history and is a well-established area of research. Erkuş and Duman [**9**] studied a Korovkin type approximation theorem via A-statistical convergence in the space $H_w(I^2)$, where $I^2 = [0,\infty)\times[0,\infty)$, which was extended for double sequences of positive linear operators of two variables in A-statistical sense by Demirci and Dirik in [**5**]. Further it was extended for double sequences of positive linear operators of two variables in $A_2^{\mathcal{I}}$-statistical and $A_2^{\mathcal{I}}$-summability sense by Dutta and Das [**7, 8**].

Our primary interest, in this paper, is to obtain a general Korovkin type approximation theorem for positive linear operators on the space $UC_*(D)$, the Banach space of all real valued uniform continuous functions on $D := [0,\infty)$ with the property that $\lim_{x\to\infty} f(x)$ exists and is finite, endowed with the supremum norm

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$\|f\|_* = \sup_{x\in D}|f(x)|$ for $f \in UC_*(D)$, using the concept of $A^{\mathcal{I}}$-statistical convergence for real sequences and test functions $1, \mathrm{e}^{-x}, \mathrm{e}^{-2x}$. We also construct an example which shows that our new result is stronger than its classical version. In the section 3, we extend the Korovkin type approximation theorem for double sequence of positive linear operators on $UC_*([0,\infty)\times[0,\infty))$.

The concept of convergence of a sequence of real numbers was extended to statistical convergence by Fast [**10**]. Further investigations started in this area after the pioneering works of Šalát [**19**] and Fridy [**11**]. A family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if:

(i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(ii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$.

If $\mathcal{I}$ is a non-trivial proper ideal in Y (i.e., $Y \notin \mathcal{I}, \mathcal{I} \neq \{\emptyset\}$), then the family of sets $F(\mathcal{I}) = \{M \subset Y : \text{there exists } A \in \mathcal{I} : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$. The notion of $\mathcal{I}$-convergence of real sequences was introduced by Kostyrko et al. [**14**] as a generalization of statistical convergence using the notion of ideals. On the other hand, statistical convergence was generalized to A-statistical convergence by Kolk ([**13**]). Later a lot of works have been done on matrix summability and A-statistical convergence (see [**3, 4, 13**]). In particular, in [**20**] and [**21**], the two above mentioned approaches were unified and the very general notion of $A^{\mathcal{I}}$-statistical convergence was introduced and studied.

Recall that a real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$ for all $m, n > N(\varepsilon)$, and denoted by $\lim\limits_{m,n} x_{mn} = L$ ([**17**]). A double sequence is called bounded if there exists a positive number M such that $|x_{mn}| \leq M$ for all $(m,n) \in \mathbb{N}\times\mathbb{N}$. A real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is statistically convergent to L if for every $\varepsilon > 0$, $\lim_{j,k} \frac{|\{m\leq j, n\leq k : |x_{mn}-L|\geq\varepsilon\}|}{jk} = 0$ ([**15, 16**]).

A non-trivial ideal $\mathcal{I}$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}$ for each $i \in \mathbb{N}$. It is evident that a strongly admissible ideal is admissible also. Let $\mathcal{I}_0 = \{A \subset \mathbb{N}\times\mathbb{N} : \text{there is } m(A) \in \mathbb{N} \text{ such that } i, j \geq m(A) \implies (i,j) \notin A\}$. Then $\mathcal{I}_0$ is a non-trivial strongly admissible ideal [**2**]. Let $A = (a_{jkmn})$ be a four dimensional summability matrix. For a given double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$, the A-transform of x, denoted by $Ax := ((Ax)_{jk})$, is given by

$$(Ax)_{jk} = \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn}x_{mn}$$

provided the double series converges in Pringsheim sense for every $(j,k) \in \mathbb{N}^2$. In 1926, Robison [**18**] presented a four dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a convergent double sequence is not necessarily bounded.

Recall that a four dimensional matrix $A = (a_{jkmn})$ is said to be RH-regular if it maps every bounded convergent double sequence into a convergent double sequence with the same limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = (a_{jkmn})$ is RH-regular if and only if:

(i) $\lim_{j,k} a_{jkmn} = 0$ for each $(m,n) \in \mathbb{N}^2$,

(ii) $\lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{jkmn} = 1$,

(iii) $\lim_{j,k} \sum_{m\in\mathbb{N}} |a_{jkmn}| = 0$ for each $n \in \mathbb{N}$,

(iv) $\lim_{j,k} \sum_{n\in\mathbb{N}} |a_{jkmn}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{jkmn}|$ is convergent for each $(j,k) \in \mathbb{N}^2$,

(vi) there exist finite positive integers M_0 and N_0 such that $\sum_{m,n>N_0} |a_{jkmn}| < M_0$ holds for every $(j,k) \in \mathbb{N}^2$.

Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix and let $K \subset \mathbb{N}^2$. Then the A-density of K is given by

$$\delta_A^{(2)}\{K\} = \lim_{j,k} \sum_{(m,n)\in K} a_{jkmn}$$

provided the limit exists. A real double sequence $x = \{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be A-statistically convergent to a number L if for every $\varepsilon > 0$,

$$\delta_A^{(2)}\{(m,n) \in \mathbb{N}^2 : |x_{mn} - L| \geq \varepsilon\} = 0.$$

2. A Korovkin type approximation theorem

Throughout the section, $\mathcal{I}$ denotes the non-trivial admissible ideal in $\mathbb{N}$. If L is a positive linear operator, then $L(f) \geq 0$ for any positive function f, and we denote the value of $L(f)$ at x by $L(f;x)$. Recall the following definition.

Definition 2.1 ([**20, 21**])**.** Let $A = (a_{nk})$ be a non-negative regular matrix. For an ideal $\mathcal{I}$ of $\mathbb{N}$, a sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $A^{\mathcal{I}}$-statistically convergent to L if for any $\varepsilon > 0$ and $\delta > 0$,

$$\Big\{n \in \mathbb{N} : \sum_{k\in K(\varepsilon)} a_{nk} \geq \delta\Big\} \in \mathcal{I},$$

where $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. In this case, we write $A^{\mathcal{I}}\text{-}st\text{-}\lim_n x_n = L$.

We now establish the following Korovkin type approximation theorem for positive linear operators on $UC_*[0,\infty)$, the Banach space of all real valued uniform continuous functions on $[0,\infty)$, with the property that $\lim_{x\to\infty} f(x)$ exists finitely for any $f \in UC_*[0,\infty)$, endowed with the supremum norm $\|f\|_* = \sup_{x\in D} |f(x)|$ for $f \in UC_*(D)$.

Theorem 2.2. *Let $\{L_n\}$ be a sequence of positive linear operators from $UC_*[0,\infty)$ into itself, and let $A = (a_{jn})$ be a non-negative regular summability matrix. Then for all $f \in UC_*[0,\infty)$,*

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(f) - f\|_* = 0$$

if and only if the following statements hold

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n \|L_n(\mathrm{e}^{-kt}) - \mathrm{e}^{-kx}\|_* = 0, \qquad k = 0,1,2.$$

Proof. Since the necessity is clear, it is enough to proof sufficiency. Our objective is to show that given $\varepsilon > 0$, there exist constants C_0, C_1, C_2 (depending on $\varepsilon > 0$) such that

$$\begin{aligned}\|L_n(f) - f\|_* \le \varepsilon + C_2\|L_n(\mathrm{e}^{-2t}) - \mathrm{e}^{-2x}\|_* \\ + C_1\|L_n(\mathrm{e}^{-t}) - \mathrm{e}^{-x}\|_* + C_0\|L_n(1) - 1\|_*.\end{aligned}$$

If this is done, then our hypothesis implies that for $\varepsilon > 0$, $\delta > 0$,

$$\{n \in \mathbb{N} : \sum_{p \in P(\varepsilon)} a_{np} \ge \delta\} \in \mathcal{I},$$

where

$$P(\varepsilon) = \{p \in \mathbb{N} : \|L_p(f) - f\|_* \ge \varepsilon\}.$$

Let $f \in UC_*[0,\infty)$. Then there is a constant M such that $| f(x) | \le M$ for each $x \in [0,\infty)$. Let ε be an arbitrary positive number. By hypothesis, we may find $\delta := \delta(\varepsilon) > 0$ such that for every $t, x \in [0,\infty)$, $| \mathrm{e}^{-t} - \mathrm{e}^{-x} | < \delta$ implies $| f(t) - f(x) | < \varepsilon$. Further note that $| f(t) - f(x) | < 2M$ for all $t, x \in [0,\infty)$.

Also if $| \mathrm{e}^{-t} - \mathrm{e}^{-x} | \ge \delta$, then

$$| f(t) - f(x) | < \frac{2M}{\delta^2}(\mathrm{e}^{-t} - \mathrm{e}^{-x})^2.$$

Then for all $t, x \in [0,\infty)$,

$$| f(t) - f(x) | < \varepsilon + \frac{2M}{\delta^2}(\mathrm{e}^{-t} - \mathrm{e}^{-x})^2.$$

Consequently, for $n \in \mathbb{N}$, using the linearity and the positivity of the operators L_n, we get

$$\begin{aligned}|L_n(f(t);x) - f(x)| &\le L_n(| f(t) - f(x)|; x) + |f(x)|\ |L_n(1;x) - 1| \\ &\le L_n(\varepsilon + \frac{2M}{\delta^2}(\mathrm{e}^{-t} - \mathrm{e}^{-x})^2; x) + |f(x)|\ |L_n(1;x) - 1| \\ &\le \varepsilon + (\varepsilon + M)|L_n(1;x) - 1| + \frac{2M}{\delta^2} L_n((\mathrm{e}^{-t} - \mathrm{e}^{-x})^2; x) \\ &\le \varepsilon + (\varepsilon + M)|L_n(1;x) - 1| + \frac{2M}{\delta^2}| \mathrm{e}^{-2x} |\ |L_n(1;x) - 1| \\ &\quad + \frac{2M}{\delta^2}|L_n(\mathrm{e}^{-2t}; x) - \mathrm{e}^{-2x} | + \frac{4M}{\delta^2}| \mathrm{e}^{-x} |\ |L_n(\mathrm{e}^{-t}; x) - \mathrm{e}^{-x} |,\end{aligned}$$

where $| \mathrm{e}^{-kt} | \le 1$ for all $t \in [0,\infty)$ and $k \in \mathbb{N}$.

Now taking supremum over $x \in [0,\infty)$, we have

$$\begin{aligned}\|L_n(f)-f\|_* &\le \varepsilon + K\{\|L_n(1)-1\|_* \\ &\quad + \|L_n(\mathrm{e}^{-t})-\mathrm{e}^{-x}\|_* + \|L_n(\mathrm{e}^{-2t})-\mathrm{e}^{-2x}\|_*\},\end{aligned} \tag{1}$$

where $K=\max\{\varepsilon+M+\frac{2M}{\delta^2},\frac{2M}{\delta^2},\frac{4M}{\delta^2}\}$. For a given $r>0$, choose $\varepsilon>0$ such that $\varepsilon<r$, let us define the following sets:

$$\begin{aligned} D &= \{n\in\mathbb{N} : \|L_n(f)-f\|_* \ge r\},\\ D_1 &= \Big\{n\in\mathbb{N} : \|L_n(1)-1\|_* \ge \frac{r-\varepsilon}{3K}\Big\},\\ D_2 &= \Big\{n\in\mathbb{N} : \|L_n(\mathrm{e}^{-t})-\mathrm{e}^{-x}\|_* \ge \frac{r-\varepsilon}{3K}\Big\},\\ D_3 &= \Big\{n\in\mathbb{N} : \|L_n(\mathrm{e}^{-2t})-\mathrm{e}^{-2x}\|_* \ge \frac{r-\varepsilon}{3K}\Big\}.\end{aligned}$$

It follows from (1) that $D\subset D_1\cup D_2\cup D_3$. Therefore, for each $n\in\mathbb{N}$, we may write

$$\sum_{p\in D} a_{np} \le \sum_{p\in D_1} a_{np} + \sum_{p\in D_2} a_{np} + \sum_{p\in D_3} a_{np}$$

which implies that for any $\sigma>0$ and $p\in D$,

$$\Big\{n\in\mathbb{N} : \sum_{p\in D} a_{np} \ge \sigma\Big\} \subseteq \bigcup_{i=1}^{3}\Big\{n\in\mathbb{N} : \sum_{p\in D_i} a_{np} \ge \frac{\sigma}{3}\Big\}.$$

From hypotheses $\{n\in\mathbb{N} : \sum_{p\in D_i} a_{np} \ge \frac{\sigma}{3}\}\in\mathcal{I}$ for $i=1,2,3$, we get

$$\bigcup_{i=1}^{3}\Big\{n\in\mathbb{N} : \sum_{p\in D_i} a_{np} \ge \frac{\sigma}{3}\Big\}\in\mathcal{I}$$

Hence

$$\Big\{n\in\mathbb{N} : \sum_{p\in D} a_{np} \ge \sigma\Big\}\in\mathcal{I}.$$

and this completes the proof. □

Remark. We now exhibit a sequence of positive linear operators $\{L_n\}$ s.t. $A^{\mathcal{I}}$-st-$\lim_n \|L_n(f)-f\|_* = 0$ but st_A-$\lim_n \|L_n(f)-f\|_* \ne 0$.

Let $\mathcal{I}$ be a non-trivial admissible ideal of $\mathbb{N}$. Choose an infinite subset $C=\{p_1<p_2<p_3<\dots\}$ from $\mathcal{I}\setminus\mathcal{I}_d$. Let $\{u_k\}_{k\in\mathbb{N}}$ be given by

$$u_k=\begin{cases}1 & \text{if } k \text{ is even},\\ 0 & \text{if } k \text{ is odd}.\end{cases}$$

Let $A=(a_{nk})$ be given by

$$a_{nk}=\begin{cases}1 & \text{if } n=p_i, k=2p_i \text{ for some } i\in\mathbb{N},\\ 1 & \text{if } n\ne p_i \text{ for any } i,\ k=2n+1,\\ 0 & \text{otherwise}.\end{cases}$$

Now for $0<\varepsilon<1$, $K(\varepsilon)=\{k\in\mathbb{N}:|u_k-0|\geq\varepsilon\}$ is the set of all even integers. Observe that

$$\sum_{k\in K(\varepsilon)} a_{nk}=\begin{cases}1 & \text{if } n=p_i \text{ for some } i\in\mathbb{N},\\ 0 & \text{if } n\neq p_i \text{ for any } i\in\mathbb{N}.\end{cases}$$

Thus for any $\delta>0$, $\left\{n\in\mathbb{N}:\sum_{k\in K(\varepsilon)}a_{nk}\geq\delta\right\}=C\in\mathcal{I}$ which shows that $\{u_k\}_{k\in\mathbb{N}}$ is $A^{\mathcal{I}}$-statistically convergent to 0 though x is not A-statistically convergent.

We now consider the following Baskakov operators $B_n\, UC_*[0,\infty)\to UC_*[0,\infty)$ defined by

$$B_nf(x)=\sum_{k=0}^{\infty}\binom{n-1+k}{k}x^k(1+x)^{-n-k}f\left(\frac{k}{n}\right).$$

Thus

$$\begin{aligned}B_n(1,x)&=1,\\ B_n(\mathrm{e}^{-u},x)&=(1+x-x\,\mathrm{e}^{-\frac{1}{n}})^{-n},\\ B_n(\mathrm{e}^{-2u},x)&=(1+x-x\,\mathrm{e}^{-\frac{2}{n}})^{-n},\end{aligned}$$

where $x\in[0,\infty)$.

Let us define $L_n(f,x)=(1+u_n)B_n(f,x)$ for any $f\in UC_*[0,\infty)$. Then

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n\|L_n(f_i)-f_i\|_*=0,\qquad i=0,1,2.$$

From previous theorem,

$$A^{\mathcal{I}}\text{-}st\text{-}\lim_n\|L_n(f)-f\|_*=0.$$

But as $st_A\text{-}\lim_n u_n\neq0$, so $st_A\text{-}\lim_n\|L_n(f)-f\|_*\neq0$.

3. A Korovkin type approximation theorem for a sequence of positive linear operators of two variables

Throughout the section, $\mathcal{I}$ denotes the non-trivial strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$. Recall the following definitions.

Definition 3.1 ([7])**.** A real double sequence $\{x_{m,n}\}_{m,n\in\mathbb{N}}$ is said to be $\mathcal{I}_2$-statistically convergent to L if for each $\varepsilon>0$ and $\delta>0$,

$$\left\{(j,k)\in\mathbb{N}^2:\frac{1}{jk}|\{m\leq j,n\leq k:|x_{mn}-L|\geq\varepsilon\}|\geq\delta\right\}\in\mathcal{I}.$$

Definition 3.2 ([7])**.** Let $A=(a_{jkmn})$ be a non-negative RH-regular summability matrix. Then a real double sequence $\{x_{mn}\}_{m,n\in\mathbb{N}}$ is said to be $A_2^{\mathcal{I}}$-statistically convergent to a number L if for every $\varepsilon>0$ and $\delta>0$,

$$\left\{(j,k)\in\mathbb{N}^2:\sum_{(m,n)\in K_2(\varepsilon)}a_{jkmn}\geq\delta\right\}\in\mathcal{I},$$

where $K_2(\varepsilon) = \{(m,n) \in \mathbb{N}^2 : \ |x_{mn} - L| \geq \varepsilon\}$.
In this case, we write $A_2^{\mathcal{I}}$-st-$\lim\limits_{m,n} x_{mn} = L$.

It should be noted that if we take $A = C(1,1)$, the double Cesáro matrix defined as follows

$$a_{jkmn} = \begin{cases} \frac{1}{jk} & \text{for } m \leq j, n \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

then $A_2^{\mathcal{I}}$-statistical convergence coincides with the notion of $\mathcal{I}_2$-statistical convergence. Again if we replace the matrix A by the identity matrix for four dimensional matrices and $\mathcal{I} = \mathcal{I}_0$, then $A_2^{\mathcal{I}}$-statistical convergence reduces to the Pringsheim convergence for double sequences. For the ideal $\mathcal{I} = \mathcal{I}_0$, $A_2^{\mathcal{I}}$-statistical convergence implies A-statistical convergence for double sequences.

Now we establish the Korovkin type approximation theorem for a double sequence of positive linear operators on $UC_*\left([0,\infty)\times[0,\infty)\right)$, the Banach space of all real valued uniformly continuous functions defined on $D := [0,\infty)\times[0,\infty)$ with the property that $\lim_{(x,y)\to(\infty,\infty)} f(x,y)$ exists finitely for any $f \in UC_*(D)$, endowed with the supremum norm $\|f\|_* = \sup_{(x,y)\in D} |f(x,y)|$ for $f \in UC_*(D)$, in $A_2^{\mathcal{I}}$-statistical sense.

Theorem 3.3. *Let $\{L_{mn}\}_{m,n\in\mathbb{N}}$ be a sequence of positive linear operators on $UC_*\left([0,\infty)\times[0,\infty)\right)$, the Banach space of all real valued uniform continuous functions defined on $[0,\infty)\times[0,\infty)$ with the property that $\lim_{(x,y)\to(\infty,\infty)} f(x,y)$ exists finitely for any $f \in UC_*\left([0,\infty)\times[0,\infty)\right)$, and let $A = (a_{jkmn})$ be a non-negative RH-regular summability matrix. Then for any $f \in UC_*\left([0,\infty)\times[0,\infty)\right)$,*

$$A_2^{\mathcal{I}}\text{-st-}\lim_{m,n} \|L_{mn}(f) - f\|_* = 0$$

is satisfied if the following holds

$$A_2^{\mathcal{I}}\text{-st-}\lim_{m,n} \|L_{mn}(f_i) - f_i\|_* = 0, \qquad i = 0,1,2,3. \tag{2}$$

where $f_0 = 1$, $f_1 = \mathrm{e}^{-x}$, $f_2 = \mathrm{e}^{-y}$, $f_3 = \mathrm{e}^{-2x} + \mathrm{e}^{-2y}$.

Proof. Assume that (2) holds. Let $f \in UC_*\left([0,\infty)\times[0,\infty)\right)$. Our objective is to show that for given $\varepsilon > 0$, there exist constants $C_0,\ C_1,\ C_2,\ C_3$ (depending on $\varepsilon > 0$) such that

$$\begin{aligned}\|L_{mn}f - f\|_* \leq \varepsilon + C_3\|L_{mn}f_3 - f_3\|_* &+ C_2\|L_{mn}f_2 - f_2\|_* \\ &+ C_1\|L_{mn}f_1 - f_1\|_* + C_0\|L_{mn}f_0 - f_0\|_*.\end{aligned}$$

If this is done, then our hypothesis implies that for any $\varepsilon > 0$, $\delta > 0$,

$$\Big\{(j,k) \in \mathbb{N}^2 : \sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} \geq \delta\Big\} \in \mathcal{I},$$

where $K_2(\varepsilon) = \{(m,n) \in \mathbb{N}^2 : \|L_{mn}f - f\|_* \geq \varepsilon\}$.

To this end, start by observing that for each $(u,v) \in \left([0,\infty)\times[0,\infty)\right)$, the function $0 \leq g_{uv} \in UC_*\left([0,\infty)\times[0,\infty)\right)$ defined by

$$g_{uv}(s,t) = (\mathrm{e}^{-s} - \mathrm{e}^{-u})^2 + ((\mathrm{e}^{-t} - (\mathrm{e}^{-v})^2$$

satisfies $g_{uv} = (\mathrm{e}^{-x})^2 + (\mathrm{e}^{-y})^2 - 2\,\mathrm{e}^{-u}\,\mathrm{e}^{-x} - 2\,\mathrm{e}^{-v}\,\mathrm{e}^{-y} + (\mathrm{e}^{-u})^2 + (\mathrm{e}^{-v})^2$. Since each L_{mn} is a positive operator, $L_{mn}g_{uv}$ is a positive function. In particular, we have for each $(u,v) \in ([0,\infty) \times [0,\infty))$,

$$\begin{aligned}
0 &\le L_{mn}g_{uv}(u,v) \\
&= \Big[L_{mn}\big(\left(\mathrm{e}^{-x}\right)^2 + \left(\mathrm{e}^{-y}\right)^2 - 2\,\mathrm{e}^{-u}\,\mathrm{e}^{-x} - 2\,\mathrm{e}^{-v}\,\mathrm{e}^{-y} + \left(\mathrm{e}^{-u}\right)^2 + \left(\mathrm{e}^{-v}\right)^2 ; u,v\big)\Big] \\
&= \Big[L_{mn}\big(\left(\mathrm{e}^{-x}\right)^2 + \left(\mathrm{e}^{-y}\right)^2 ; u,v\big) - \left(\mathrm{e}^{-u}\right)^2 - \left(\mathrm{e}^{-v}\right)^2\Big] \\
&\quad - 2\,\mathrm{e}^{-u}\left[L_{mn}\left(\mathrm{e}^{-x}; u,v\right) - \mathrm{e}^{-u}\right] - 2\,\mathrm{e}^{-v}\left[L_{mn}\left(\mathrm{e}^{-y}; u,v\right) - \mathrm{e}^{-v}\right] \\
&\quad + \Big\{\left(\mathrm{e}^{-u}\right)^2 + \left(\mathrm{e}^{-v}\right)^2\Big\}\left[L_{mn}f_0 - f_0\right] \\
&\le \|L_{mn}f_3 - f_3\|_* + 2\,\mathrm{e}^{-u}\,\|L_{mn}f_1 - f_1\|_* + 2\,\mathrm{e}^{-v}\,\|L_{mn}f_2 - f_2\|_* \\
&\quad + \Big\{\left(\mathrm{e}^{-u}\right)^2 + \left(\mathrm{e}^{-v}\right)^2\Big\}\|L_{mn}f_0 - f_0\|_*.
\end{aligned}$$

Let $f \in UC_*([0,\infty)\times[0,\infty))$. Then there exists a constant M such that $|f(x,y)| \le M$ for each $(x,y) \in ([0,\infty)\times[0,\infty))$. Let $\varepsilon > 0$ be arbitrary. Then by the uniform continuity of f on $([0,\infty)\times[0,\infty))$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $|\,\mathrm{e}^{-x} - \mathrm{e}^{-u}\,| < \delta$ and $|\,\mathrm{e}^{-y} - \mathrm{e}^{-v}\,| < \delta$, then

$$|f(x,y) - f(u,v)| < \varepsilon + \frac{2M}{\delta^2}\Big[\left(\mathrm{e}^{-x} - \mathrm{e}^{-u}\right)^2 + \left(\mathrm{e}^{-y} - \mathrm{e}^{-v}\right)^2\Big]$$

for all $(x,y),(u,v) \in [0,\infty)\times[0,\infty)$. Since each L_{mn} is positive and linear it follows that

$$\begin{aligned}
-\varepsilon L_{mn}f_0 - \frac{2M}{\delta^2}L_{mn}g_{uv} &\le L_{mn}f - f(u,v)L_{mn}f_0 \\
&\le \varepsilon L_{mn}f_0 + \frac{2M}{\delta^2}L_{mn}g_{uv}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|L_{mn}(f;u,v) - f(u,v)L_{mn}(f_0;u,v)| &\le \varepsilon + \varepsilon\left[L_{mn}(f_0;u,v) - f_0(u,v)\right] \\
&\quad + \frac{2M}{\delta^2}L_{mn}g_{uv} \\
&\le \varepsilon + \varepsilon\|L_{mn}f_0 - f_0\|_* + \frac{2M}{\delta^2}L_{mn}g_{uv}.
\end{aligned}$$

In particular, note that

$$\begin{aligned}
|L_{mn}(f;u,v) - f(u,v)| \le\ & |L_{mn}(f;u,v) - f(u,v)L_{mn}(f_0;u,v)| \\
& + |f(u,v)|\,|L_{mn}(f_0;u,v) - f_0(u,v)| \\
\le\ & \varepsilon + (M+\varepsilon)\|L_{mn}f_0 - f_0\|_* + \frac{2M}{\delta^2}L_{mn}g_{uv},
\end{aligned}$$

which implies

$$\begin{aligned}
\|L_{mn}f - f\|_* \le\ & \varepsilon + C_3\|L_{mn}f_3 - f_3\|_* + C_2\|L_{mn}f_2 - f_2\|_* \\
& + C_1\|L_{mn}f_1 - f_1\|_* + C_0\|L_{mn}f_0 - f_0\|_*,
\end{aligned}$$

where there exist A and B such that $C_0 = \left[\frac{2M}{\delta^2}\{(\mathrm{e}^{-A})^2 + (\mathrm{e}^{-B})^2\} + M + \varepsilon\right]$, $C_1 = \frac{4M}{\delta^2}\,\mathrm{e}^{-A}$, $C_2 = \frac{4M}{\delta^2}\,\mathrm{e}^{-B}$ and $C_3 = \frac{2M}{\delta^2}$, i.e.,

$$\|L_{mn}f - f\|_* \le \varepsilon + C\sum_{i=0}^{3}\|L_{mn}f_i - f_i\|_*, \qquad i = 0, 1, 2, 3,$$

where $C = \max\{C_0,\ C_1,\ C_2,\ C_3\}$.

For a given $\gamma > 0$, choose $\varepsilon > 0$ such that $\varepsilon < \gamma$. Now let

$$U = \{(m, n) : \|L_{mn}f - f\|_* \ge \gamma\}$$

and

$$U_i = \Big\{(m, n) : \|L_{mn}f_i - f_i\|_* \ge \frac{\gamma - \varepsilon}{4C}\Big\}, \qquad i = 0, 1, 2, 3.$$

It follows that $U \subset \bigcup_{i=0}^{3} U_i$ and consequently for all $(j, k) \in \mathbb{N}^2$,

$$\sum_{(m,n)\in U} a_{jkmn} \le \sum_{i=0}^{3}\ \sum_{(m,n)\in U_i} a_{jkmn},$$

which implies that for any $\sigma > 0$ and $(m, n) \in U$,

$$\Big\{(j, k) \in \mathbb{N}^2 : \sum_{(m,n)\in U} a_{jkmn} \ge \sigma\Big\} \subseteq \bigcup_{i=0}^{3} \Big\{(j, k) \in \mathbb{N}^2 : \sum_{(m,n)\in U_i} a_{jkmn} \ge \frac{\sigma}{3}\Big\}.$$

Therefore, from hypothesis, $\Big\{(j, k) \in \mathbb{N}^2 : \sum_{(m,n)\in U} a_{jkmn} \ge \sigma\Big\} \in \mathcal{I}$, and this completes the proof of the theorem. □

Remark. We now show that our theorem is stronger than the A-statistical version [**6**] (and so the classical version). Let $\mathcal{I}$ be a non-trivial strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$. Choose an infinite subset $C = \{(p_i, q_i) : i \in \mathbb{N}\}$ from $\mathcal{I} \setminus \mathcal{I}_d$, where $\mathcal{I}_d$ denotes the set of all subsets of $\mathbb{N} \times \mathbb{N}$ with natural density zero, such that $p_i \ne q_i$ for all i, $p_1 < p_2 < \dots$ and $q_1 < q_2 < \dots$

Let $\{u_{mn}\}_{m,n\in\mathbb{N}}$ be given by

$$u_{mn} = \begin{cases} 1 & m, n \text{ are even,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $A = (a_{jkmn})$ be given by

$$a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i,\ k = q_i,\ m = 2p_i,\ n = 2q_i \text{ for some } i \in \mathbb{N}, \\ 1 & \text{if } (j, k) \ne (p_i, q_i) \text{ for any } i, m = 2j + 1,\ n = 2k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Now for $0 < \varepsilon < 1$, $K_2(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |u_{mn} - 0| \ge \varepsilon\} = \{(m, n) : m, n \text{ are even}\}$. Observe that

$$\sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} = \begin{cases} 1 & \text{if } j = p_i,\ k = q_i \text{ for some } i \in \mathbb{N}, \\ 0, & \text{if } (j, k) \ne (p_i, q_i) \text{ for any } i \in \mathbb{N}. \end{cases}$$

Thus for any $\delta > 0$,

$$\Big\{(j,k) \in \mathbb{N}\times\mathbb{N} : \sum_{(m,n)\in K_2(\varepsilon)} a_{jkmn} \geq \delta\Big\} = C \in \mathcal{I}$$

which shows that $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is $A_2^{\mathcal{I}}$-statistically convergent to 0. Evidently this sequence is not A-statistically convergent to 0.

Let $\mathcal{K} = [0,\infty) \times [0,\infty)$. We consider the following Baskakov operators $B_{mn}\, UC_*(\mathcal{K}) \to UC_*(\mathcal{K})$ defined by

$$B_{mn}(f;x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} f\left(\frac{j}{n},\frac{k}{n}\right)\binom{m-1+j}{j}\binom{n-1+k}{k} \times (1+x)^{-m-j}(1+y)^{-n-k}x^j y^k,$$

Now we consider the double sequence $\{L_{mn}\}_{m,n\in\,\mathbb{N}}$ of positive linear operators defined by $L_{mn}(f;x,y) = (1+u_{mn})B_{mn}(f;x,y)$.
Then observe that

$$L_{mn}(f_0;x,y) = (1+u_{mn})f_0(x,y),$$
$$L_{mn}(f_1;x,y) = (1+u_{mn})\Big(1+x-x\,\mathrm{e}^{-\frac{1}{m}}\Big)^{-m},$$
$$L_{mn}(f_2;x,y) = (1+u_{mn})\Big(1+y-y\,\mathrm{e}^{-\frac{1}{n}}\Big)^{-n},$$
$$L_{mn}(f_3;x,y) = (1+u_{mn})\left[\Big(1+x-x\,\mathrm{e}^{-\frac{1}{m}}\Big)^{-m} + \Big(1+y-y\,\mathrm{e}^{-\frac{1}{n}}\Big)^{-n}\right],$$

Then

$$A_2^{\mathcal{I}}\text{-}st\text{-}\lim_{m,n} \|L_{mn}(f_i) - f_i\|_* = 0, \qquad i = 0,1,2,3. \tag{3}$$

Therefore, by previous theorem, for any $f \in UC_*(\mathcal{K})$,

$$A_2^{\mathcal{I}}\text{- st -}\lim_{m,n} \|L_{mn}(f) - f\|_* = 0.$$

But since $\{u_{mn}\}_{m,n\in\mathbb{N}}$ is not usual convergent and not A-statistical convergent, so we can say that the classical version and A-statistical version of the previous theorem do not work for the operator defined above.

4. Conclusion

We conclude this article by pointing out some important features of this study. The result that we have encountered, for a sequence $\{L_n\}_{n\in\mathbb{N}}$ of positive linear operators on $UC_*[0,\infty)$, established the necessary and sufficient conditions for the $A^{\mathcal{I}}$-statistically convergence of $\{L_n(f)\}_{n\in\mathbb{N}}$ to a function f by using the test functions $f_0 = 1,\ f_1 = \mathrm{e}^{-x},\ f_2 = \mathrm{e}^{-2x}$. The same type result (Theorem 3.3) is also established for a sequence of positive linear operators of two variables by using the test functions $f_0 = 1,\ f_1 = \mathrm{e}^{-x},\ f_2 = \mathrm{e}^{-y}, f_2 = \mathrm{e}^{-2x} + \mathrm{e}^{-2y}$. The examples in

Section 2 and Section 3 show that our new results are stronger than its A-statistical version ([**6**, Theorem 2.2]) and consequently stronger than its classical version.

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