Acta Math. Univ. Comenianae
Vol. LXXXIX, 1 (2020), pp. 143–152

ON BANACH AND KANNAN TYPE RESULTS IN CONE $b_v(s)$-METRIC SPACES OVER BANACH ALGEBRA

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Abstract. In this paper, the concept of a cone $b_v(s)$-metric space over Banach algebra is introduced as a generalization of metric spaces, rectangular metric spaces, b-metric spaces, rectangular b-metric spaces, v-generalized metric spaces, cone b-metric spaces over Banach algebra and rectangular cone b-metric spaces over Banach algebra. We also give Banach and, Kannan fixed point results in cone $b_v(s)$-metric spaces over Banach algebra. Some examples are provided as well.

1. Introduction and preliminaries

Bakhtin [**5**] and Czerwik [**8**] introduced b-metric spaces (a generalization of metric spaces) and proved the Banach contraction principle in this framework. In the last period, many authors obtained fixed point results for single-valued or set-valued functions in the setting of b-metric spaces, see [**2, 3, 4, 18**].

In a b-metric space (X, d) with coefficient $s \geq 1$, the modified triangular inequality is

$$d(x,z) \leq s[d(x,y) + d(y,z)]$$

for all $x, y, z \in X$. In 2000, Branciari [**6**] introduced the concept of rectangular metric spaces (in short RMS) by replacing the sum on the right hand-side of the triangular inequality in the definition of a metric space by a three-term expression and proved an analogue of the Banach Contraction Principle on such spaces. Here, $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$. In [**11**], George et al. introduced the concept of a rectangular b-metric space, generalizing the concept of rectangular metric spaces and b-metric spaces. Here, there exists a real number $s \geq 1$ such that $d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)]$ for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$. Note that a rectangular b-metric space is not necessarily Hausdorff. Moreover, the analogue of Banach contraction principle on rectangular b-metric spaces was provided by Mitrović [**21**]. Branciari [**6**] also introduced the following concept.

Received October 21, 2018; revised November 16, 2018.
2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 55M20.
Key words and phrases. Fixed points; b-metric space; rectangular metric space; rectangular b-metric space; rectangular cone b-metric space over Banach algebra.

Definition 1.1 ([**6**])**.** Let X be a set, d be a function from $X \times X$ into $[0, \infty)$, and $v \in \mathbb{N}$. Then (X, d) is said to be a v-generalized metric space if the following hold:
(N1) $d(x, y) = 0$ if and only if $x = y$,
(N2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(N3) $d(x, y) \leq d(x, u_1) + d(u_1, u_2) + \cdots + d(u_v, y)$ for all $x, u_1, u_2, \ldots, u_v, y \in X$ such that $x, u_1, u_2, \ldots, u_v, y$ are all different.

Recently, Suzuki et al. [**26**] gave the analogue of the Banach contraction principle in v-generalized metric spaces. Very recently, Mitrović and Radenović [**22**] introduced the notion of $b_v(s)$-metric spaces and established some fixed point theorems in such spaces.

Definition 1.2 ([**22**])**.** Let X be a non-empty set. Let d be a function from $X \times X$ into $[0, \infty)$ and $v \in \mathbb{N}$. Then (X, d) is said to be a $b_v(s)$-metric space if for all $x, y \in X$ and for all distinct points $u_1, u_2, \ldots, u_v \in X$, each of them different from x and y the following holds:
(B1) $d(x, y) = 0$ if and only if $x = y$,
(B2) $d(x, y) = d(y, x)$,
(B3) there exists a real number $s \geq 1$ such that

$$d(x, y) \leq s[d(x, u_1) + d(u_1, u_2) + \cdots + d(u_v, y)].$$

Note that

- a $b_1(1)$-metric space is an usual metric space,
- a $b_1(s)$-metric space is a b-metric space with coefficient s of Bakhtin and Czerwik,
- a $b_2(1)$-metric space is a rectangular metric space,
- a $b_2(s)$-metric space is a rectangular b-metric space with coefficient s of George et al.,
- a $b_v(1)$-metric space is a v-generalized metric space of Branciari.

On the other hand, by replacing the set of real numbers which forms the domain of distance functions with a complete normed space, Huang and Zang [**15**] obtained cone metric spaces. Further, some generalizations of cone metric spaces, also appeared in literature. Recently, Liu and Xu [**19**] reported the concept of cone metric spaces over Banach algebra and proved the contraction principle in such spaces. The concept defined by Liu and Xu [**19**] was further generalized by Huang and Radenović [**13, 14**] by introduction of cone b-metric spaces over Banach algebra. This concept was again generalized by George et al. [**10**] by introduction of rectangular cone b-metric spaces over Banach algebra.

In the following, we always suppose that $\mathcal{A}$ is a Banach algebra with a unit e, P is a solid cone in $\mathcal{A}$, and $\preceq$ is a partial ordering with respect to P.

Definition 1.3 ([**27**])**.** Let P be a solid cone in a Banach algebra $\mathcal{A}$. A sequence $\{x_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$, there exists a natural number n_0 such that $x_n \ll c$ for all $n \geq n_0$.

Lemma 1.4 ([**13**]). *Let $\{x_n\}$ and $\{y_n\}$ be two c-sequences in a solid cone P. If $a, b \in P$ are two given vectors, then $\{ax_n + by_n\}$ is also a c-sequence.*

Lemma 1.5 ([**24**]). *Let $\mathcal{A}$ be a Banach algebra with a unit e and $k \in \mathcal{A}$, then $\lim_{n\to\infty} \|k^n\|^{\frac{1}{n}}$ exists and the spectral radius $r(k)$ satisfies*

$$r(k) = \lim_{n\to\infty} \|k^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|k^n\|^{\frac{1}{n}}.$$

If $r(k) < |\lambda|$, then $\lambda e - k$ is invertible in $\mathcal{A}$. Moreover,

$$(\lambda e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}},$$

where λ is a constant.

Lemma 1.6 ([**25**]). *Let $P \subset \mathcal{A}$ be a cone.*

(a) *If $a, b \in \mathcal{A}, c \in P$ and $a \preceq b$, then $ca \preceq cb$.*
(b) *If $a, k \in P$ are such that $r(k) < 1$, and $a \preceq ka$, then $a = \theta$.*
(c) *If $k \in P$ and $r(k) < 1$, then for any fixed $m \in \mathbb{N}$ we have $r(k^m) < 1$.*

Lemma 1.7 ([**16**]). *If $u \preceq v$ and $v \ll w$, then $u \ll w$.*

Lemma 1.8 ([**13**]). *Let $\mathcal{A}$ be a Banach algebra with a unit e. Let $k \in \mathcal{A}$ and $r(k) < 1$. Then $\{k^n\}$ is a c-sequence.*

Lemma 1.9 ([**24**]). *Let $\mathcal{A}$ be a Banach algebra with a unit e and $a, b \in \mathcal{A}$. If a commutes with b, then*

$$r(a + b) \leq r(a) + r(b), \ r(ab) \leq r(a)r(b).$$

Lemma 1.10 ([**14**]). *Let $\mathcal{A}$ be a Banach algebra with a unit e and $k \in \mathcal{A}$. If λ is a constant and $r(k) < |\lambda|$, then*

$$r((\lambda e - k)^{-1}) \leq \frac{1}{|\lambda| - r(k)}.$$

The concept defined by Liu and Xu [**19**] was further generalized by Huang and Radenović [**13, 14**] by introduction of cone b-metric spaces over Banach algebra.

Definition 1.11 ([**14**]). Let X be a nonempty set and $s \in P$ with $e \preceq s$. Suppose that the mapping $d \colon X \times X \to \mathcal{A}$ satisfies:

(i) $\theta \prec d(x, y)$ for all $x, y \in X$, with $x \neq y$, and $d(x, y) = \theta$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, y) \preceq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. Then d is called a cone b-metric on X and (X, d) is called a cone b-metric space over Banach algebra $\mathcal{A}$.

In [**12**], the following theorem was proved.

Theorem 1.12 ([**12**]). *Let (X, d) be a b-complete cone b-metric space over Banach algebra with $s \in P$ and $e \preceq s$. Suppose that $T \colon X \to X$ is a mapping such that for all $x, y \in X$,*

$$d(Tx, Ty) \preceq kd(x, y),$$

where $k \in P$ is a generalized Lipschitz constant with $r(k) < 1$. Then T has a unique fixed point in X.

The rectangular cone b-metric space over Banach algebra was introduced by George et al. [**10**] as a generalization of a metric space and many of its generalizations (b-metric space, rectangular metric space, rectangular b-metric space, cone metric space, rectangular cone metric space, cone b-metric space over Banach algebra).

Definition 1.13 ([**10**])**.** Let X be a nonempty set. Let $d\colon X \times X \to \mathcal{A}$ satisfies

(RCbM1) $d(x, y) = \theta$ if and only if $x = y$,
(RCbM2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(RCbM3) there exists $s \in P$ with $e \preceq s$ such that

$$d(x, y) \preceq s[d(x, u) + d(u, v) + d(v, y)]$$

for all $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$.

Then d is called a rectangular cone b-metric on X and (X, d) is called a rectangular cone b-metric space over Banach algebra (in short, RCbMS-BA) with coefficient s. If $s = e$, we say that (X, d) is a rectangular cone metric space over Banach algebra (in short, RCMS-BA).

Theorem 1.14 ([**10**])**.** *Let (X, d) be a complete RCbMS-BA over $\mathcal{A}$ with $s \in P$ such that $e \preceq s$. Given $T\colon X \to X$. If there exist $\lambda \in P$ and $r(\lambda) < 1$ such that*

$$d(Tx, Ty) \preceq \lambda d(x, y)$$

for all $x, y \in X$, then T has a unique fixed point.

Remark 1.15. Note that in Theorem 3.5. in the paper [**10**], there is a printing error. Namely, instead of $\theta \preceq s$, there must be condition $s \geq e$.

In this paper, we initiate the concept of cone $b_v(s)$-metric spaces over Banach algebra. We also provide some related fixed point results for Banach and Kannan contractions types.

2. Main results

The concept of cone $b_v(s)$-metric spaces over Banach algebra is given as follows.

Definition 2.1. Let X be a nonempty set. Let $d : X \times X \to \mathcal{A}$ satisfies

(Cb_v1) $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ if and only if $x = y$,
(Cb_v2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(Cb_v3) there exists $s \in P$ with $e \preceq s$ such that

$$d(x, y) \preceq s[d(x, u_1) + d(u_1, u_2) + \cdots + d(u_v, y)]$$

for all $x, y \in X$ and all distinct elements $u_1, u_2, \ldots, u_v \in X \setminus \{x, y\}$.

Then (X, d) is called a cone $b_v(s)$-metric space over Banach algebra with coefficient s such that $s \in P$ and $e \preceq s$. In the case that $s = e$, we say that (X, d) is a Branciari cone metric space over Banach algebra.

Definitions of convergent sequence, Cauchy sequence and completeness in cone $b_v(s)$-metric space over Banach algebra go on the same line as those one of cone b-metric spaces over Banach algebra given in the papers [**13, 14**] of Huang and Radenović.

Definition 2.2. Let (X, d) be a cone $b_v(s)$-metric space over Banach algebra $\mathcal{A}$, $x \in X$, and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to x whenever for every $c \gg \theta$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$,
(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \gg \theta$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$,
(iii) (X, d) is complete if every Cauchy sequence is convergent.

Our first main result is the analogue of the Banach contraction principle on cone $b_v(s)$-metric spaces over Banach algebra.

Theorem 2.3. *Let (X, d) be a complete cone $b_v(s)$-metric space over Banach algebra $\mathcal{A}$. Let $k \in P$ be such that k commutes with s and $r(k) < 1$. Let $T\colon X \to X$ be such that*

$$d(Tx, Ty) \preceq kd(x, y) \tag{2.1}$$

for all $x, y \in X$, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and $x_n = Tx_{n-1} = T^n x_0$. From (2.1), we have

$$\begin{aligned} d(x_{m+p}, x_{n+p}) &= d(Tx_{m+p-1}, Tx_{n+p-1}) \\ &\preceq kd(x_{m+p-1}, x_{n+p-1}) \preceq k^2 d(x_{m+p-2}, x_{n+p-2}) \\ &\vdots \\ &\preceq k^p d(x_m, x_n). \end{aligned}$$

So,

$$d(x_{m+p}, x_{n+p}) \preceq k^p d(x_m, x_n) \tag{2.2}$$

for all $m, n, p \in \mathbb{N}$.

If $x_n = x_{n+1}$, then x_n is a fixed point of T and the proof holds. Now, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. We will prove that $x_n \neq x_{n+p}$ for all $n \geq 0, p \geq 1$. Namely, if $x_n = x_{n+p}$ for some $n \geq 0$ and $p \geq 1$, we have $Tx_n = Tx_{n+p}$ and $x_{n+1} = x_{n+p+1}$. Then (2.1) implies that

$$d(x_{n+1}, x_n) = d(x_{n+p+1}, x_{n+p}) \preceq k^p d(x_{n+1}, x_n).$$

From Lemma 1.6, we obtain that $d(x_{n+1}, x_n) = \theta$, i.e., $x_{n+1} = x_n$ is a contradiction. Thus, we obtain that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. Let us consider the following three cases:

- $v = 1$,
- $v = 2$,
- $v \geq 3$.

Let $v = 1$. Then (X, d) is a cone $b_1(s)$-metric space over Banach algebra. From condition (C$b_1$3), we have

$$d(x, y) \preceq s[d(x, u_1) + d(u_1, y)] \tag{2.3}$$

for all $x, y, u_1 \in X$. From (2.3), we obtain

$$d(x, y) \preceq s[d(x, u_1) + s(d(u_1, u_2) + d(u_2, y))] \tag{2.4}$$

for all $x, y, u_1, u_2 \in X$. Since $s \in P, e \preceq s$, from Lemma 1.6 (a), we have that $d(x, u_1) \preceq sd(x, u_1)$ and $sd(x, u_1) \preceq s^2 d(x, u_1)$. So,

$$d(x, y) \preceq s^2[d(x, u_1) + d(u_1, u_2) + d(u_2, y)] \tag{2.5}$$

for all $x, y, u_1, u_2 \in X$. Since $r(k) < 1$, there exists $p_1 \in \mathbb{N}$ such that

$$r^2(s) r^{p_1}(k) < 1. \tag{2.6}$$

We note that $r(s)$ exists because of Lemma 1.5. Using the inequality (2.5), we get

$$d(x_m, x_n) \preceq s^2[d(x_m, x_{m+p_1}) + d(x_{m+p_1}, x_{n+p_1}) + d(x_{n+p_1}, x_n)].$$

Using (2.2), from the above inequality, we get

$$d(x_m, x_n) \preceq s^2[k^m d(x_0, x_{p_1}) + k^{p_1} d(x_m, x_n) + k^n d(x_{p_1}, x_0)].$$

Therefore,

$$(e - s^2 k^{p_1}) d(x_m, x_n) \preceq s^2[k^m d(x_0, x_{p_1}) + k^n d(x_{p_1}, x_0)]$$

Since k commutes with s, by Lemma 1.9 and Lemma 1.10, we obtain that

$$r(s^2 k^{p_1}) \leq r^2(s) r^{p_1}(k) < 1$$

and $(e - s^2 k^{p_1})$ is invertible. Therefore,

$$d(x_n, x_m) \preceq (e - s^2 k^{p_1})^{-1}[k^m d(x_0, x_{p_1}) + k^n d(x_{p_1}, x_0)].$$

Now, Lemma 1.4 and Lemma 1.8 implicate that $\{x_n\}$ is a Cauchy sequence.

Let $v = 2$. Then similar to the case $v = 1$, only in the inequality (2.5) instead of s^2, we have s we get that $\{x_n\}$ is a Cauchy sequence.

Let $v \geq 3$. Similar to earlier, since $r(k) < 1$, there exists $p_0 \in \mathbb{N}$ such that

$$r(s) r(k)^{p_0} < 1. \tag{2.7}$$

From (2.2), we obtain

$$\begin{aligned} d(x_m, x_{m+p_0}) &\preceq k^m d(x_0, x_{p_0}), \\ d(x_{m+p_0}, x_{n+p_0}) &\preceq k^{p_0} d(x_m, x_n), \\ d(x_{n+p_0}, x_{n+p_0+1}) &\preceq k^{n+p_0} d(x_0, x_1), \\ &\vdots \\ d(x_{n+p_0+v-3}, x_{n+p_0+v-2}) &\preceq k^{n+p_0+v-3} d(x_0, x_1), \\ d(x_{n+p_0+v-2}, x_n) &\preceq k^n d(x_{p_0+v-2}, x_1). \end{aligned}$$

So, we get

$$\begin{aligned}d(x_m,x_n) &\preceq s[d(x_m,x_{m+p_0})+d(x_{m+p_0},x_{n+p_0}))+d(x_{n+p_0},x_{n+p_0+1})\\&\quad+d(x_{n+p_0+1},x_{n+p_0+2})+\cdots+d(x_{n+p_0+v-3},x_{n+p_0+v-2})\\&\quad+d(x_{n+p_0+v-2},x_n))]\\&\preceq s[k^m d(x_0,x_{p_0})+k^{p_0}d(x_m,x_n))+k^{n+p_0}d(x_0,x_1)\\&\quad+k^{n+p_0+1}d(x_0,x_1)+\cdots+k^{n+p_0+v-3}(d(x_0,x_1)\\&\quad+k^n d(x_{p_0+v-2},x_0))].\end{aligned}$$

It follows that

$$(e-sk^{p_0})d(x_m,x_n)\preceq c_1k^m+c_2k^n, \tag{2.8}$$

where the elements c_1 and c_2 are given with $c_1=sd(x_0,x_{p_0})$ and $c_2=s[d(x_0,x_1)\times(k^{p_0}+\cdots+k^{p_0+v-3})+d(x_{p_0+v-2},x_0)]$. Since k commutes with s, by Lemma 1.9 and Lemma 1.10, we obtain that

$$r(sk^{p_0})\leq r(s)r(k)^{p_0}<1$$

and $(e-sk^{p_0})$ is invertible. Therefore, we conclude that

$$d(x_n,x_m)\preceq(e-sk^{p_0})^{-1}(c_1k^m+c_2k^n).$$

From Lemma 1.4 and Lemma 1.8, $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $x^*\in X$ such that $\{x_n\}$ converges to x^*. Now, we obtain that x^* is the unique fixed point of T. Namely, for any $n\in\mathbb{N}$, we have

$$\begin{aligned}d(x^*,Tx^*) &\preceq s[d(x^*,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+\cdots\\&\quad+d(x_{n+v-2},x_{n+v-1})+d(x_{n+v-1},x_{n+v})+d(x_{n+v},Tx^*)]\\&\preceq s[d(x^*,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+\cdots\\&\quad+d(x_{n+v-2},x_{n+v-1})+d(x_{n+v-1},x_{n+v})+d(Tx_{n+v-1},Tx^*)]\\&\leq s[d(x^*,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+\cdots\\&\quad+d(x_{n+v-2},x_{n+v-1})+d(x_{n+v-1},x_{n+v})+kd(x_{n+v-1},x^*)].\end{aligned}$$

Since $\{d(x^*,x_n)\}$ is a c-sequence and $d(x_n,x_{n+1})\preceq k^n d(x_0,x_1)$, we also get that $\{dx_n,x_{n+1})\}$ is a c-sequence. We deduce $d(x^*,Tx^*)=\theta$, i.e., $Tx^*=x^*$.

For uniqueness, let y^* be another fixed point of T. It follows from (2.1) that

$$d(x^*,y^*)=d(Tx^*,Ty^*)\preceq kd(x^*,y^*).$$

Using Lemma 1.6, we have $d(x^*,y^*)=\theta$, i.e., $x^*=y^*$. □

We present the following example illustrating Theorem 2.3.

Example 2.4. (The case of a non-normal cone) Let $\mathcal{A}=C^1_{\mathbb{R}}[0,1]$ and $\|a\|=\|a\|_\infty+\|a'\|_\infty$ be its norm. Consider the usual pointwise multiplication as its multiplication. Clearly, $\mathcal{A}$ is a Banach algebra with the unit $e(t)=1$ for all $t\in[0,1]$. Put $P=\{a\in\mathcal{A}:a=a(t)\geq 0,t\in[0,1]\}$. Then P is a non-normal cone (well-known). Let $X = A \cup B$, where $A = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ and $B = [1,2]$. Define $d\colon X \times X \to P$ as

$$\begin{cases} d\left(\frac{1}{2}, \frac{1}{3}\right)(t) = d\left(\frac{1}{4}, \frac{1}{5}\right), & (t) = 0.03t, \\ d\left(\frac{1}{2}, \frac{1}{5}\right)(t) = d\left(\frac{1}{4}, \frac{1}{3}\right), & (t) = 0.03t, \\ d\left(\frac{1}{2}, \frac{1}{4}\right)(t) = d\left(\frac{1}{3}, \frac{1}{5}\right), & (t) = 0.6t, \\ d(x,y)(t) = |x-y|^2 t, & \text{otherwise.} \end{cases}$$

Clearly, (X,d) is a cone $b_2(4)$ −metric space over Banach algebra $\mathcal{A}$ with a non-normal cone. But, (X,d) is neither a cone metric space, nor a rectangular cone metric space. Again, consider $T\colon X \to X$ as $Tx = \begin{cases} \frac{1}{4} \text{ if } x \in A, \\ \frac{1}{5} \text{ if } x \in B. \end{cases}$ Then T satisfies the conditions of Theorem 2.3 and has a unique fixed point, which is $x = \frac{1}{4}$.

Remark 2.5. 1. Theorem 2.3 improves [**12**, Theorem 2.1] of Huang, Radenović and Deng (see Theorem 1.12).

2. Theorem 2.3 improves the result of George et al. [**10**] (see Theorem 1.14).

3. Also, Theorem 2.3 improves the results of Mitrović and Radenović [**22**], Suzuki [**26**], and Mitrović [**21**].

Our second main result is a fixed point theorem of Kannan type [**17**].

Theorem 2.6. *Let (X,d) be a complete cone $b_v(s)$-metric space over Banach algebra $\mathcal{A}$ and $T\colon X \to X$ be a mapping satisfying*

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)] \tag{2.9}$$

for all $x, y \in X$, where $k \in P$ such that $r(k) < \frac{1}{2}$. Then T has a unique fixed point x^* and for any $x_0 \in X$ the sequence $\{T^n x_0\}$ converges to x^* if one of the following conditions is satisfied:
(i) $r(sk) < 1$, (ii) $r(s) < 2$.

Proof. Let $x_0 \in X$ be arbitrary and $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\preceq k[d(Tx_n, x_n) + d(Tx_{n-1}, x_{n-1})] \\ &\preceq kd(x_{n+1}, x_n) + kd(x_n, x_{n-1}). \end{aligned}$$

So,

$$(e-k)d(x_{n+1}, x_n) \preceq kd(x_n, x_{n-1}). \tag{2.10}$$

Since $r(k) < \frac{1}{2}$, by Lemma 1.5, we obtain that $(e-k)$ is invertible. Hence,

$$d(x_n, x_{n+1}) \preceq (e-k)^{-1} kd(x_{n-1}, x_n).$$

Let $h = (e-k)^{-1}k$. Since k commutes with $(e-k)^{-1}$, from Lemma 1.5 and Lemma 1.10, we have

$$r(h) = r(k(e-k)^{-1}) \leq r(k) r((e-k)^{-1}) \leq \frac{r(k)}{1 - r(k)} < 1. \tag{2.11}$$

Therefore,

$$d(x_n, x_{n+1}) \preceq h^n d(x_0, x_1). \tag{2.12}$$

Now, for $n, m \in \mathbb{N}$ with $m > n$, from (2.12), we have

$$d(x_n, x_m) \preceq k[d(x_n, x_{n-1}) + d(x_m, x_{m-1})] \preceq k[h^{n-1}d(x_1, x_0) + h^{m-1}d(x_1, x_0)] \\ \preceq k[h^{n-1} + h^{m-1}]d(x_1, x_0).$$

Thus, $\{x_n\}$ is a Cauchy sequence in X using Lemma 1.4 and Lemma 1.8. By completeness of (X, d), there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{2.13}$$

Now, we obtain that x^* is the unique fixed point of T.

<u>*Case 1*</u>: Let $r(sk) < 1$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x^*, Tx^*) &\preceq s[d(x^*, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots \\ &\quad + d(x_{n+v-2}, x_{n+v-1}) + d(x_{n+v-1}, x_{n+v}) + d(x_{n+v}, Tx^*)] \\ &\preceq s[d(x^*, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots \\ &\quad + d(x_{n+v-2}, x_{n+v-1}) + d(x_{n+v-1}, x_{n+v}) + d(Tx_{n+v-1}, Tx^*)] \\ &\leq s[d(x^*, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots \\ &\quad + d(x_{n+v-2}, x_{n+v-1}) + d(x_{n+v-1}, x_{n+v}) \\ &\quad + k(d(x_{n+v-1}, Tx_{n+v-1}) + d(x^*, Tx^*)]. \end{aligned}$$

Since $\{d(x^*, x_n)\}$ is a c-sequence and $d(x_n, x_{n+1}) \preceq k^n d(x_0, x_1)$, we also get that $\{dx_n, x_{n+1})$ is a c-sequence, so we have $(e - sk)d(x^*, Tx^*) \preceq u_n$, where $\{u_n\}$ is a c-sequence. Since $r(sk) < 1$, we obtain $d(x^*, Tx^*) \preceq (e - sk)^{-1}u_n$. So, $Tx^* = x^*$.

<u>*Case 2*</u>: $r(s) < 2$. Clearly, condition (ii) implies condition (i).

For uniqueness, let y^* be another fixed point of T. It follows from (2.9) that $d(x^*, y^*) = d(Tx^*, Ty^*) \preceq k[d(x^*, Tx^*) + d(y^*, Ty^*)] = \theta$. Therefore, we must have $d(x^*, y^*) = \theta$, i.e., $x^* = y^*$. □

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