Acta Math. Univ. Comenianae
Vol. LXXXV, 1 (2016), pp. 21–28

A RESULT ON FUNCTIONAL EQUATIONS IN SEMIPRIME RINGS AND STANDARD OPERATOR ALGEBRAS

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ABSTRACT. Let X be a real or complex Banach space, $\mathcal{L}(X)$ be the algebra of all bounded linear operators of X into itself and $\mathcal{A}(X) \subset \mathcal{L}(X)$ be a standard operator algebra. Suppose there exist linear mappings $D, G\colon \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relations $2D(A^n) = D(A^{n-1})A + A^{n-1}G(A) + G(A)A^{n-1} + AG(A^{n-1})$ and $2G(A^n) = G(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1})$ for all $A \in \mathcal{A}(X)$. Then there exists a fixed $B \in \mathcal{L}(X)$ such that $D(A) = G(A) = [A, B]$ holds for all $A \in \mathcal{A}(X)$.

1. Introduction

Throughout, R represents an associative ring with center $Z(R)$. As usual, we write $[x, y]$ for $xy - yx$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. Let A be an algebra over the real or complex field and let B be a subalgebra of A. A linear mapping $D\colon B \to A$ is called a linear derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in B$. In case we have a ring R, an additive mapping $D\colon R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists such $a \in R$ that $D(x) = [x, a]$ holds for all $x \in R$. Every derivation is a Jordan derivation, but the converse is in general not true. A classical result of Herstein [**6**] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [**1**]. Cusack [**5**] generalized Herstein's result to 2-torsion free semiprime rings (see also [**2**] for an alternative proof). An additive mapping $D\colon R \to R$, where R is an arbitrary ring, is called a Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$.

Let X be a real or complex Banach space and let $\mathcal{L}(X)$ and $\mathcal{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank

Received Juni 26, 2014.
2010 *Mathematics Subject Classification.* Primary 46K15, 16N60, 16W10.
Key words and phrases. Prime ring; standard operator algebra; derivation; Jordan derivation; functional identity.

operators in $\mathcal{L}(X)$, respectively. An algebra $\mathcal{A}(X) \subset \mathcal{L}(X)$ is said to be a standard if $\mathcal{F}(X) \subset \mathcal{A}(X)$. Let us point out that any standard operator algebra is prime, which is a consequence of a Hahn-Banach theorem.

M. Brešar [**3**] proved the following result.

Theorem A. *Let R be a 2-torsion free semiprime ring and $D\colon R \to R$ be a Jordan triple derivation. In this case D is a derivation.*

One can easily prove that any Jordan derivation D on an arbitrary 2-torsion free ring R is Jordan triple derivation, which means that Theorem A generalizes Cusack's generalization of Herstein's theorem we have mentioned above.

Motivated by Theorem A, Vukman [**14**] recently conjectured that in case there exists an additive mapping $D\colon R \to R$, where R is a 2-torsion free semiprime ring satisfying the relation

$$2D(xyx) = D(xy)x + xyD(x) + D(x)yx + xD(yx) \tag{1}$$

for all $x, y \in R$, then D is derivation. Note that in case a ring has an identity element, the proof of Vukman's conjecture is immediate. In this case substituting $y = e$ in the relation (1), where e stands for an identity element, gives that D is a Jordan derivation, and then it follows from Cusack's generalization of Herstein theorem that D is a derivation. The substitution $y = x^{n-2}$ in relations (1) gives

$$2D(x^n) = D(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1}). \tag{2}$$

Recently, Širovnik [**8**] obtained the following result which is related to functional equation (2).

Theorem B. *Let X be a real or complex Banach space and $\mathcal{A}(X)$ be a standard operator algebra on X. Suppose there exists linear mapping $D\colon \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relation*

$$2D(A^n) = D(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1})$$

for all $A \in \mathcal{A}(X)$ and a fixed integer $n \geq 2$. In this case D is of the form $D(A) = [A, B]$ for all $A \in \mathcal{A}(X)$ and a fixed $B \in \mathcal{L}(X)$, such that D is a linear derivation.

In the present paper, we prove the result related to the functional identity given above on operator algebra and semiprime ring. More precisely we prove the following theorem.

Theorem 1. *Let X be a real or complex Banach space, let $\mathcal{A}(X)$ be a standard operator algebra on X. Suppose there exist linear mappings $D, G\colon \mathcal{A}(X) \to \mathcal{L}(X)$ satisfying the relations*

$$2D(A^n) = D(A^{n-1})A + A^{n-1}G(A) + G(A)A^{n-1} + AG(A^{n-1}),$$
$$2G(A^n) = G(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1})$$

for all $A \in \mathcal{A}(X)$, where $n \geq 2$ is a fixed integer. Then there exists a fixed $B \in \mathcal{L}(X)$ such that $D(A) = G(A) = [A, B]$ holds for all $A \in \mathcal{A}(X)$.

Proof. We have

$$2D(A^n) = D(A^{n-1})A + A^{n-1}G(A) + G(A)A^{n-1} + AG(A^{n-1}), \tag{3}$$

$$2G(A^n) = G(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1}) \tag{4}$$

for all $A \in \mathcal{A}(X)$, where $n \geq 2$ is a fixed integer. Subtracting the above two relations, we obtain

$$2T(A^n) = T(A^{n-1})A - A^{n-1}T(A) - T(A)A^{n-1} - AT(A^{n-1}), \tag{5}$$

where $T = D - G$. First we restrict T on $\mathcal{F}(X)$. Let A be from $\mathcal{F}(X)$ and $P \in \mathcal{F}(X)$ be a projection with $AP = PA = A$. Then from the relation (5), we obtain

$$T(P) + PT(P) = 0. \tag{6}$$

Multiplying the relation (6) by P from both side, we obtain

$$PT(P)P = 0. \tag{7}$$

Multiplying the relation (6) by P from left, we get

$$PT(P) = 0. \tag{8}$$

Multiplying the relation (6) by P from right and using (7), we get

$$T(P)P = 0. \tag{9}$$

Putting $(A + P)$ for A in (5), we get

$$\begin{aligned}
2\sum_{i=0}^{n} \tbinom{n}{i} T(A^{n-i}P^i) = &\Big(\sum_{i=0}^{n-1} \tbinom{n-1}{i} T(A^{n-1-i}P^i)\Big)(A+P) \\
&- \Big(\sum_{i=0}^{n-1} \tbinom{n-1}{i} A^{n-1-i}P^i\Big)T(A+P) \\
&- T(A+P)\Big(\sum_{i=0}^{n-1} \tbinom{n-1}{i} A^{n-1-i}P^i\Big) \\
&- (A+P)\Big(\sum_{i=0}^{n-1} \tbinom{n-1}{i} T(A^{n-1-i}P^i)\Big).
\end{aligned}$$

We rearrange the above relation in sense of collecting together terms involving equal number of factors of P, we obtain

$$\sum_{i=1}^{n-1} f_i(A, P) = 0,$$

where $f_i(A,P)$ stands for

$$\begin{aligned} f_i(A,P) = \tbinom{n}{i}T(A^{n-i}P^i) - \tbinom{n-1}{i}\Big(&T(A^{n-1-i}P^i)A - (A^{n-1-i}P^i)T(A) \\ &- T(A)(A^{n-1-i}P^i) - AT(A^{n-1-i}P^i)\Big) - \tbinom{n-1}{i-1}\Big(T(A^{n-i}P^i)P \\ &- (A^{n-i}P^i)T(P) - T(P)(A^{n-i}P^i) - PT(A^{n-i}P^i)\Big). \end{aligned}$$

Replacing A by $A+2P$, $A+3P$,..., $A+(n-1)P$ in turn in the equation (5) and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(A,P)$, $i=1,2,\dots,n-1$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \dots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular

$$\begin{aligned} f_{n-1}(A,P) = 2\tbinom{n}{n-1}T(A) - \tbinom{n-1}{n-1}\Big(&T(P)A - PT(A) - T(A)P - AT(P)\Big) \\ - \tbinom{n-1}{n-2}\Big(&T(A)P - AT(P) - T(P)A - PT(A)\Big) = 0. \end{aligned}$$

Using the relations (8) and (9) in above expression, we obtain

$$2nT(A) = (n-2)T(A)P - nPT(A). \tag{10}$$

Multiplying the relation (10) by P from both sides, we get

$$PT(A)P = 0. \tag{11}$$

Right multiplication of (10) by P gives (using (11))

$$T(A)P = 0. \tag{12}$$

Left multiplication of (10) by P gives (using (11))

$$PT(A) = 0. \tag{13}$$

Using relations (12) and (13) in the relation (10), we obtain

$$T(A) = 0$$

for all $A \in \mathcal{F}(X)$. From the relation (10) one can conclude that D maps $\mathcal{F}(X)$ into itself. We therefore have a linear mapping T, which maps $\mathcal{F}(X)$ into itself and vanishes for all $A \in \mathcal{F}(X)$. It remains to prove that $T(A) = 0$ for all $A \in \mathcal{A}(X)$ as well. For this purpose we introduce $T_1 \colon \mathcal{A}(X) \to \mathcal{L}(X)$ by $T_1(A) = 0$ and consider a mapping $T_0 = T - T_1$. The mapping T_0 is obviously linear satisfying the relation (5) and vanishes on $\mathcal{F}(X)$. It is our aim to prove that T_0 vanishes on $\mathcal{A}(X)$ as well. Let $A \in \mathcal{A}(X)$, P be a one-dimensional projection and let us introduce $S \in \mathcal{A}(X)$ by $S = A + PAP - (AP + PA)$. Obviously, since $S - A \in \mathcal{F}(X)$, we have $T_0(S) = T_0(A)$. Besides, $SP = PS = 0$. Therefore by the relation (5), we have

$$\begin{aligned} &T_0(S^{n-1})S - S^{n-1}T_0(S) - T_0(S)S^{n-1} - ST_0(S^{n-1}) \\ &= 2T_0(S^n) \\ &= 2T_0(S^n + P) = 2T_0((S+P)^n) \\ &= T_0((S+P)^{n-1})(S+P) - (S+P)^{n-1}T_0(S+P) \\ &\quad - T_0(S+P)(S+P)^{n-1} - (S+P)T_0((S+P)^{n-1}) \\ &= T_0(S^{n-1})S + T_0(S^{n-1})P - S^{n-1}T_0(S) - PT_0(S) \\ &\quad - T_0(S)S^{n-1} - T_0(S)P - ST_0(S^{n-1}) - PT_0(S^{n-1}). \end{aligned}$$

From the above relation, it follows that

$$T_0(S^{n-1})P - PT_0(S) - T_0(S)P - PT_0(S^{n-1}) = 0.$$

Since $T_0(S) = T_0(A)$, we can rewrite the above relation as

$$T_0(A^{n-1})P - PT_0(A) - T_0(A)P - PT_0(A^{n-1}) = 0. \tag{14}$$

This implies

$$PT_0(A) + T_0(A)P = T_0(A^{n-1})P - PT_0(A^{n-1}). \tag{15}$$

Putting $2A$ for A in the relation (14), we obtain

$$2^{n-1}T_0(A^{n-1})P - 2PT_0(A) - 2T_0(A)P - 2^{n-1}PT_0(A^{n-1}) = 0. \tag{16}$$

This implies

$$2(PT_0(A) + T_0(A)P) = 2^{n-1}(T_0(A^{n-1})P - PT_0(A^{n-1})). \tag{17}$$

Comparing (15) and (17), and using (15), we obtain

$$(PT_0(A) + T_0(A)P) = (2^{n-1} - 1)(PT_0(A) + T_0(A)P).$$

That implies

$$PT_0(A) + T_0(A)P = 0. \tag{18}$$

Multiplying above relation by P from both sides, we obtain

$$PT_0(A)P = 0.$$

Right multiplication by P in the relation (18) gives $PT_0(A)P + T_0(A)P = 0$, which is reduced by the above relation to

$$T_0(A)P = 0.$$

Since P is an arbitrary one-dimensional projection, it follows from the above relation that $T_0(A) = 0$ for all $A \in \mathcal{A}(X)$, which was our intention to prove. Therefore, we have $D = G$. This ascertainment enables us to combine (3) and (4) into one relation, which is

$$2D(A^n) = D(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1}).$$

Now using Theorem B, we can conclude that $D(A) = G(A) = [A, B]$ for all $A \in \mathcal{A}(X)$ and a fixed $B \in \mathcal{L}(X)$, which means that D and G are linear derivations. $\square$

We proceed with the following purely algebraic conjecture.

Conjecture 2. *Let $n \geq 2$ be a fixed interger and R be a semiprime ring with suitable torsion restrictions. Suppose there exist additive mappings $D, G\colon R \to R$ satisfying the relations*

$$2D(x^n) = D(x^{n-1})x + x^{n-1}G(x) + G(x)x^{n-1} + xG(x^{n-1}),$$
$$2G(x^n) = G(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1}).$$

then D and G are derivations and $D = G$.

Theorem 3. *Let R be $(n-1)!$-torsion free semiprime unital ring and $D, G\colon R \to R$ be two additive mappings satisfying the relations*

$$2D(x^n) = D(x^{n-1})x + x^{n-1}G(x) + G(x)x^{n-1} + xG(x^{n-1}),$$
$$2G(x^n) = G(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1})$$

for all $x \in R$, where $n \geq 2$ is a fixed integer. Then D and G are derivations and $D = G$.

Proof. We have

$$2D(x^n) = D(x^{n-1})x + x^{n-1}G(x) + G(x)x^{n-1} + xG(x^{n-1}), \tag{19}$$
$$2G(x^n) = G(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1}) \tag{20}$$

for all $x \in R$, where $n \geq 2$ is a fixed integer. Subtracting the two relations of equation, we obtain

$$2T(x^n) = T(x^{n-1})x - x^{n-1}T(x) - T(x)x^{n-1} + xT(x^{n-1}), \tag{21}$$

where $T = D - G$. We will denote the identity element of the ring R by e. Putting e for x in the above relation gives

$$T(e) = 0. \tag{22}$$

Let y be any element of center $Z(R)$. Putting $x + y$ for x in the relation (21), we get

$$\begin{aligned} 2\sum_{i=0}^{n} \tbinom{n}{i} T(x^{n-i}y^i) = {} & \Bigg(\sum_{i=0}^{n-1} \tbinom{n-1}{i} T(x^{n-1-i}y^i)\Bigg)(x+y) \\ & - \Bigg(\sum_{i=0}^{n-1} \tbinom{n-1}{i} x^{n-1-i}y^i\Bigg) T(x+y) \\ & - T(x+y)\Bigg(\sum_{i=0}^{n-1} \tbinom{n-1}{i} x^{n-1-i}y^i\Bigg) \\ & - (x+y)\Bigg(\sum_{i=0}^{n-1} \tbinom{n-1}{i} T(x^{n-1-i}y^i)\Bigg). \end{aligned}$$

Using (21) and rearranging the above relation in sense of collecting terms involving equal number of factors of y together, we obtain

$$\sum_{i=1}^{n-1} f_i(x,y) = 0,$$

where $f_i(x,y)$ stands for the expression of terms involving i factors of y, that is

$$\begin{aligned} f_i(x,y) = \tbinom{n}{i} T(x^{n-i}y^i) - \tbinom{n-1}{i}\Big(T(x^{n-1-i}y^i)x - (x^{n-1-i}y^i)T(x) \\ - T(x)(x^{n-1-i}y^i) - xT(x^{n-1-i}y^i)\Big) - \tbinom{n-1}{i-1}\Big(T(x^{n-i}y^i)y \\ - (x^{n-i}y^i)T(y) - T(y)(x^{n-i}y^i) - yT(x^{n-i}y^i)\Big). \end{aligned}$$

Replacing x by $x+2y$, $x+3y$,..., $x+(n-1)y$ in turn in the relation (21) and expressing the resulting system of $n-1$ homogeneous equations of the variables $f_i(x,y)$, $i = 1, 2, \dots, n-1$, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & (n-1)^2 & \dots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of this matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular, putting the identity element e for y, we obtain

$$\begin{aligned} f_{n-1}(x,e) = 2\tbinom{n}{n-1}T(x) - \tbinom{n-1}{n-1}\Big(T(e)x + eT(x) + T(x)e + xT(e)\Big) \\ - \tbinom{n-1}{n-2}\Big(T(x)e + xT(e) + T(e)x + eT(x)\Big) = 0. \end{aligned}$$

Using (22) in the above relation, it reduces to

$$2nT(x) = 2T(x) - 2(n-1)T(x).$$

That implies $4(n-1)T(x) = 0$. Using torsion restriction, we get $T(x) = 0$ for all $x \in R$, which was our intention to prove. Therefore, we have $D = G$. This ascertainment enables us to combine (19) and (20) into one relation, which is

$$2D(x^n) = D(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1}).$$

Using ([**8**], Theorem 2) we conclude that D and G are derivations and $D = G$. That completes the proof of the theorem. $\square$

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