A RESULT ON FUNCTIONAL EQUATIONS IN SEMIPRIME RINGS AND STANDARD OPERATOR ALGEBRAS

N. U. REHMAN and T. BANO

Abstract. Let \( X \) be a real or complex Banach space, \( \mathcal{L}(X) \) be the algebra of all bounded linear operators of \( X \) into itself and \( \mathcal{A}(X) \subset \mathcal{L}(X) \) be a standard operator algebra. Suppose there exist linear mappings \( D, G: \mathcal{A}(X) \rightarrow \mathcal{L}(X) \) satisfying the relations
\[
2D(A^n) = D(A^{n-1})A + A^{n-1}G(A) + G(A)A^{n-1} + AG(A^{n-1})
\]
and
\[
2G(A^n) = G(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1})
\]
forsome\( A \in \mathcal{A}(X) \). Then there exists a fixed \( B \in \mathcal{L}(X) \) such that \( D(A) = G(A) = [A, B] \) holds for all \( A \in \mathcal{A}(X) \).

1. Introduction

Throughout, \( R \) represents an associative ring with center \( Z(R) \). As usual, we write \([x, y]\) for \( xy - yx \). Given an integer \( n \geq 2 \), a ring \( R \) is said to be \( n \)-torsion free if for
\( x \in R, nx = 0 \) implies \( x = 0 \). Recall that a ring \( R \) is prime if for \( a, b \in R, aRb = (0) \) implies \( a = 0 \) or \( b = 0 \), and is semiprime in case \( aRa = (0) \) implies \( a = 0 \). Let \( A \) be an algebra over the real or complex field and let \( B \) be a subalgebra of \( A \). A linear mapping \( D: B \rightarrow A \) is called a linear derivation in case
\[
D(xy) = D(x)y + xD(y)
\]
holds for all pairs \( x, y \in B \). In case we have a ring \( R \), an additive mapping \( D: R \rightarrow R \) is called a derivation if
\[
D(xy) = D(x)y + xD(y)
\]
holds for all pairs \( x, y \in R \) and is called a Jordan derivation in case
\[
D(x^2) = D(x)x +xD(x)
\]
is fulfilled for all \( x \in R \). A derivation \( D \) is inner in case there exists such \( a \in R \) that \( D(x) = [x, a] \) holds for all \( x \in R \). Every derivation is a Jordan derivation, but the converse is in general not true. A classical result of Herstein [6] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein’s result can be found in [1]. Cusack [5] generalized Herstein’s result to 2-torsion free semiprime rings (see also [2] for an alternative proof). An additive mapping \( D: R \rightarrow R \), where \( R \) is an arbitrary ring, is called a Jordan triple derivation in case
\[
D(xy) = D(x)yx + xD(y)x + xyD(x)
\]
holds for all pairs \( x, y \in R \).

Let \( X \) be a real or complex Banach space and let \( \mathcal{L}(X) \) and \( \mathcal{F}(X) \) denote the algebra of all bounded linear operators on \( X \) and the ideal of all finite rank operators on \( X \).
operators in \( L(X) \), respectively. An algebra \( \mathcal{A}(X) \subset L(X) \) is said to be a standard if \( \mathcal{F}(X) \subset \mathcal{A}(X) \). Let us point out that any standard operator algebra is prime, which is a consequence of a Hahn-Banach theorem.

M. Brešar [3] proved the following result.

**Theorem A.** Let \( R \) be a 2-torsion free semiprime ring and \( D: R \to R \) be a Jordan triple derivation. In this case \( D \) is a derivation.

One can easily prove that any Jordan derivation \( D \) on an arbitrary 2-torsion free ring \( R \) is Jordan triple derivation, which means that Theorem A generalizes Cusack’s generalization of Herstein’s theorem we have mentioned above.

Motivated by Theorem A, Vukman [14] recently conjectured that in case there exists an additive mapping \( D: R \to R \), where \( R \) is a 2-torsion free semiprime ring satisfying the relation

\[
2D(xy) = D(xy)x + xyD(x) + D(x)yx + xD(yx)
\]

for all \( x, y \in R \), then \( D \) is derivation. Note that in case a ring has an identity element, the proof of Vukman’s conjecture is immediate. In this case substituting \( y = e \) in the relation (1), where \( e \) stands for an identity element, gives that \( D \) is a Jordan derivation, and then it follows from Cusack’s generalization of Herstein theorem that \( D \) is a derivation. The substitution \( y = x^{n-2} \) in relations (1) gives

\[
2D(x^n) = D(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1}).
\]

Recently, Širovnik [8] obtained the following result which is related to functional equation (2).

**Theorem B.** Let \( X \) be a real or complex Banach space and \( \mathcal{A}(X) \) be a standard operator algebra on \( X \). Suppose there exists linear mapping \( D: \mathcal{A}(X) \to L(X) \) satisfying the relation

\[
2D(A^n) = D(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1})
\]

for all \( A \in \mathcal{A}(X) \) and a fixed integer \( n \geq 2 \). In this case \( D \) is of the form \( D(A) = [A, B] \) for all \( A \in \mathcal{A}(X) \) and a fixed \( B \in L(X) \), such that \( D \) is a linear derivation.

In the present paper, we prove the result related to the functional identity given above on operator algebra and semiprime ring. More precisely we prove the following theorem.

**Theorem 1.** Let \( X \) be a real or complex Banach space, let \( \mathcal{A}(X) \) be a standard operator algebra on \( X \). Suppose there exist linear mappings \( D, G: \mathcal{A}(X) \to L(X) \) satisfying the relations

\[
2D(A^n) = D(A^{n-1})A + A^{n-1}G(A) + G(A)A^{n-1} + AD(A^{n-1}),
\]

\[
2G(A^n) = G(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1})
\]

for all \( A \in \mathcal{A}(X) \), where \( n \geq 2 \) is a fixed integer. Then there exists a fixed \( B \in L(X) \) such that \( D(A) = G(A) = [A, B] \) holds for all \( A \in \mathcal{A}(X) \).
Proof. We have

\begin{align*}
(3) & \quad 2D(A^n) = D(A^{n-1})A + A^{n-1}G(A) + G(A)A^{n-1} + A(G(A^{n-1})) , \\
(4) & \quad 2G(A^n) = G(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1})
\end{align*}

for all \(A \in \mathcal{A}(X)\), where \(n \geq 2\) is a fixed integer. Subtracting the above two relations, we obtain

\begin{align*}
(5) & \quad 2T(A^n) = T(A^{n-1})A - A^{n-1}T(A) - T(A)A^{n-1} - AT(A^{n-1}),
\end{align*}

where \(T = D - G\). First we restrict \(T\) on \(\mathcal{F}(X)\). Let \(A\) be from \(\mathcal{F}(X)\) and \(P \in \mathcal{F}(X)\) be a projection with \(AP = PA = A\). Then from the relation (5), we obtain

\begin{align*}
(6) & \quad T(P) + PT(P) = 0.
\end{align*}

Multiplying the relation (6) by \(P\) from both side, we obtain

\begin{align*}
(7) & \quad PT(P)P = 0.
\end{align*}

Multiplying the relation (6) by \(P\) from left, we get

\begin{align*}
(8) & \quad PT(P) = 0.
\end{align*}

Multiplying the relation (6) by \(P\) from right and using (7), we get

\begin{align*}
(9) & \quad T(P)P = 0.
\end{align*}

Putting \((A + P)\) for \(A\) in (5), we get

\begin{align*}
2 \sum_{i=0}^{n} {^n \choose i} T(A^{n-i}P^i) & = \left( \sum_{i=0}^{n-1} {n-1 \choose i} T(A^{n-1-i}P^i) \right) (A + P) \\
& - \left( \sum_{i=0}^{n-1} {n-1 \choose i} A^{n-1-i}P^i \right) T(A + P) \\
& - T(A + P) \left( \sum_{i=0}^{n-1} {n-1 \choose i} A^{n-1-i}P^i \right) \\
& - (A + P) \left( \sum_{i=0}^{n-1} {n-1 \choose i} T(A^{n-1-i}P^i) \right) .
\end{align*}

We rearrange the above relation in sense of collecting together terms involving equal number of factors of \(P\), we obtain

\begin{align*}
\sum_{i=1}^{n-1} f_i(A, P) = 0,
\end{align*}
where \( f_i(A, P) \) stands for
\[
f_i(A, P) = \binom{n}{i} T(A^{n-i} P^i) - \binom{n-1}{i} \left( T(A^{n-1-i} P^i) A - (A^{n-1-i} P^i) T(A) \right) - T(A)(A^{n-1-i} P^i) - AT(A^{n-1-i} P^i) - \binom{n-1}{i} \left( T(A^{n-i} P^i) P \right) - (A^{n-1} P^i) T(P) - T(P)(A^{n-1} P^i) - PT(A^{n-1} P^i).
\]
Replacing \( A \) by \( A + 2P, A + 3P, \ldots, A + (n-1)P \) in the above expression, we obtain
\[
P \frac{\partial f}{\partial A} |_{A=0} = (n-i) \binom{n}{i} T(A^{n-i} P^i) - \binom{n-1}{i} \left( T(A^{n-1-i} P^i) A - (A^{n-1-i} P^i) T(A) \right) - T(A)(A^{n-1-i} P^i) - AT(A^{n-1-i} P^i) - \binom{n-1}{i} \left( T(A^{n-i} P^i) P \right) - (A^{n-1} P^i) T(P) - T(P)(A^{n-1} P^i) - PT(A^{n-1} P^i).
\]
Expressing the resulting system of \( n - 1 \) homogeneous equations of variables \( f_i(A, P), \ i = 1, 2, \ldots, n - 1 \), we see that the coefficient matrix of the system is a van der Monde matrix
\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
2 & 2^2 & \cdots & 2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
n-1 & (n-1)^2 & \cdots & (n-1)^{n-1}
\end{pmatrix}
\]
Since the determinant of matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular
\[
f_{n-1}(A, P) = 2 \binom{n}{n-1} T(A) - \binom{n-1}{n-1} \left( T(P) A - PT(A) - T(A) P - AT(P) \right) - \binom{n-1}{n-2} \left( T(A) P - AT(P) - T(P) A - PT(A) \right) = 0.
\]
Using the relations (8) and (9) in above expression, we obtain
\[
2nT(A) = (n-2)T(A) P - nPT(A).
\]
Multiplying the relation (10) by \( P \) from both sides, we get
\[
PT(A)P = 0. \tag{11}
\]
Right multiplication of (10) by \( P \) gives (using (11))
\[
T(A)P = 0. \tag{12}
\]
Left multiplication of (10) by \( P \) gives (using (11))
\[
PT(A) = 0. \tag{13}
\]
Using relations (12) and (13) in the relation (10), we obtain
\[
T(A) = 0
\]
for all \( A \in \mathcal{F}(X) \). From the relation (10) one can conclude that \( D \) maps \( \mathcal{F}(X) \) into itself. We therefore have a linear mapping \( T \), which maps \( \mathcal{F}(X) \) into itself and vanishes for all \( A \in \mathcal{F}(X) \). It remains to prove that \( T(A) = 0 \) for all \( A \in \mathcal{A}(X) \) as well. For this purpose we introduce \( T_1 : \mathcal{A}(X) \to \mathcal{L}(X) \) by \( T_1(A) = 0 \) and consider a mapping \( T_0 = T - T_1 \). The mapping \( T_0 \) is obviously linear satisfying the relation (5) and vanishes on \( \mathcal{F}(X) \). It is our aim to prove that \( T_0 \) vanishes on \( \mathcal{A}(X) \) as well. Let \( A \in \mathcal{A}(X) \), \( P \) be a one-dimensional projection and let us introduce \( S \in \mathcal{A}(X) \) by \( S = A + PAP - (AP + PA) \). Obviously, since \( S - A \in \mathcal{F}(X) \), we
have \( T_0(S) = T_0(A) \). Besides, \( SP = PS = 0 \). Therefore by the relation (5), we have
\[
T_0(S^{n-1})S - S^{n-1}T_0(S) - T_0(S)S^{n-1} - ST_0(S^{n-1}) = 2T_0(S^n) = 2T_0(S^n + P) = 2T_0((S + P)^n) = T_0((S + P)^{n-1})(S + P) - (S + P)^{n-1}T_0(S + P) - T_0(S + P)(S + P)^{n-1} - (S + P)T_0((S + P)^{n-1}) = T_0(S^{n-1})S + T_0(S^{n-1})P - S^{n-1}T_0(S) - PT_0(S) - T_0(S)S^{n-1} - T_0(S)P - ST_0(S^{n-1}) - PT_0(S^{n-1}).
\]
From the above relation, it follows that
\[
T_0(S^{n-1})P - PT_0(S) - T_0(S)P - PT_0(S^{n-1}) = 0.
\]
Since \( T_0(S) = T_0(A) \), we can rewrite the above relation as
\[
T_0(A^{n-1})P - PT_0(A) - T_0(A)P - PT_0(A^{n-1}) = 0.
\]
This implies
\[
PT_0(A) + T_0(A)P = T_0(A^{n-1})P - PT_0(A^{n-1}).
\]
Putting \( 2A \) for \( A \) in the relation (14), we obtain
\[
2^{n-1}T_0(A^{n-1})P - 2PT_0(A) - 2T_0(A)P - 2^{n-1}PT_0(A^{n-1}) = 0.
\]
This implies
\[
2(PT_0(A) + T_0(A)P) = 2^{n-1}(T_0(A^{n-1})P - PT_0(A^{n-1})).
\]
Comparing (15) and (17), and using (15), we obtain
\[
(PT_0(A) + T_0(A)P) = (2^{n-1} - 1)(PT_0(A) + T_0(A)P).
\]
That implies
\[
PT_0(A) + T_0(A)P = 0.
\]
Multiplying above relation by \( P \) from both sides, we obtain
\[
PT_0(A)P = 0.
\]
Right multiplication by \( P \) in the relation (18) gives \( PT_0(A)P + T_0(A)P = 0 \), which is reduced by the above relation to
\[
T_0(A)P = 0.
\]
Since \( P \) is an arbitrary one-dimensional projection, it follows from the above relation that \( T_0(A) = 0 \) for all \( A \in \mathcal{A}(X) \), which was our intention to prove. Therefore, we have \( D = G \). This ascertainment enables us to combine (3) and (4) into one relation, which is
\[
2D(A^n) = D(A^{n-1})A + A^{n-1}D(A) + D(A)A^{n-1} + AD(A^{n-1}).
\]
Now using Theorem B, we can conclude that \( D(A) = G(A) = [A, B] \) for all \( A \in \mathcal{A}(X) \) and a fixed \( B \in \mathcal{L}(X) \), which means that \( D \) and \( G \) are linear derivations. \( \square \)
We proceed with the following purely algebraic conjecture.

**Conjecture 2.** Let \( n \geq 2 \) be a fixed integer and \( R \) be a semiprime ring with suitable torsion restrictions. Suppose there exist additive mappings \( D,G: R \to R \) satisfying the relations
\[
2D(x^n) = D(x^{n-1})x + x^{n-1}G(x) + G(x)x^{n-1} + xG(x^{n-1}), \\
2G(x^n) = G(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1}).
\]
then \( D \) and \( G \) are derivations and \( D = G \).

**Theorem 3.** Let \( R \) be \((n-1)!\)-torsion free semiprime unital ring and \( D,G: R \to R \) be two additive mappings satisfying the relations
\[
2D(x^n) = D(x^{n-1})x + x^{n-1}G(x) + G(x)x^{n-1} + xG(x^{n-1}), \\
2G(x^n) = G(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1})
\]
for all \( x \in R \), where \( n \geq 2 \) is a fixed integer. Then \( D \) and \( G \) are derivations and \( D = G \).

**Proof.** We have
\[
2D(x^n) = D(x^{n-1})x + x^{n-1}G(x) + G(x)x^{n-1} + xG(x^{n-1}), \\
2G(x^n) = G(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1})
\]
for all \( x \in R \), where \( n \geq 2 \) is a fixed integer. Subtracting the two relations of equation, we obtain
\[
2T(x^n) = T(x^{n-1})x - x^{n-1}T(x) - T(x)x^{n-1} + xT(x^{n-1}),
\]
where \( T = D - G \). We will denote the identity element of the ring \( R \) by \( e \). Putting \( e \) for \( x \) in the above relation gives
\[
T(e) = 0.
\]
Let \( y \) be any element of center \( Z(R) \). Putting \( x+y \) for \( x \) in the relation (21), we get
\[
2 \sum_{i=0}^{n} \binom{n}{i} T(x^{n-i}y^i) = \left( \sum_{i=0}^{n-1} \binom{n-1}{i} T(x^{n-1-i}y^i) \right)(x+y) \\
- \left( \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}y^i \right) T(x+y) \\
- T(x+y) \left( \sum_{i=0}^{n-1} \binom{n-1}{i} x^{n-1-i}y^i \right) \\
- (x+y) \left( \sum_{i=0}^{n-1} \binom{n-1}{i} T(x^{n-1-i}y^i) \right).
\]
Using (21) and rearranging the above relation in sense of collecting terms involving equal number of factors of \( y \) together, we obtain

\[ \sum_{i=1}^{n-1} f_i(x, y) = 0, \]

where \( f_i(x, y) \) stands for the expression of terms involving \( i \) factors of \( y \), that is

\[
\begin{align*}
f_i(x, y) &= \binom{n}{i} T(x^{n-i}y^i) - \binom{n-1}{i} \left( T(x^{n-1-i}y^i)x - (x^{n-1-i}y^i)T(x) \right) \\
&\quad - T(x)(x^{n-1-i}y^i) - xT(x^{n-1-i}y^i) - \binom{n-1}{i-1} \left( T(x^{n-i}y^i) \right) \\
&\quad - (x^{n-i}y^i)T(y) - T(y)(x^{n-i}y^i) - yT(x^{n-i}y^i).
\end{align*}
\]

Replacing \( x \) by \( x + 2y, x + 3y, \ldots, x + (n - 1)y \) in turn in the relation (21) and expressing the resulting system of \( n - 1 \) homogeneous equations of the variables \( f_i(x, y), i = 1, 2, \ldots, n - 1 \), we see that the coefficient matrix of the system is a Vandermonde matrix

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
2 & 2^2 & \cdots & 2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
(n-1) & (n-1)^2 & \cdots & (n-1)^{n-1}
\end{bmatrix}
\]

Since the determinant of this matrix is different from zero, it follows immediately that the system has only a trivial solution. In particular, putting the identity element \( e \) for \( y \), we obtain

\[
f_{n-1}(x, e) = 2 \binom{n}{n-1} T(x) - \binom{n-1}{n-2} \left( T(e)x + eT(x) + T(x)e + xT(e) \right) \\
- \binom{n-1}{n-1} \left( T(x)e + xT(e) + T(e)x + eT(x) \right) = 0.
\]

Using (22) in the above relation, it reduces to

\[
2nT(x) = 2T(x) - 2(n - 1)T(x).
\]

That implies \( 4(n - 1)T(x) = 0 \). Using torsion restriction, we get \( T(x) = 0 \) for all \( x \in \mathcal{R} \), which was our intention to prove. Therefore, we have \( D = G \). This ascertainment enables us to combine (19) and (20) into one relation, which is

\[
2D(x^n) = D(x^{n-1})x + x^{n-1}D(x) + D(x)x^{n-1} + xD(x^{n-1}).
\]

Using ([8], Theorem 2) we conclude that \( D \) and \( G \) are derivations and \( D = G \).

That completes the proof of the theorem. \( \square \)

**References**


N. U. Rehman, Department of Mathematics, Aligarh Muslim University, Aligarh-202002 India, e-mail: rehman100@gmail.com

T. Bano, Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India, e-mail: tarannumdlw@gmail.com