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ESTIMATING THE SHORT RATE FROM TERM STRUCTURES IN THE CHAN-KAROLYI-LONGSTAFF-SANDERS MODEL

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Abstract. Short rate models are formulated in terms of a stochastic differential equation for the instantaneous interest rate, so called short rate. The interest rates with other maturities, forming the term structure of interest rates, are then determined by bond prices which are solutions to the partial differential equation. We study the Chan-Karolyi-Longstaff-Sanders model and estimate the dependence of volatility on the short rate using the observed term structures together with estimating the unobservable short rate process. Starting with minimizing the sum of squares of erros, we make two approximations of this original optimization problem: Firstly, we relax the constraints regarding zero short rates and allow only positive vales. Secondly, the partial differential equation for the bond prices has a closed form solution only in special cases, so we use an analytical approximation formula in a convenient form. Finally, we apply the proposed algorithm to real data.

1. Introduction

An interest rate is a rate charged for the use of money. It is related to a discount bond, which is a security that pays its holder a unit amount of money at the specified time T (called maturity of the bond). The price $P(t,T)$ at time t of a discount bond with maturity T and the corresponding interest rate $R(t,T)$ are, in case of continuous compounding, connected by a formula

$$P(t,T) = \mathrm{e}^{-R(t,T)(T-t)}, \quad \text{i.e.,} \quad R(t,T) = -\frac{\log P(t,T)}{T-t}.$$

Interest rates at a certain time t with different maturities T form the so called term structure of interest rates. Its beginning (i.e., the limit $\lim_{T\to t^+} R(t,T)$) is called the instantaneous interest rate or the short rate.

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Short rate models are formulated in terms of a stochastic differential equation for the short rate r which has, in general, the form

$$\mathrm{d}r = \mu(r,t)\mathrm{d}t + \sigma(r,t)\mathrm{d}w, \tag{1}$$

where w is a Wiener process. After specifying the market price of risk $\lambda = \lambda(r,t)$ which provides an expected rise of the bond return for the unit rise of risk, the bond price $P = P(r,t)$ (as a function of r and t, treating T as a parameter) satisfies the partial differential equation

$$\frac{\partial P}{\partial t} + (\mu(r,t) - \lambda(r,t)\sigma(r,t))\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2(r,t)\frac{\partial^2 P}{\partial r^2} - rP = 0 \tag{2}$$

for all $r > 0, t \in (0,T)$, with terminal condition$P(r,T) = 1$ for all $r > 0$. See, e.g., [**2**] and [**11**] for more details on short rate models.

In particular, if the drift function $\mu(r,t)$ in (1) is defined to be $\mu(r,t) = \kappa(\theta - r)$ for some constants $\kappa, \theta > 0$, the model displays a mean-reversion feature: the short rate is attracted to its long-term level θ. The function $\sigma(r,t)$ then describes the character of random fluctuations added to this trend. Many previously suggested models can be nested within the choice $\sigma(r,t) = \sigma r^\gamma$, which was popularized by an influencing paper [**4**]. Specifically, it nests two models which for a suitable choice of market price of risk admit a closed form solution for the bond pricing partial differential equation (2): Vasicek model [**20**] assuming constant volatility and market price of risk (i.e., $\gamma = 0$ and $\lambda(r,t) = \lambda$) and Cox-Ingersoll-Ross (CIR hereafter) model [**6**] assuming volatility and market price of risk being proportional to the square root of the short rate (i.e., $\gamma = 1/2$ and $\lambda(r,t) = \lambda\sqrt{r}$). The constant volatility of the short rate in the Vasicek model leads to its normal distribution, which implies the possibility of negative interest rates. This can be indeed observed, for example, at Euribor market (see [**23**]), but, for example, Canadian rates have small values without reaching zero or becoming negative (see [**22**]). The latter case corresponds to the case of $\gamma > 0$. The CIR model with its tractability (explicit expression for the transition density of the short rate as well as the bond prices) makes a popular choice. Empirical results suggest that it is meaningful to study also the models with the general parameter γ, cf. the original paper [**4**], for example, [**7**] for an application of Nowman method based on quasi maximum likelihood or a recent paper [**21**] for an approach using Bayesian analysis. Note that all these studies use a certain market time series which are considered to be a proxy to the short rate process r.

Note that in the Vasicek and the CIR models, as described above, the bond price is a function of four parameters $\kappa, \theta, \sigma, \lambda$. However, actually they depend only on three independent combinations of parameters. This is closely related to the so called risk neutral probability measure which is an equivalent measure to the real probability measure in which the rates are observed. Both models under the risk neutral measure have the form

$$\mathrm{d}r = (\alpha + \beta r)\mathrm{d}t + \sigma r^\gamma \mathrm{d}w \tag{3}$$

and the partial differential equation (2), after the transformation $\tau = T - t$, can be written as

$$(4) \qquad -\frac{\partial P}{\partial \tau} + (\alpha + \beta r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 P}{\partial r^2} - rP = 0$$

for all $r > 0, \tau \in (0, T)$, with initial condition $P(r, 0) = 1$ for all $r > 0$. For a more detailed treatment of the risk neutral measure, we again refer the reader to books [**2**] and [**11**]. In what follows, we consider the risk neutral formulation (3) of the Chan-Karolyi-Longstaff-Sanders (CKLS hereafter) model and the corresponding partial differential equation (4).

Solving the partial differential equation (4) allows us to compute the bond prices and interest rates for the given set of parameters. In practice, we are often faced with an inverse problem: on the market we observe the interest rates and we are interested in estimating the model parameters. Since the solution to the equation (4) depends on the short rate r, this variable is needed to compute the bond prices on the given day, as implied by the model. However, it is not observable in the market; it is only a theoretical construction. In [**9**], this problem was addressed for the case of Vasicek model. The special form of the bond price enables to estimate the evolution of the short rate together with the model parameters by means of a simple optimization problem. In the inner optimization problem the objective is a quadratic function and the outer optimization is only a one variable optimization problem.

This approach cannot be directly generalized to the CKLS model. The explicit form of the bond price is known only in the case of the CIR model and even in this case, it has a more complicated form, and so the previous procedure cannot be applied. Papers [**19**] and [**18**] calibrated the CIR model using the exact solution and even when taking the short rate as known, they used computationally complex evolution algorithms to minimize the objective function. For the other choices of parameter γ in (3), besides $\gamma = 0$ (Vasicek) and $\gamma = 1/2$ (CIR), the explicit solution to (4) is not even available.

In the absence of a closed form solution, there are several possible approaches to obtain an approximation: Monte Carlo simulations (cf. [**8**] for an overview of Monte Carlo methods applied to financial mathematics), numerical solution of the PDE (e.g., [**13**], [**14**], [**15**]), analytical approximation formulae (e.g., [**3**], [**17**], [**16**], [**10**]). When used in the context of calibration, it has to be taken into account that every evaluation of the objective function requires the computation of the bond prices for all the maturities and short rates from the data set. This favours analytical approximation formulae which are the least computationally complex.

In this paper, we use the approximation formula from [**17**]. Its simple form allows us to propose and implement a calibration procedure for estimating short rates in the CKLS model based on the term structure data. The paper is organized as follows: In the next section, we formulate the optimization problem and its modification leading to a computationally simpler problem. Afterwards, we propose the calibration algorithm The third section considers alternative approximations of the bond prices and discusses their effect on calibration procedure.

In the fourth section, we show a sample calibration on simulated data. The fifth section contains the application of the algorithm to real data. We complete the paper with concluding remarks in the fifth section.

2. Formulation of the optimization problem and its approximation

Let us consider the data of the interest rates R_{ij} for $i = 1, \dots, n$ and $j = 1, \dots, m$. Here, R_{ij} is the interest rate with maturity τ_j observed on the i-th day. Following [**19**], [**18**], [**16**] (cf. also [**5**] for a similar idea applied to a convergence model and [**12**] where this approach was used in a model with cyclic behaviour separately for each maturity), we define the objective function by

$$\begin{aligned} F &= \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} (R(r_i, \tau_j) - R_{ij}))^2 \\ &= \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{w_{ij}}{\tau_j^2} (\log P(r_i, \tau_j) + R_{ij}\tau_j)^2, \end{aligned} \tag{5}$$

where w_{ij} are the weights. In what follows, we consider $w_{ij} = 1$, i.e., uniform weights. We note that this is not the only meaningful choice, assigning more weight to fitting interest rates with longer maturities was proposed in [**19**], [**18**]. However, later we introduce an approximation to compute the bond prices which is more precise for shorter maturities, and therefore, we have chosen the uniform weights as a certain compromise. The algorithm can be easily changed for any choice of weights.

The function F is to be minimized with respect to the parameters of the model and the short rate values r_i. If $\gamma = 0$, there are no restrictions on r_i, but in the case of $\gamma > 0$, they have to be nonnegative. Furthermore, depending on the parameters, the zero might be unattainable and the short rate has to be strictly positive. In later steps we require strictly positive values of the short rate for all parameter values, therefore we do not go into details now, but we refer the interested reader to, for example, [**1**] for a discussion on the behaviour of short rate processes at the zero boundary.

Since we do not have a closed form expression for $\log P$, we replace it by an approximation formula from [**17**], which is based on substituting the constant volatility in the Vasicek bond price formula by the instantaneous volatility from the CKLS model considered. We recall (see [**20**]) that in the Vasicek model we have

$$\ln P^{vas}(r, \tau) = \left(\frac{\alpha}{\beta} + \frac{\sigma^2}{2\beta^2}\right) \left(\frac{1 - e^{\beta\tau}}{\beta} + \tau\right) + \frac{\sigma^2}{4\beta^3} \left(1 - e^{\beta\tau}\right)^2 + \frac{1 - e^{\beta\tau}}{\beta} r.$$

The approximation, therefore, reads as

$$\ln P^{ap}(r, \tau) = \left(\frac{\alpha}{\beta} + \frac{\sigma^2 r^{2\gamma}}{2\beta^2}\right) \left(\frac{1 - e^{\beta\tau}}{\beta} + \tau\right) + \frac{\sigma^2 r^{2\gamma}}{4\beta^3} \left(1 - e^{\beta\tau}\right)^2 + \frac{1 - e^{\beta\tau}}{\beta} r. \tag{6}$$

It can be shown (cf. [**17**, Theorem 4]) that $\log P^{ap} - \log P = O(\tau^4)$ as $\tau \to 0^+$. The paper [**17**] also provides numerical examples of the performance of the approximation.

For the calibration purposes, it is useful to write (6) as

$$\ln P^{ap}(r,\tau) = c_0(\tau)r + c_1(\tau)\alpha + c_2(\tau)\sigma^2 r^{2\gamma}, \tag{7}$$

where

$$c_0 = \frac{1-\mathrm{e}^{\beta\tau}}{\beta}, \qquad c_1 = \frac{1}{\beta}\left(\frac{1-\mathrm{e}^{\beta\tau}}{\beta} + \tau\right),$$
$$c_2 = \frac{1}{2\beta^2}\left(\frac{1-\mathrm{e}^{\beta\tau}}{\beta} + \tau + \frac{(1-\mathrm{e}^{\beta\tau})^2}{2\beta}\right).$$

By inserting (7) into (5), the objective function takes the form

$$F = \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{w_{ij}}{\tau_j^2}\left(c_0(\tau_j)r_i + c_1(\tau_j)\alpha + c_2(\tau_j)\sigma^2 r_i^{2\gamma} + R_{ij}\tau_j\right)^2. \tag{8}$$

If $\gamma = 0$, we have a quadratic function of variables $\alpha, \sigma^2, r_1, \ldots, r_n$ which was studied in [**9**] dealing with estimation of the short rate in Vasicek model (note that the approximation formula (7) which we use is exact in that case). In general, there is a nontrivial nonlinearity.

The main idea of our procedure lies in the substitution

$$y_i = \sigma^2 r_i^{2\gamma} \tag{9}$$

in the objective function (8). It results in the new objective function

$$\tilde{F} = \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{w_{ij}}{\tau_j^2}\left(c_0(\tau_j)r_i + c_1(\tau_j)\alpha + c_2(\tau_j)y_i + R_{ij}\tau_j\right)^2, \tag{10}$$

which we minimize with respect to α, β, σ^2 (the model parameters), $r_1, \ldots, r_n$ (short rates), and $y_1, \ldots, y_n$ (auxiliary variables). Both r_i and y_i are nonnegative, but we see that a possibility of zero short rates for certain combinations of parameters means a further condition: $r_i = 0$ if and only if $y_i = 0$. Without this condition, for each β, we would solve a quadratic optimization problem with certain variables restricted to be nonnegative, and find the optimal value of $\tilde{F}$ for the given β. What we propose here is to disregard those values of β which, when using unconstrained optimization, lead to some r_i or y_i that are zero (in practical implementation the condition of being equal to zero is replaced by being less that a chosen small number ε). We call them unfeasible. Therefore, the inner optimization problem is a simple unrestricted quadratic optimization and moreover, feasible values of β typically produce a reasonable range where we look for the optimal β. This is a useful feature since the dependence of either optimal $\tilde{F}$ or $\log \tilde{F}$ on β is not necessarily convex. In practical realization of the calibration, we minimize $\log F$ in the outer optimzation problem. We remark that this step is independent of the choice of parameter γ, and therefore, it is performed only once.

Note that the variables r_i and y_i are not independent, the substitution (9) implies that the ratio $y_i/r_i^{2\gamma}$ is equal to a constant σ^2. By treating them as independent variables in the estimation procedure, we expect the values y_i to be approximations of $\sigma^2 r_i^{2\gamma}$ when using real data or data simulated from the exact bond price formula. Also, the ratios $y_i/r_i^{2\gamma}$ should provide a good approximation of σ^2, by using their mean or median values as an estimate. In order to do this, we need the value of γ. We estimate it as the value which gives the lowest value of variation coefficient (defined as a ratio of standard deviation and mean) for the values of the auxiliary variable $y_i/r_i^{2\gamma}$ (since theoretically it should be constant).

3. Choice of the approximation formula and the resulting optimization problem

We note that the simple structure of the optimization problem (10) which, for fixed values of γ and β, is a quadratic function of r_i, y_i $(i = 1, 2, \ldots, n)$ and α, is a consequence of the approximation formula (6). In particular, for fixed values of γ and β, and after a substitution $y = \sigma^2 r^{2\gamma}$, the logarithm of the approximated bond price is a linear function of r, α, and y. In this section, we review some other possible approximation formulae. We note that out of the approximations considered, only the approximation (6) has this desirable property.

Firstly, we recall that the approximation formula (6) is based on the closed form solution for the bond price in the Vasicek model. Another one factor model with a closed form solution is the model by Cox-Ingersoll-Ross [**6**] which corresponds to the choice of $\gamma = 1/2$ in the stochastic differential equation (3). Using the same idea, i.e., choosing the volatility parameter σ so that the instantaneous volatilities of the CIR model and the CKLS model considered are equal, we arrive to a new approximation formula. The logarithm of the approximated bond price is then given by the CIR bond price formula (see [**6**])

$$\ln P^{\mathrm{cir}}(r,\tau) = \frac{2\alpha}{\sigma^2}\ln\left(\frac{2\phi\,\mathrm{e}^{(\phi-\beta)\tau/2}}{(\phi-\beta)(\mathrm{e}^{\phi\tau}-1)+2\phi}\right) - \frac{2(\mathrm{e}^{\phi\tau}-1)}{(\phi-\beta)(\mathrm{e}^{\phi\tau}-1)+2\phi}r$$

with $\phi = \sqrt{\beta^2 + 2\sigma^2}$, where we substitute σ by $\sigma r^{\gamma-1/2}$. Using the same reasoning as in [**16**] and [**17**], we can derive the order of accuracy of this approximation. It turns out that it is the same as in the approximation based on Vasick model, i.e., the difference of logarithms of exact and approximate solution is of order $O(\tau^4)$ as $\tau \to 0^+$. However, the new approximation based on the CIR model lacks the simple structure of the Vasicek-based approximation (6).

Another approximation, due to Choi and Wirjanto, was suggested in [**3**] and subsequently analyzed in [**17**], where it is proved that the difference of logarithms of exact and approximate solutions is of order $O(\tau^5)$ as $\tau \to 0^+$. It means that the accuracy is higher by one order, compared to the accuracy of the approximation which we use here. However, the approximation reads as (see [**3**])

$$\ln P^{ap,ch-w}(\tau,r) = -rB + \frac{\alpha}{\beta}(\tau - B) + (r^{2\gamma} + q\tau)\frac{\sigma^2}{4\beta}\left[B^2 + \frac{2}{\beta}(\tau - B)\right] - q\frac{\sigma^2}{8\beta^2}\left[B^2(2\beta\tau - 1) - 2B\left(2\tau - \frac{3}{\beta}\right) + 2\tau^2 - \frac{6\tau}{\beta}\right],$$

where $q(r) = \gamma(2\gamma-1)\sigma^2 r^{2(2\gamma-1)} + 2\gamma r^{2\gamma-1}(\alpha+\beta r)$ and $B(\tau) = (\mathrm{e}^{\beta\tau} - 1)/\beta$, and therefore, we cannot make a substitution similar to that one which we are going to use. An alternative might be a definition of two auxiliary variables $y = \sigma^2 r^{2\gamma}, z = \sigma^2 q(r)$. For fixed values of γ and β, the logarithm of the approximated bond price is then linear in α, r, y, z. As we can see, this leads to an increase of parameters estimated in the inner optimization step by n (we recall that n is the number of days for which we have data). Furthermore, in the estimation of σ and γ, we need to take two criteria into account, coming from definitions of auxiliary variables y and z, which is not just a direct analogy of our approach.

Finally, we derive a new approximation. We consider a possibility of using an approximation based on Vasicek model, but aiming for an approximation with a higher order of accuracy. We show that this is indeed possible. However, the formula which leads to an error of order $O(\tau^5)$ as $\tau \to 0^+$ has already a too complicated form. Let us consider replacing the volatility σ^2 in the Vasicek bond pricing formula by a general function $d(r,\tau)$ which may depend on the instantaneous short rate and the time remaining to maturity of the bond which we are pricing. We use the methodology from [**16**] and [**17**], where the main step is the analysis of the equation for $g = f^{ap} - f^{ex}$, which is shown to be

$$-\frac{\partial g}{\partial \tau} + \frac{1}{2}\sigma^2 r^{2\gamma}\left[\left(\frac{\partial g}{\partial r}\right)^2 + \frac{\partial^2 g}{\partial r^2}\right] + (\alpha+\beta r)\frac{\partial g}{\partial r} = h(r,\tau) - \sigma^2 r^{2\gamma}\frac{\partial f^{ex}}{\partial r}\frac{\partial g}{\partial r}.$$

Here, f^{ex} and f^{ap} are the logarithms of the exact and approximated bond prices, respectively, and

$$h(\tau,r) = -\frac{\partial f^{ap}}{\partial \tau} + \frac{1}{2}\sigma^2 r^{2\gamma}\left[\left(\frac{\partial f^{ap}}{\partial r}\right)^2 + \frac{\partial^2 f^{ap}}{\partial r^2}\right] + (\alpha+\beta r)\frac{\partial f^{ap}}{\partial r} - r.$$

If the function h is $O(\tau^{\omega-1})$, then the order of g, i.e., the order accuracy of the approximation formula, is $O(\tau^\omega)$. If we write the function $d(r,\tau)$ in a series form $d(r,\tau) = \sum_{i=0}^{\infty} d_i(r)\tau^i$, then by a direct substitution, we obtain the function h as

$$h(r,\tau) = \frac{1}{2}\left(\sigma^2 r^{2\gamma} - d_0\right)\tau^2 + \frac{1}{12}\left(\sigma^2 r^{2\gamma} d_0'' + 2(\alpha+\beta r)d_0' + 6\beta\left(\sigma^2 r^{2\gamma} - d_0\right) - 8d_1\right)\tau^3 + O(\tau^3).$$

It follows that we need to set $d_0 = \sigma^2 r^{2\gamma}$ in order to have the function h of order at least $O(\tau^3)$ and subsequently the approximation of order at least $O(\tau^4)$. The approximation which we use in this paper corresponds to taking d as a constant function of τ, i.e., $d_i(r) = 0$ for $i \geq 1$. However, taking $d_1 = \frac{1}{8}\left[\sigma^2 r^{2\gamma} d_0'' + 2(\alpha+\beta r)d_0'\right]$

with $d_0 = \sigma^2 r^{2\gamma}$ leads to an approximation of order $O(\tau^5)$. The simplest approximation of this form then takes the remaining d_i for $i \geq 2$, equal to zero. However, replacing the parameter σ^2 by

$$\sigma^2 \mapsto \sigma^2 r^{2\gamma} + \frac{1}{8}\left[2\gamma(\gamma-1)\sigma^4 r^{2\gamma-2} + 4\gamma(\alpha+\beta r)\sigma^2 r^{2\gamma-1}\right]\tau \tag{11}$$

once again leads to a more complicated approximation formula. We may substitute the right hand side of (11) for y, which leads to the same first step of optimization as we present in this paper. However, the second step would need a suitable adjustment.

4. A sample calibration using simulated data

We illustrate this idea on simulated data. We consider the CIR model with parameters $\alpha = 0.00315$, $\beta = -0.0555$, $\sigma = 0.0894$. The parameters are the same as the risk neutral parameters used in [**3**] and [**16**] to check the proposed approximations of the bond prices. We note that the approximation from [**16**] is used in our calibration. In this case, we use these parameters as real probability measure parameters which is equivalent to assuming market price of risk. A different market price of risk with the same risk neutral parameters would lead to the same term structures but different distribution properties of the short rate (speed of mean reversion, limiting value). For a simulation of the short rate, we use Euler-Maruyama discretization. We note that the way of simulating them as well as the choice of the market price of risk have no effect on the calibration precision since the calibration is based only on fitting the term structures. We simulate one trading year of daily data, where we consider 252 trading days. Afterwards, we simulate the term structures using the exact closed form formula. We use the exact closed form formula to simulate the term structures. We consider 12 maturities randing from 1 to 12 quarters.

We illustrate the calibration procedure from the previous section in a series of figures:

- In Figure 1, we show the dependence of $\tilde{F}$ and $\log \tilde{F}$ as a function of β. We distinguish the infeasible values of β (the higher value of the objective function in these cases in the figure is only illustrative). Minimizing this function with respect to β gives the estimate of β. The corresponding optimal value of α is given by estimated coefficients in the linear regression and in the same way we obtain also the estimated evolution of the short rate.
- As explained above, the parameter γ is obtained by minimizing the coefficient of variation of the auxiliary variable $y_i/r_i^{2\gamma}$ (where both r_i and y_i were obtained in the previous step by a linear regression), which is displayed in Figure 2. For a comparison, the figure also shows a histogram of values of the auxiliary variable obtained for the optimal γ and another selected

value of γ. Finally, optimal γ allows us to estimate σ^2 as the median of the corresponding auxiliary variable.
- Figure 3 shows a comparison of the fitted yields with their observed values and comparison of the estimated short rate with its exact behaviour. The fitted yields are computed using the approximation formula (6) evaluated with the estimated parameters and short rates values.

We note that results from this sample calibration are typical to those obtained in another simulation with new simulated data. Therefore, the two quantities of interest can be obtained with satisfactory results: We have a precise estimate of the short rate and a reasonable estimate of the parameter γ which provides an alternative to estimating it using a proxy for the short rate.

Additionally, we add noise to these term structures, which is simulated using independent normally distributed variables with zero mean and standard deviation equal to 3×10^{-5}, and repeat the calibration for new input data. Figure 4 shows a comparison of the fitted yields with their observed values (which were an input to calibration) and the exact values (which were treated as unknown during the calibration, but were used to generate the observed values by adding noise to them). Here we note that larger errors in fitting the observed term structures can be caused by noise in the data, while comparison with exact values gives lower errors.

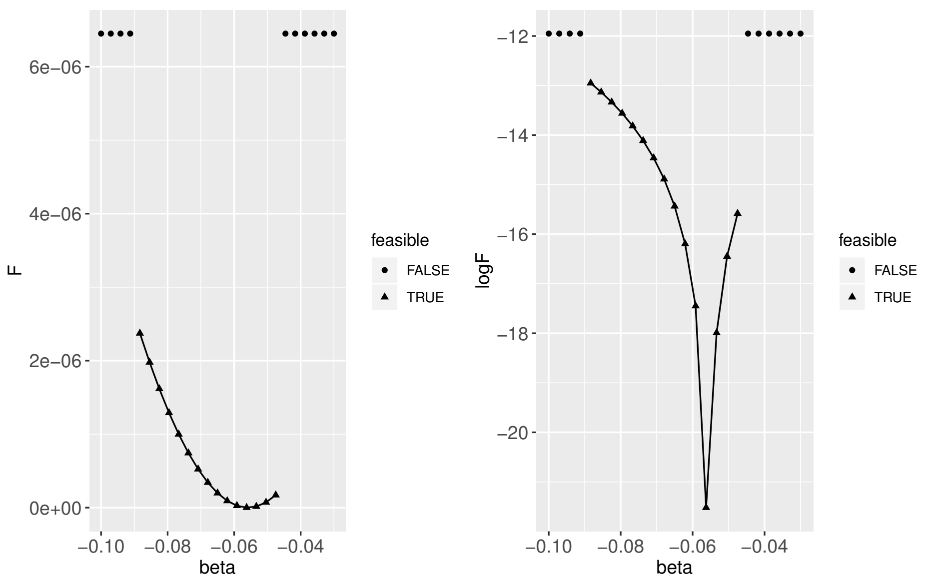

Figure 1. Optimal value of modified objective function F (left) and its logarithm (right) as a function of parameter β for the simulated data.

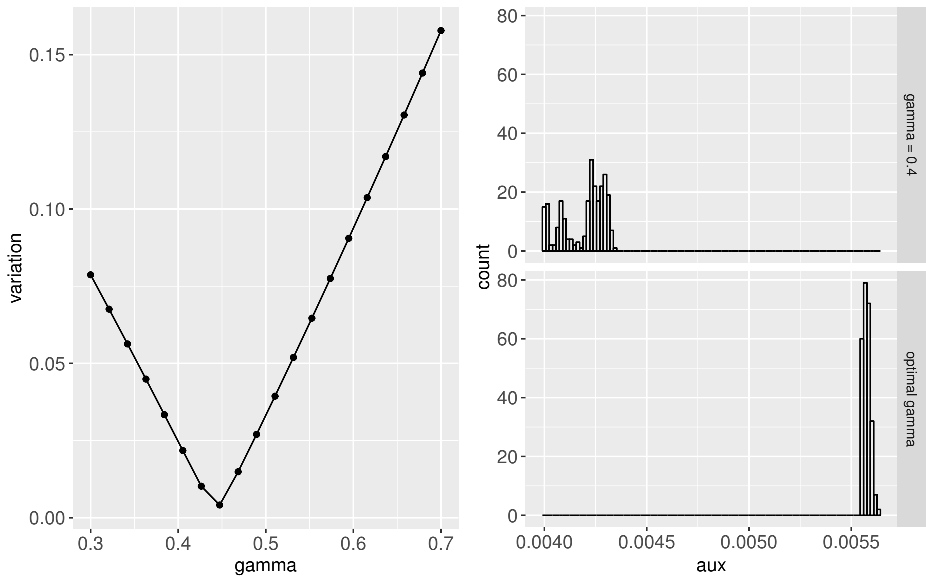

Figure 2. Estimation of the parameter γ by minimizing the coefficient of variation of the auxiliary variable for the simulated data: dependence of coefficient of variation on γ (left) and comparison of histograms for auxiliary variable computed with optimal γ and $\gamma = 0.4$ (right). Note higher variability of the auxiliary variable for a nonoptimal value of γ.

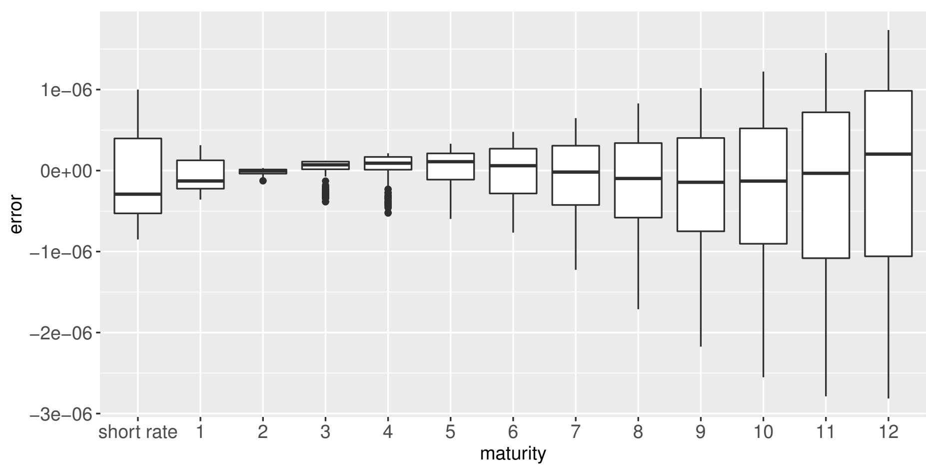

Figure 3. Assessing the calibration using simulated data: error of fitted yields, compared with real yields. The first boxplot shows the accuracy of estimated short rate which as an unobservable variable does not enter the calibration but is only used to generate the term structures.

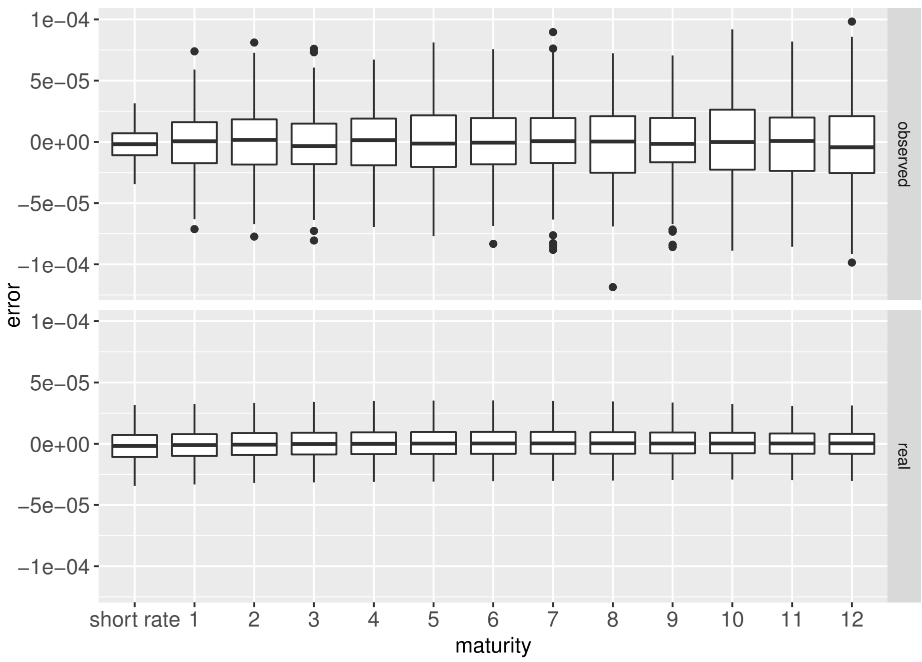

Figure 4. Assessing the calibration using simulated data: error of fitted yields, compared with real yields (without the added noise) and observed yields (with the added noise) for the simulated data.

5. Application to real data

We apply the proposed algorithm to calibration of Canadian zero coupon yield curves with maturities from 1 to 12 quarters from 2017 year [**22**]. We follow the steps of the calibration procedure as described above. Here we note that for certain input data, the dependence of either optimal (optimal for given β) $\tilde{F}$ or $\log \tilde{F}$ on β is not necessarily convex, as shown in Figure 5. We give answers to the following two problems that we are interested in:

- estimation of the unobservable short rate,
- estimation of the form of volatility in terms of estimated parameter γ.

They are shown in Figure 6. We also show the quality of fitting the observed yield curves in Figure 7 by boxplots of errors from yield curves fitting, similarly as in case of the simulated data, but this time we naturally have no error for the short rate, as it is an unobservable quantity.

We can notice that the behaviour of the estimated short rate changes around the day 100. Before, at the beginning of the year, it has low values and around this time we see a start of an increasing trend. This motivates us to divide the data into two parts, the first one consisting of the first 100 days and the second

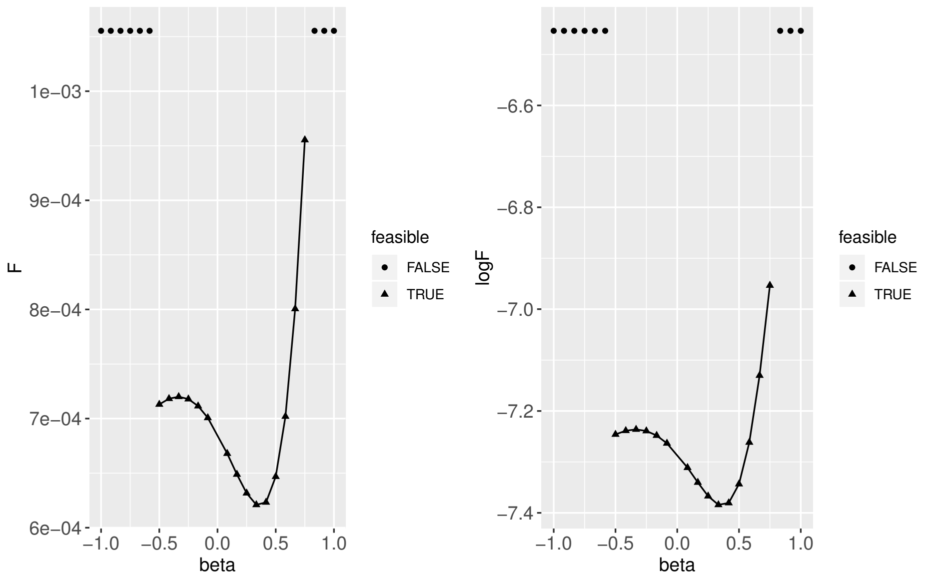

Figure 5. Optimal value of modified objective function F (left) and its logarithm (right) as a function of parameter β for the Canada data. We note the nonconvexity of both functions.

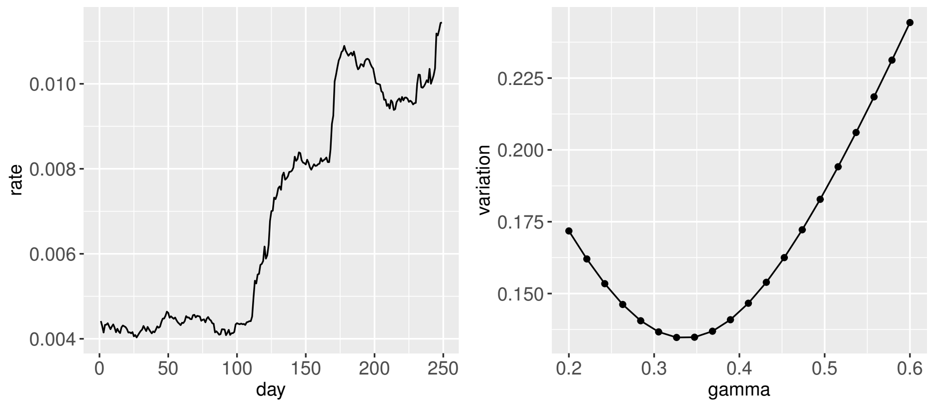

Figure 6. Results from calibration of Canadian data: estimated short rate (left) and optimization of the parameter γ by minimizing the coefficient of variation (right).

containing the last 100 days (the middle of the data set is omitted). Interestingly, we get very different estimates of the parameter γ. In the first period it equals approximately 1.486, while in the second period it is order 10^{-5} which is, when restricted to nonnegative values, is essencially a zero, a Vasicek model. The shape of the estimated short rate remains similar, although a little shifted upwards in

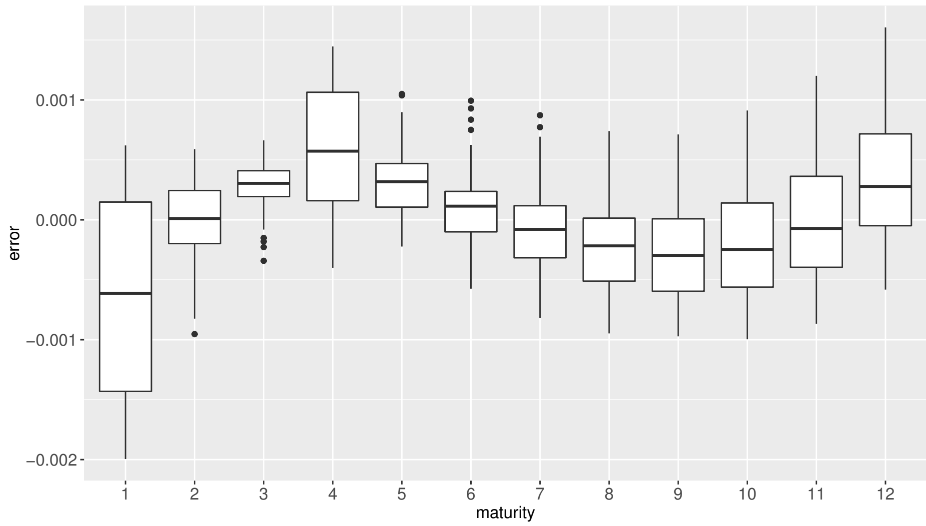

Figure 7. Assessing the calibration using Canadian data: error of fitted yields, compared with observed yields.

the first period and with a larger downwards shift in the second period. The fit is better in the first period of the data, as it is shown in Figure 8.

6. Conclusions

We have proposed an algorithm for estimating the unobservable short rate together with the risk neutral parameters of the CKLS process based on fitting the market yield curves. In particular, we address the question of the correct form of the volatility which is usually studied using a proxy for the short rate, selected from the available market data. Several approximations used in the estimation make the procedure quick and at the same time they give good results when tested on simulated data.

We see three ways for the future research: Firstly, the approximations discussed in Section 3 might be useful, in spite of their more complicated form. The approximation based on the CIR model yields positive interest rates, regardless of the model parameters. The other two approximations have a higher order of accuracy. It might be useful to try to find a simple procedure for calibration also in these cases. Secondly, our estimation is based solely on fitting the term structures and for every day we estimate the value of the short rate. This may cause an overfitting of the model, and therefore, it would be useful to observe its predictive power. In order to make predictions, we need not only risk neutral parameters (which we can obtain using our methodology) but the parameters under the real probability measure or equivalently, the market price of risk which allows us to

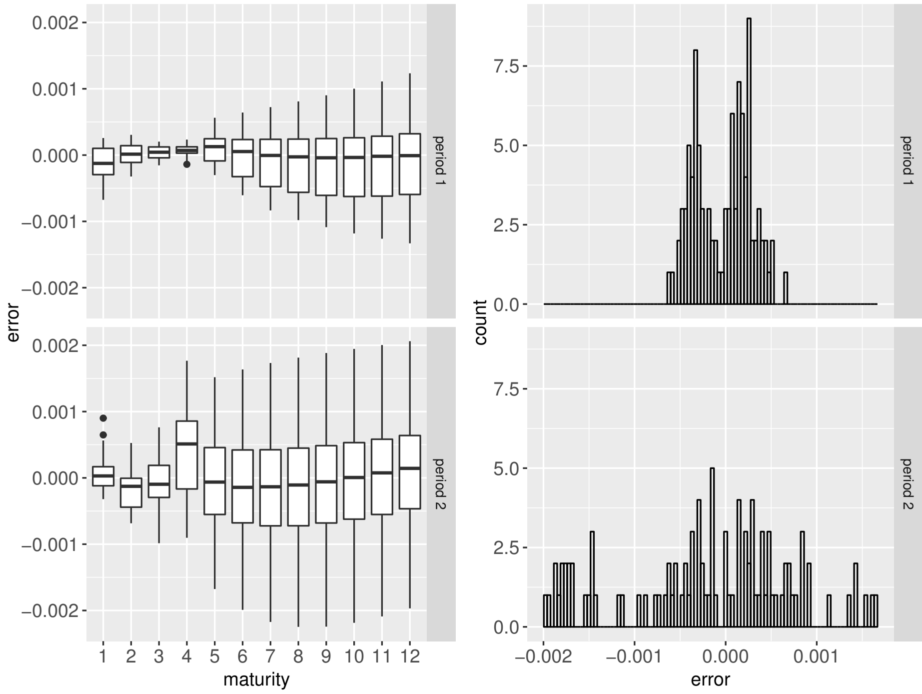

Figure 8. Assessing the calibration using Canadian data in the two periods: comparison of boxplots of the errors (left) and comparison of histograms of errors for interest rates with maturity of 1.5 year, i.e., 6 quarters (right).

compute real parameters from the risk neutral ones. Therefore, a natural task is to find a meaningful parametrization of the market price of risk and its subsequent estimation. Finally, another possibility lies in studying other models for the given data, such as multifactor models, estimating the short rate in these models and comparing the results. We have seen in the simulation that errors in fitting the observed yields can be caused by a noise in them, but naturally it can be also caused by a misspecification of the model. Since multifactor models bring even more parameters to estimate, the question of a possible overfitting needs to be studied too, This brings us again to constriction of predictions.

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