Acta Math. Univ. Comenianae
Vol. LXXXV, 1 (2016), pp. 29–42

COFIBRATIONS IN THE CATEGORY OF NONCOMMUTATIVE CW COMPLEXES

V. MILANI, S. M. H. MANSOURBEIGI and A.-A. REZAEI

Abstract. Cofibration in the category of noncommutative CW complexes is defined. The C*-algebraic counterparts of topological mapping Cylinder and mapping cone are presented as examples of noncommutative CW complex cofibres. As a generalization, the concepts of noncommutative mapping cylindrical and conical telescope are introduced to provide more examples of noncommutative CW complex cofibres. Their properties and K-theoretic behavior are also studied in detail. It is seen that they carry the properties similar to the topological properties of their CW complex counterparts.

1. Introduction

The category of C*-algebras and *-homorphisms can be interpreted as the noncommutative counterpart of the category of topological spaces and continuous maps [**1, 2, 8**]. Its origin goes back to the Gelfand duality. The results of the paper [**7**] known as the Gelfand-Naimark theorem provide a duality between the topology of locally compact spaces and the algebraic structure of commutative C*-algebras. The duality creates a dictionary between the two categories. Topological constructions such as cofibrations, mapping cylinder and mapping cone are translated into their C*-algebraic counterparts [**12, 13**]. In the absence of commutativity, the dictionary may still contain noncommutative CW complexes (NCCW complexes) as the C*-algebraic version of the topological CW complexes defined in [**6**]. The noncommutativity comes from the fact that noncommutative CW complexes are algebras of matrix-valued continuous functions. In [**11**], we studied some of the geometric properties of noncommutative CW complexes. In this paper, we are motivated by noncommutative constructions through NCCW complex examples and study their topological properties. In this regard the paper is organized as follows.

Section 2 is a review of basic tools: extensions, pullbacks, NCCW complexes and their primary properties. Section 3 is devoted to the study of cofibrations and cofibres in the category of NCCW complexes. In this section we explain the C*-algebraic counterparts of the topological mapping cylinder and mapping cone.

Received July 8, 2014.
2010 *Mathematics Subject Classification.* Primary 46Lxx, 57Txx, 57Q05, 57Q12.
Key words and phrases. C*-algebra; Cofibration; Cofibre; CW complex; K-group; mapping cone(cylinder); mapping conical(cylindrical) telescope; noncommutative CW complexes.

We apply the pull back point of view of Pedersen [**12**] to the class of NCCW complexes. We show that the C*-algebraic mapping cylinder and mapping cone defined in [**13**] are obtained through the pull back constructions and are examples of NCCW complexes. We also calculate their dimensions. Our approach is different from that of Diep in [**3, 4, 5**] who defined the noncommutative mapping cylinder and cone in a different way as NCCW complexes. The key concept of this section is the construction of examples for NCCW complex cofibrations and cofibres. In Section 4, we generalize the concepts of the previous section to provide more examples of NCCW complex cofibres and introduce noncommutative mapping cylindrical and conical telescope. Their properties and K-theoretic behavior are also studied in detail. We will see that they carry the properties similar to the topological properties of their CW complex counterparts.

All throughout the paper $\mathcal{C}^*_{\text{alg}}$ is the category of C*-algebras and *-homomorphisms as morphisms. The kernel of a morphism $\varphi\colon A \to B$ in this category is the embedding $\ker\varphi \to A$. We also use Pedersen abbreviation [**12**] "NCCW complexes" for noncommutative CW complexes. The category of topological spaces and continuous maps is denoted by **Top**.

2. Background

The notion of NCCW complexes was first introduced by Pedersen et al [**6**]. They are in fact the C*-algebraic counterpart of topological CW complexes. We discuss the category of noncommutatice CW complexes from [**6**] and refer to [**9**] for details on topological CW complexes. First we review basic tools of pullbacks from [**10, 12**].

Notation. The following notations are used throughout this paper.

$$\mathbb{I} = [0,1], \quad \mathbb{I}^n = [0,1]^n, \quad \mathbb{I}^n_0 = (0,1)^n.$$

For a C*-algebra A and a compact space X, $XA = C(X \to A)$ is the C*-algebra of continuous functions on X with values in A, and if X is locally compact, $C_0(X \to A)$ is the C*-algebra of continuous functions on X vanishing at infinity. Also we denote $C(X \to \mathbb{C})$ by $C(X)$ and

$$\mathbb{I}A = C(\mathbb{I} \to A), \qquad \mathbb{I}^n A = C(\mathbb{I}^n \to A)$$
$$\mathbb{I}^n_0 A = C_0(\mathbb{I}^n_0 \to A), \qquad S^n A = C(S^n \to A).$$

Here $\mathbb{I}^{n+1} \setminus \mathbb{I}^{n+1}_0$ is identified with the unit sphere S^n.

(All the above sets together with the usual pointwise operations, and supremum norm are C*-algebras).

Definition 2.1. For a C*-algebra A and for $t \in \mathbb{I}$, the morphism

$$ev(t)\colon \mathbb{I}A \to A, \qquad ev(t)f := f(t)$$

is called the evaluation map.

Definition 2.2 ([**8**])**.** Two morphisms $\alpha, \beta\colon A \to B$ are said to be homotopic if there exists a morphism $H\colon A \to \mathbb{I}B$ called homotopy such that $ev(1)H = \alpha$

and $ev(0)H = \beta$. A is a retraction of B if there are morphisms $i\colon A \to B$ and $r\colon B \to A$ such that $ri = \mathrm{id}_A$ and ir is homotopic to the id_B. The C*-algebra A is contractible if its identity map id_A is homotopic to the zero map.

Definition 2.3 ([**10, 12**])**.** Let A and C be two C*-algebras. An *extension* for A with respect to C is a C*-algebra B together with two morphisms α and β for which the following sequence is exact

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

Definition 2.4 ([**10, 12**])**.** A *pull-back* for the C*-algebra C via morphisms $\alpha_1\colon A_1 \to C$ and $\alpha_2\colon A_2 \to C$ is the C*-subalgebra PB of $A_1 \oplus A_2$ defined by

$$PB := \{a_1 \oplus a_2 \in A_1 \oplus A_2 : \alpha_1(a_1) = \alpha_2(a_2)\}.$$

From now on the pull-back decomposition, notation $PB := A_1 \bigoplus_C A_2$ is used all throughout this paper.

Remark 2.5. Since α_1 and α_2 are continuous maps, PB is closed in $A_1 \oplus A_2$, and so it is a C*-subalgebra.

Remark 2.6. It follows from the definition that the pull-back satisfies the following universality properties:

• It commutes the following diagram

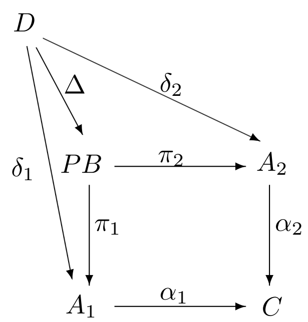
$$\begin{array}{ccc} PB & \xrightarrow{\pi_2} & A_2 \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ A_1 & \xrightarrow{\alpha_1} & C \end{array}$$

(π_1 and π_2 are projections onto the first and second coordinates).

• For any C*-algebra D and any two C*-morphisms $\delta_1\colon D \to A_1$ and $\delta_2\colon D \to A_2$ satisfying the following commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\delta_2} & A_2 \\ {\scriptstyle\delta_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ A_1 & \xrightarrow{\alpha_1} & C \end{array}$$

there exists a unique C*-morphism $\Delta\colon D \to PB$ which commutes the diagram



Definition 2.7 ([**6**])**.** The *noncommutative CW complexes* are defined by induction on their dimension as follows.
A NCCW complex of dimension zero is defined to be a finite linear dimensional C*-algebra A_0 corresponding to the decomposition $A_0 = \bigoplus_k M_{n(k)}$ of finite dimensional matrix algebras.
In dimension n, a NCCW complex is defined as a sequence of C*-algebras $\{A_0, A_1, \dots A_n\}$, where each A_k is obtained inductively from the previous one by the following pullback construction

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{I}_0^k F_k & \longrightarrow & A_k & \xrightarrow{\pi} & A_{k-1} & \longrightarrow & 0 \\ & & \| & & \downarrow{\rho_k} & & \downarrow{\sigma_k} & & \\ 0 & \longrightarrow & \mathbb{I}_0^k F_k & \longrightarrow & \mathbb{I}_0^k F_k & \xrightarrow{\partial} & S^{k-1} F_k & \longrightarrow & 0. \end{array}$$

In the above diagram, the rows are extensions, F_k is some C*-algebra of finite dimension, ∂ – the boundary map – is the restriction morphism, σ_k the connecting morphism can be any morphism, and finally, ρ_k and π are projections onto the first and second factors in the pullback decomposition

$$A_k = \mathbb{I}_0^k F_k \bigoplus_{S^{k-1} F_k} A_{k-1}.$$

Example 2.8. $C(\mathbb{I})$ is a 1-dimensional NCCW complex. To see this, let $F_1 = \mathbb{C}$, $A_0 = \mathbb{C} \oplus \mathbb{C}$ and $A_1 = C(\mathbb{I})$, then we have $\mathbb{I}_0^1 F_1 = C_0((0,1))$, $\mathbb{I}^1 F_1 = C(\mathbb{I})$, and $S^0 F_1 = \mathbb{C} \oplus \mathbb{C}$. The pullback construction diagram becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0((0,1)) & \longrightarrow & C(\mathbb{I}) & \xrightarrow{\pi} & \mathbb{C} \oplus \mathbb{C} & \longrightarrow & 0 \\ & & \| & & \downarrow{\rho_1} & & \downarrow & & \\ 0 & \longrightarrow & C_0((0,1)) & \longrightarrow & C(\mathbb{I}) & \xrightarrow{\partial} & \mathbb{C} \oplus \mathbb{C} & \longrightarrow & 0. \end{array}$$

where $C(\mathbb{I})$ is identified with

$$\begin{aligned} C(\mathbb{I}) &\simeq \{f \oplus (\lambda \oplus \mu) \in C(\mathbb{I}) \oplus (\mathbb{C} \oplus \mathbb{C}) : f(0) = \lambda, f(1) = \mu\} \\ &= C(\mathbb{I}) \bigoplus_{\mathbb{C} \oplus \mathbb{C}} (\mathbb{C} \oplus \mathbb{C}). \end{aligned}$$

For each $f \in C(\mathbb{I})$ and $\lambda, \mu \in \mathbb{C}$,

$$\begin{aligned} \pi(f \oplus (\lambda \oplus \mu)) &= \lambda \oplus \mu \\ \rho_1(f \oplus (\lambda \oplus \mu)) &= f \\ \partial f &= f(0) \oplus f(1). \end{aligned}$$

Now the sequence $\{A_0 = \mathbb{C} \oplus \mathbb{C}, A_1 = C(I)\}$ makes $C(\mathbb{I})$ into a 1-dimensional NCCW complex. In a similar way we can see that both $C_0((0,1])$ and $C_0((0,1))$ are 1-dimensional NCCW complexes. We go back to this construction from a different point of view in the following sections.

Remark 2.9 ([**12**]). It follows from the definition of NCCW complexes that for each n-dimensional NCCW complex A_n, a decreasing family of closed ideals, called canonical ideals, corresponds

$$A_n = I_0 \supset I_1 \supset \cdots \supset I_{n-1} \supset I_n \neq 0,$$

where $I_n = \mathbb{I}_0^n F_n$ and for each $k \geq 1$, $I_k/I_{k+1} = \mathbb{I}_0^k F_k$. Moreover, for each $0 \leq k \leq n-1$, A_n/I_{k+1} is a k-dimensional NCCW complex.

Example 2.10. For the NCCW complex of $C(\mathbb{I})$, the canonical ideals are

$$C(\mathbb{I}) = I_0 \supset I_1 = \mathbb{I}_0^1 F_1 = C_0((0,1)).$$

The noncommutative analog of simplicial maps between CW complexes are defined as follows.

Definition 2.11 ([**6**]). A *simplicial morphism* from the n-dimensional NCCW complex A_n into the m-dimensional NCCW complex B_m is a mapping $\alpha\colon A_n \to B_m$ satisfying the following two conditions:

• If

$$A_n = I_0 \supset I_1 \supset \cdots \supset I_{n-1} \supset I_n \neq 0$$
$$B_m = J_0 \supset J_1 \supset \cdots \supset J_{m-1} \supset J_m \neq 0$$

are the sequences of canonical ideals for A_n and B_m, then $\alpha(I_k) \subset J_k$ for all k. Particularly $\alpha(I_k) = 0$ for $k > m$.

• For $0 \leq k \leq n$, if $I_k/I_{k-1} = \mathbb{I}_0^k F_k$, $J_k/J_{k-1} = \mathbb{I}_0^k G_k$ and $\tilde{\alpha_k}\colon \mathbb{I}_0^k F_k \longrightarrow \mathbb{I}_0^k G_k$ is the homomorphism induced by α, then there exist a morphism $\varphi_k\colon F_k \longrightarrow G_k$ and a homeomorphism i_k of $\mathbb{I}_0^k$ such that $\tilde{\alpha_k} = i_k^* \otimes \varphi_k$, where $i_k^*\colon C_0(\mathbb{I}_0^k) \longrightarrow C_0(\mathbb{I}_0^k)$ is induced by i_k. Here $\mathbb{I}_0^k F_k$ is identified with $C_0(\mathbb{I}_0^k) \otimes F_k$ and the same for $\mathbb{I}_0^k G_k$.

The category of NCCW complexes and simplicial morphisms is denoted by $\mathcal{C}_{nccw}$.

Remark 2.12. Some facts on simplicial morphisms are stated here. For more details on the proof, see [**12**].

1. The kernel and the image of a simplicial morphism are NCCW complexes.
2. The pullback of a NCCW complex C via simplicial morphisms $\alpha\colon A_n \to C$ and $\beta\colon B_m \to C$ is a NCCW complex of dimension $\max\{n, m\}$.
3. If A_n and B_m are NCCW complexes of dimensions n, m, respectively, then their tensor product $A_n \otimes B_m$ is a NCCW complex of dimension $n + m$. Moreover, for each $k \leq n$, $l \leq m$, the quotient morphism

   $$A_n \otimes B_m \to A_k \otimes B_l$$

   onto the tensor product of subcomplexes is a simplicial morphism.

3. Cofibrations in the category NCCW complexes

In this section, we review the concept of cofibration in the category $\mathcal{C}^*_{\text{alg}}$ from [**13, 14**] and modify it to define cofibrations on the category $\mathcal{C}_{nccw}$. We see how the C*-algebraic mapping cylinder and mapping cone defined in [**13**] are obtained through the pull back constructions of [**12**] and study their relation with cofibrations and cofibres in the category $\mathcal{C}_{nccw}$.

The notion of cofibration in the category $\mathcal{C}^*_{\text{alg}}$ is the C*-algebraic translation of the concept of fibration for the category **Top**. It is defined in the following way.

Definition 3.1 ([**13, 14**])**.** A morphism $f\colon A \to B$ in the category $\mathcal{C}^*_{\text{alg}}$ is called a cofibration if it satisfies the following property.

Given a C*-algebra D, a homotopy $g\colon D \to \mathbb{I}B$ and a morphism $g_0\colon D \to A$ lifting g at zero, i.e., $fg_0(x) = g(x)(0)$, then there exists a homotopy $h\colon D \to \mathbb{I}A$ lifting g, i.e., $h(x)(0) = g_0(x)$ and $f(h(x)(t)) = g(x)(t)$ for $x \in D$, $t \in \mathbb{I}$.

Remark 3.2. A cofibration is surjective [**14**].

Definition 3.3. A cofibration in the category of NCCW complexes $\mathcal{C}_{nccw}$ is a cofibration in the category $\mathcal{C}^*_{\text{alg}}$ which is also simplicial. The kernel of a cofibration is called a cofiber.

Proposition 3.4. *For each NCCW complex A of dimension n, $\mathbb{I}A$ is a NCCW complex of dimension $n+1$. Moreover, the evaluation map $ev(1)\colon \mathbb{I}A \to A$ defined by $f \mapsto f(1)$ is a simplicial morphism and a cofibration (the same is true for $ev(t_0)$ for each $t_0 \in \mathbb{I}$).*

Proof. $\mathbb{I}A$ can be identified with the tensor product $C(\mathbb{I}) \otimes A$. So it is a NCCW complex of dimension $n+1$. A is embedded in $\mathbb{I}A$ and we can regard A as a NCCW subcomplex of $\mathbb{I}A$. $ev(1)$ (and also $ev(t_0)$ for each $t_0 \in \mathbb{I}$), is a quotient morphism, and so by the property $\S_3$ of remark 2.12, it is simplicial morphism.

To show that it is a cofibration, let D be a C*-algebra and $G\colon D \to \mathbb{I}A$ be a homotopy with a lift $g_0\colon D \to \mathbb{I}A$ at zero, i.e., $(ev(1)g_0)(d) = G(d)(0)$ for $d \in D$. Define the homotopy $H\colon D \to \mathbb{I}(\mathbb{I}A)$ by

$$H(d)(t) = f_t$$

for $d \in D$ and $t \in \mathbb{I}$. Let $g_0(d) = f$, then $f(1) = G(d)(0)$. f_t must be defined so that

$$\begin{aligned} ev(1)(H(d)(t)) = G(d)(t) \quad &\Longrightarrow \quad f_t(1) = G(d)(t) \\ H(d)(0) = g_0(d) \quad &\Longrightarrow \quad f_0 = f. \end{aligned}$$

Set

$$\hat{f}_t(s) = \begin{cases} G(d)(t-2s) & 0 \leq s \leq \dfrac{t}{2} \\ f\Big(\dfrac{2-2s}{2-t}\Big) & \dfrac{t}{2} \leq s \leq 1. \end{cases}$$

Now set $f_t(s) = \hat{f}_t(1-s)$.

In the same way we can see $ev(t_0)$ is a cofibration. □

Example 3.5. $C_0((0,1])$ is a cofiber. To see this, we notice that $\mathbb{C}$ is a zero dimensional NCCW complex with the only nonzero ideal $\mathbb{C}$. Now by Proposition 3.4, $C(\mathbb{I}) = \mathbb{I}\mathbb{C}$ is a 1=dimensional NCCW complex. Moreover, the evaluation map

$$ev(0)\colon C(\mathbb{I}) \longrightarrow \mathbb{C}, \qquad f \longmapsto f(0),$$

is a simplicial cofibration. Its kernel is $C_0((0,1])$ which is the cofiber of this cofibration. Also from property $\S_1$ of remark 2.12, $C_0((0,1])$ is a NCCW complex. Its dimension is one, because it is not of finite linear dimension(and so it is not a 0-dimensional NCCW complex). $C_0([0,1))$ being identical to $C_0((0,1])$ is a NCCW complex of dimension one. In a similar way, $C_0((0,1))$ is the kernel of the simplicial morphism

$$\beta\colon C_0((0,1]) \longrightarrow \mathbb{C}, \qquad f \longmapsto f(1)$$

and so it is a 1-dimensional NCCW complex cofiber.

Definition 3.6 ([**13**])**.** For a morphism $\alpha\colon A \to B$ in the category $\mathcal{C}^*_{\text{alg}}$, the *noncommutative mapping cylinder* is defined by

$$\mathrm{Cyl}(\alpha) := \{a \oplus f \in A \oplus \mathbb{I}B : f(1) = \alpha(a)\}.$$

Proposition 3.7. *For an arbitrary morphism $\alpha\colon A \to B$ between C*-algebras, the induced morphism $p\colon \mathrm{Cyl}(\alpha) \to B$ defined by $p(a,f) = f(0)$ for $(a,f) \in \mathrm{Cyl}(\alpha)$, is a cofibration in the category $\mathcal{C}^*_{\text{alg}}$.*

Proof. See [**13**]. □

Proposition 3.8. *If A_n and B_m are NCCW complexes of dimensions n, m, respectively, and if $\alpha\colon A_n \to B_m$ is a simplicial morphism, then $\mathrm{Cyl}(\alpha)$ is a NCCW complex of dimension $\max\{n, m+1\}$.*

Proof. $\mathrm{Cyl}(\alpha)$ satisfies the following pullback diagram

$$\begin{array}{ccc} \mathrm{Cyl}(\alpha) & \xrightarrow{\pi_2} & \mathbb{I}B_m \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle ev(1)} \\ A_n & \xrightarrow{\alpha} & B_m, \end{array}$$

where π_1 and π_2 are projections onto the first and second coordinates.

$\mathbb{I}B_m$ is a NCCW complex of dimension $m+1$, $ev(1)$ and α are simplicial morphisms in the pullback diagram for $\mathrm{Cyl}(\alpha)$. Now it follows from Remark 2.12 that $\mathrm{Cyl}(\alpha)$ is a NCCW complex of dimension $\max\{n, m+1\}$. □

Definition 3.9 ([**9, 13**])**.** For a C*-algebra A, the *noncommutative cone* over A, CA, and the *noncommutative suspension* of A, SA, respectively, are defined by

$$CA := \{f \in \mathbb{I}A; f(0) = 0\}$$
$$SA := \{f \in \mathbb{I}A; f(0) = f(1) = 0\}.$$

Remark 3.10. It follows that:

- $SA \subset CA \subset \mathbb{I}A$.
- CA is contractible.
- SA is contractible only if A is contractible.

See[**15**] for details.

Proposition 3.11. *For every NCCW complex A of dimension n, the cone CA and the suspension SA are NCCW complexes of dimension $n+1$.*

Proof. This follows from Example 3.5, property §3 of Remark 2.12, and the following identifications

$$CA = C_0((0,1]) \to A) \simeq C_0((0,1]) \otimes A$$
$$SA = C_0((0,1)) \to A) \simeq C_0((0,1)) \otimes A.$$

□

Definition 3.12 ([**13**])**.** For a morphism $\alpha\colon A \to B$, the *noncommutative mapping cone of* α is defined by

$$\mathrm{Cone}(\alpha) := \{a \oplus f \in A \oplus CB : f(1) = \alpha(a)\}.$$

Proposition 3.13. *If* $\alpha\colon A_n \to B_m$ *is a morphism between NCCW complexes of dimension* n *and* m*, then* Cone(α) *is a cofiber in the category* $\mathcal{C}_{nccw}$*. Moreover, if* α *is a simplicial morphism, then* Cone(α) *is a NCCW complex of dimension* $\max\{n, m+1\}$*.*

Proof. By Propositions 3.7 and 3.8, the morphism $p\colon \mathrm{Cyl}(\alpha) \to B_m$ induced by α is a cofibration with Cone(α) as its kernel. So Cone(α) is a cofiber for this cofibration. Also Cone(α) satisfies the following pullback diagram

$$\begin{array}{ccc} \mathrm{Cone}(\alpha) & \xrightarrow{\pi_2} & CB_m \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle ev(1)} \\ A_n & \xrightarrow{\alpha} & B_m. \end{array}$$

where π_1 and π_2 are projections onto the first and second coordinates.

Since CB_m is a NCCW complex of dimension $m+1$, $ev(1)$ and α are simplicial morphisms in the pullback diagram for Cone(α), it follows from Remark 2.12 that Cone(α) is a NCCW complex of dimension $\max\{n, m+1\}$. □

Remark 3.14. Since $CB \subset \mathbb{I}B$, so for any *-morphism α, we have the inclusion Cone$(\alpha) \subset$ Cyl(α). Also for the zero morphism $0\colon A \to B$, we have

$$\mathrm{Cone}(0) = \{a \oplus f \in A \oplus CB : f(1) = 0\} = A \oplus C_0((0,1) \to b) = A \oplus SB$$
$$\mathrm{Cyl}(0) = \{a \oplus f \in A \oplus \mathbb{I}B : f(1) = 0\} = A \oplus C_0([0,1) \to B) \simeq A \oplus CB.$$

Proposition 3.15. *For the morphism* $\alpha\colon A \to B$ *between C*-algebras, A is a retraction of* Cyl(α)*.*

Proof. Let $\pi\colon \mathrm{Cyl}(\alpha) \to A$ be given by $\pi(a \oplus f) := a$ and $\eta\colon A \to \mathrm{Cyl}(\alpha)$ by $\eta(a) := a \oplus \alpha(a)$, where $\alpha(a)$ denotes the constant map $t \mapsto \alpha(a)$. Then $\pi \circ \eta = \mathrm{id}_A$. Now let the homotopy $\psi\colon \mathrm{Cyl}(\alpha) \to \mathbb{I}\,\mathrm{Cyl}(\alpha)$ be defined by

$$\psi(a \oplus f)\colon \mathbb{I} \longrightarrow \mathrm{Cyl}(\alpha), \qquad t \longmapsto a \oplus f_t,$$

where $f_t\colon \mathbb{I} \to B$ is defined as $f_t(s) = f((1-s)t + s)$ for each $s \in \mathbb{I}$. Then we can see that

$$\psi(a \oplus f)(0) = a \oplus f = \mathrm{id}_{\mathrm{Cyl}(\alpha)}(a \oplus f)$$
$$\psi(a \oplus f)(1) = a \oplus f(1) = a \oplus \alpha(a) = \eta \circ \pi(a \oplus f)),$$

and so $\eta \circ \pi$ and $\mathrm{id}_{\mathrm{Cyl}(\alpha)}$ are homotopic. □

4. Noncommutative mapping cylindrical and conical telescope

In this part we generalize the notions of noncommutative mapping cylinder and noncommutative mapping cone. Conditions for the existence of their NCCW complex structure are specified. It is seen that they provide examples of NCCW complex cofibers. Their topological properties are studied in detail.

Definition 4.1. For a sequence of length n of C*-algebras

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1},$$

the *NC mapping cylindrical telescope* is defined by

$$\begin{aligned} T_n := \{a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus \mathbb{I}A_2 \oplus \cdots \oplus \mathbb{I}A_{n+1} : \\ f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a), k = 1, 2, \ldots, n\}. \end{aligned}$$

Since each α_k is continuous, T_n is a closed subalgebra of the C*-algebra $A_1 \oplus \mathbb{I}A_2 \oplus \cdots \oplus \mathbb{I}A_{n+1}$, and hence is itself a C*-algebra. Also we note that for $n = 1$, $T_1 = \mathrm{Cyl}(\alpha_1)$.

Proposition 4.2. *For each $n \geq 2$, T_{n-1} is a retraction of T_n.*

Proof. The proof is done by induction on n. For $n = 1$, let $\beta_2 \colon T_1 \to A_3$ be defined by $\beta_2(a \oplus f_2) = \alpha_2 \circ \alpha_1(a)$. Then β_2 is a C*-morphism and from Proposition 3.15, T_1 is a retraction of $\mathrm{Cyl}(\beta_2)$. But $\mathrm{Cyl}(\beta_2) = T_2$, since

$$\begin{aligned} \mathrm{Cyl}(\beta_2) &= \{(a \oplus f_2) \oplus f_3 \in T_1 \oplus \mathbb{I}A_3 : \beta_2(a \oplus f_2) = f_3(1)\} \\ &= \{a \oplus f_2 \oplus f_3 \in A_1 \oplus \mathbb{I}A_2 \oplus \mathbb{I}A_3 : f_2(1) = \alpha_1(a), f_3(1) = \alpha_2 \circ \alpha_1(a)\} \\ &= T_2. \end{aligned}$$

Now by induction, if we define $\beta_n \colon T_{n-1} \to A_{n+1}$ by $\beta_n(a \oplus f_2 \oplus \cdots \oplus f_n) = \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1(a)$, we see that T_{n-1} is a retraction of T_n. □

Corollary 4.3. *For the sequence of Definition 4.1, A_1 is a retraction of T_n.*

Proof. From the Proposition 4.2, T_n retracts to $T_1 = \mathrm{Cyl}(\alpha_1)$ and since by Proposition 3.15, $\mathrm{Cyl}(\alpha_1)$ retracts to A_1, so A_1 is a retraction of T_n. □

Corollary 4.4. *For the sequence of Definition 4.1, $K_0(A_1) = K_0(T_1) = \cdots = K_0(T_n)$.*

Proposition 4.5. *For the sequence of Definition 4.1, if $\alpha_m \circ \alpha_{m-1} \circ \cdots \circ \alpha_{m-k} = 0$ for some $k < m \leq n$, then*

$$T_n \simeq T_{m-1} \oplus \Big(\bigoplus_{i=m+1}^{n+1} CA_i \Big).$$

Proof.

$$\begin{aligned}T_n &= \{a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus \mathbb{I}A_2 \oplus \cdots \oplus \mathbb{I}A_{n+1} : \\ &\qquad f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a),\ k = 1, 2, \ldots, n\} \\ &= \{a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus \mathbb{I}A_2 \oplus \cdots \oplus \mathbb{I}A_{n+1} : \\ &\qquad f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a),\ 1 \leq k < m,\ f_{k+1}(1) = 0,\ m \leq k \leq m\} \\ &\simeq T_{m-1} \oplus CA_{m+1} \oplus \cdots \oplus CA_{n+1}.\end{aligned}$$

□

Corollary 4.6. *For the exact sequence $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1}$ of length n, $T_n \simeq T_{n-1} \oplus CA_{n+1}$ and in particular,*

$$T_n \simeq \mathrm{Cyl}(\alpha_1) \oplus \Big(\bigoplus_{i=3}^{n+1} CA_i\Big).$$

Proposition 4.7. *If $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1}$ is a sequence of simplicial morphisms between NCCW complexes of dimensions $m_1, m_2, \ldots, m_{n+1}$ then T_n is a NCCW complex of dimension* $\max\{m_1, 1 + m_2, 1 + m_3, \ldots, 1 + m_{n+1}\}$.

Proof. Since $\alpha_1 : A_1 \to A_2$ is a simplicial morphism, by Proposition 3.8, $T_1 = \mathrm{Cyl}(\alpha_1)$ is a NCCW complex of dimension $\max\{m_1, 1 + m_2\}$. Let $\beta_2 : T_1 \to A_3$ be the *-morphisms of Proposition 4.2. Since α_1 and α_2 are simplicial morphisms, then so is β_2, and $T_2 = \mathrm{Cyl}(\beta_2)$ is a NCCW complex of dimension $\max\{\max\{m_1, 1 + m_2\}, 1 + m_3\} = \max\{m_1, 1 + m_2, 1 + m_3\}$. Inductively, the morphism

$$\beta_n : T_{n-1} \longrightarrow A_{n+1}, \qquad a \oplus f_2 \oplus \cdots \oplus f_n \longmapsto \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1(a)$$

is simplicial, and so $T_n = \mathrm{Cyl}(\beta_n)$ is a NCCW complex of dimension $\max\{m_1, 1 + m_2, 1 + m_3, \ldots, 1 + m_{n+1}\}$. □

Definition 4.8. Let $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1}$ be a sequence of morphisms between C*-algebras. The *NC mapping conical telescope* is defined as

$$\begin{aligned}T_nC := \{&a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus CA_2 \oplus \cdots \oplus CA_{n+1} : \\ &f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a), k = 1, 2, \ldots, n\}.\end{aligned}$$

As in the case T_n, for $n = 1$, $T_1C = \mathrm{Cone}(\alpha_1)$.

Proposition 4.9. *For the sequence of Definition 4.1, if $\alpha_m \circ \alpha_{m-1} \circ \cdots \circ \alpha_{m-k} = 0$ for some $k < m \leq n$, then*

$$T_nC = T_{m-1}C \oplus \Big(\bigoplus_{i=m+1}^{n+1} SA_i\Big).$$

Proof.

$$\begin{aligned} T_nC = \{&a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus CA_2 \oplus \cdots \oplus CA_{n+1} : \\ &f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a),\ k = 1, 2, \ldots, n\} \\ = \{&a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus CA_2 \oplus \cdots \oplus CA_{n+1} : \\ &f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a),\ 1 \le k < m,\ f_{k+1}(1) = 0,\ m \le k \le m\} \\ = T&_{m-1}C \oplus SA_{m+1} \oplus \cdots \oplus SA_{n+1}. \end{aligned}$$

□

Corollary 4.10. *If the sequence of definition 4.1 is exact, then $T_nC = T_{n-1} \oplus SA_{n+1}$, and in particular,*

$$T_nC = \mathrm{Cone}(\alpha_1) \oplus \Big(\bigoplus_{i=3}^{n+1} SA_i\Big).$$

Proposition 4.11. *For each sequence of length $n > 1$, there is an exact sequence*

$$0 \longrightarrow SA_{n+1} \longrightarrow T_nC \longrightarrow T_{n-1} \longrightarrow 0$$

Proof. Let $i\colon SA_{n+1} \to T_nC$ be the inclusion morphism $f \mapsto 0 \oplus 0 \oplus \cdots \oplus f$ and $\pi\colon T_nC \to T_{n-1}C$ be the projection $a \oplus f_2 \oplus \cdots \oplus f_{n+1} \mapsto a \oplus f_2 \oplus \cdots \oplus f_n$, then $ker\pi = i(SA_{n+1})$. □

Proposition 4.12. *For each sequence of length n, there exists an exact sequence*

$$0 \longrightarrow T_nC \longrightarrow T_n \longrightarrow \bigoplus_{k=2}^{n+1} A_k \longrightarrow 0.$$

Proof. Let $i\colon T_nC \to T_n$ be the obvious inclusion and $p\colon T_n \to \bigoplus_{k=2}^{n+1} A_k$ be defined by $p(a \oplus f_2 \oplus \cdots \oplus f_{n+1}) := f_2(0) \oplus \cdots \oplus f_{n+1}(0)$. Then $\ker p = T_nC$. □

Proposition 4.13. *Let*

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \tag{4.1}$$

be an arbitrary sequence of length n of C^*-algebras, then:

- *All the morphisms*

$$p_j\colon T_n \to A_j, \qquad p_j(a \oplus f_2 \oplus \cdots \oplus f_{n+1}) = f_j(0)$$

for $j = 2, \ldots, n+1$ are cofibrations.

- *The morphism $p\colon T_n \to A_2 \oplus \cdots \oplus A_{n+1}$ defined by*

$$p(a \oplus f_2 \oplus \cdots \oplus f_{n+1}) = (f_2(0) \oplus \cdots \oplus f_{n+1}(0))$$

is a cofibration.

- *T_nC is a cofiber of the cofibration p.*

Proof. Fix j. Let a homotopy $G\colon D \to \mathbb{I}A_j$ and a lift $g_0\colon D \to T_n$, of G at zero be given. So $p_j g_0(d) = G(d)(0)$ for $d \in D$. Write $g_0(d) = (a \oplus f_2 \oplus, \dots, \oplus f_{n+1})$ so that for $k = 1, 2, \dots, n$,

$$f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a).$$

From the lifting property of g_0, $f_j(0) = G(d)(0)$. Let the lift $H\colon D \to \mathbb{I}T_n$ for G, be defined by

$$H(d)(t) = (a \oplus k_{2,t} \oplus, \dots, \oplus k_{n+1,t}).$$

To fulfil the homotopy lifting properties H must satisfy

- $p_j(H(d)(t)) = G(d)(t)$ for $d \in D$ and $t \in \mathbb{I}$, i.e., $k_{j,t}(0) = G(d)(t)$.
- $H(d)(0) = g_0(d)$, i.e., $k_{i,0} = f_i$ for $i = 2, \dots, n+1$.

To complete the definition of the homotopy H, for $i = 2, \dots, n+1$, set

$$k_{i,t}(s) = \begin{cases} G(d)(t-2s) & 0 \le s \le \dfrac{t}{2} \\ f_i\Big(\dfrac{2s-t}{2-t}\Big) & \dfrac{t}{2} \le s \le 1 \end{cases}$$

for $s \in \mathbb{I}$.

So p_js are cofibrations. This completes the first part of the proposition.

For the second part as in part one, let

$$\mathcal{G}\colon D \to \mathbb{I}(A_2 \oplus \cdots \oplus A_{n+1})$$

be a homotopy with $\hat{g}_0\colon D \to T_n$ as its lift at zero. Define the homotopy

$$\mathcal{H}\colon D \to \mathbb{I}T_n, \qquad \mathcal{H}(d)(t) = (a \oplus k_{2,t} \oplus \cdots \oplus k_{n+1,t})$$

where

$$\hat{g}_0(d) = (a \oplus f_2 \oplus \cdots \oplus f_{n+1})$$

and for $k = 1, 2, \dots, n$,

$$f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a).$$

Let $\mathcal{G}(d)(t) = (g_{2,t} \oplus \cdots \oplus g_{n+1,t})$. For $i = 2, \dots, n+1$, define

$$k_{i,t}(s) = \begin{cases} g_{i,t}(t-2s) & 0 \le s \le \dfrac{t}{2} \\ f_i\Big(\dfrac{2s-t}{2-t}\Big) & \dfrac{t}{2} \le s \le 1. \end{cases}$$

Then $\mathcal{H}$ satisfies the properties of the homotopy lift for $\mathcal{G}$.

The third part of the proposition follows from the fact that the mapping conical telescope T_nC is the kernel of the cofibtation p. So it is a cofiber by definition. □

The following proposition studies the K-theoretic behaviour of mapping cylindrical and conical telescope. For more details on K-theory and C*-algebras, we refer to [**15**].

Proposition 4.14. *For each sequence of length n, there exists a cyclic six term exact Sequence,*

$$\begin{array}{ccccc} K_0(T_nC) & \longrightarrow & K_0(T_n) & \longrightarrow & \bigoplus_{i=2}^{n+1} K_0(A_i) \\ \uparrow & & & & \downarrow \\ \bigoplus_{i=2}^{n+1} K_1(A_i) & \longleftarrow & K_1(T_n) & \longleftarrow & K_1(T_nC). \end{array}$$

Proof. Since T_nC is an ideal in T_n, $T_n/T_nC = \bigoplus_{i=2}^{n+1} A_i$, $K_0\big(\bigoplus_{i=2}^{n+1} A_i\big) = \bigoplus_{i=2}^{n+1} K_0(A_i)$ and $K_1\big(\bigoplus_{i=2}^{n+1} A_i\big) = \bigoplus_{i=2}^{n+1} K_1(A_i)$, the exactness of the six term sequence follows from [**15**, 9.3.2]. □

Proposition 4.15. *If* $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1}$ *is a sequence of simplicial morphisms between NCCW complexes of dimensions* $m_1, m_2, \dots, m_{n+1}$*, then* T_nC *is a NCCW complex of dimension* $\max\{m_1, 1+m_2, 1+m_3, \dots, 1+m_{n+1}\}$.

Proof. Since $\alpha_1 \colon A_1 \to A_2$ is a simpicial morphism, by Proposition 3.13 , $T_1C = \mathrm{Cone}(\alpha_1)$ is a NCCW complex of dimension $\max\{m_1, 1+m_2\}$. Let $\beta_2 \colon T_1C \to A_3$ be defined as $\beta(a \oplus f_2) = \alpha_2 \circ \alpha_1(a)$. Since α_1 and α_2 are simplicial morphisms, then so is β_2, and $T_2C = \mathrm{Cone}(\beta_2)$ is a NCCW complex of dimension $\max\{\max\{m_1, 1+m_2\}, 1+m_3\} = \max\{m_1, 1+m_2, 1+m_3\}$. Inductively, the morphism

$$\beta_n \colon T_{n-1}C \longrightarrow A_{n+1}, \qquad a \oplus f_2 \oplus \cdots \oplus f_n \mapsto \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1(a)$$

is simplicial, and so $T_nC = \mathrm{Cone}(\beta_n)$ is a NCCW complex of dimension $\max\{m_1, 1+m_2, 1+m_3, \dots, 1+m_{n+1}\}$. □

References

1. Connes A., *Noncommutative geometry* Academic press, California 1994.

2. Cuntz J., *Quantum spaces and their noncommutative topology*, AMS notices **8** (2001), 793–799.

3. Diep D. N., *Category of noncommutative CW complexes I.* arxiv:math.QA/0707.0191 (2007).

4. Diep D. N., *Category of noncommutative CW complexes II.*, arxiv:math.QA/0211048.

5. Diep D. N., *Category of Noncommutative CW complexes III*, arxiv:math/0211047 (2007).

6. Eilers S., Loring T. and Pedersen G. K., *Stability of anticommutation relations: an application of NCCW complexes*, J. Reine Angew. Math. **499** (1998), 101–143.

7. Gelfand I. M. and Naimark M. A., *On the embedding of normal rings on to the ring of operators in Hilbert spaces*, Mat. Sbornik **12** (1943), 197–213.

8. Gracia-Bondia, J. M., Varilly J. C. and Figueroa H., *Elements of noncommutative geometry*, Birkhauser, Boston 2001.

9. Hatcher A., *Algebraic topology*, Cambridge University Press 2002.

10. Loring T., *Lifting solutions to perturbing problems in C*-algebras*, AMS monographs 1997.

11. Milani V., Rezaei A. and Mansourbeigi S. M. H., *Morse theory for C*-algebras: a geometric interpretation of some noncommutative manifolds*, Applied General Topology **12(2)** (2011), 175–185.

12. Pedersen G. K. *Pullback and pushout constructions in C*-algebras theory*, J. Funct. Analysis **167** (1999), 243–344.

13. Pop I. and Tofan A., *Cofibrations and bicofibrations for C*-algebras*, Rev. Math. Complut **21(2)** (2008), 529–552.

14. Schochet C., *Topological methods for C*-algebras III, Axiomatic homology*, Pacific Journal of Math.**114(2)** (1984).

15. Wegge-Olsen N. E., *K-theory and C*-algebras: a friendly approach*, Oxford University Press 1993.

V. Milani, Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran and School of Mathematics, Georgia Institute of Theology, Atlanta, GA USA, *e-mail*: v-milani@sbu.ac.ir

S. M. H. Mansourbeigi, Computer Science Department, College of Engineering, Utah State University, UT, USA and Department of Engineering and Manufacturing, Suffolk County Community College, NY USA, *e-mail*: smansourbeigi@aggiemail.usu.edu

A.-A. Rezaei, Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, Iran, *e-mail*: a_rezaei@kashanu.ac.ir