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ON INCLUSION PROBLEMS INVOLVING CAPUTO AND HADAMARD FRACTIONAL DERIVATIVES

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Abstract. In this paper, we study the existence of solutions to new inclusion problems involving both Caputo and Hadamard fractional derivatives, and separated boundary conditions. We apply the modern tools of the fixed point theory to study the cases when the multi-valued map (the right hand-side of the inclusions) takes convex as well as non-convex values. Examples illustrating the abstract results are also presented.

1. Introduction

Recently, a great interest there has been shown in the area of differential equations and inclusions with non-integer order, since fractional order models are found to be more informative and practical than the ones with integer-order. Fractional order differential operators can describe the memory and hereditary properties of systems and processes in view of their nonlocal characteristic. Many books and monographs are devoted to the development of fractional calculus, for instance, see [**4, 18, 20, 23, 25, 27**] and references therein. For applications of fractional calculus in other fields such as science, engineering and technology, we refer the reader to [**10, 15, 20, 22**]. Differential equations, and inclusions equipped with various boundary conditions are widely investigated by many researchers; for details and development of the topic, for example, see the papers [**1, 2, 3, 5, 13, 24**] and the references cited therein.

In a recent work [**26**], the authors studied the existence and uniqueness of solutions for the boundary value problems:

$$(1)\qquad \begin{cases} {}^C D^p({}^H D^q x)(t) = f(t, x(t)), & t \in J := [a, b], \\ \alpha_1 x(a) + \alpha_2({}^H D^q x)(a) = 0, & \beta_1 x(b) + \beta_2({}^H D^q x)(b) = 0, \end{cases}$$

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and

$$
(2) \qquad \begin{cases} {}^H D^q({}^C D^p x)(t) = f(t, x(t)), & t \in J, \\ \alpha_1 x(a) + \alpha_2({}^C D^p x)(a) = 0, & \beta_1 x(b) + \beta_2({}^C D^p x)(b) = 0, \end{cases}
$$

where ${}^C D^p$ and ${}^H D^q$ are the Caputo and Hadamard fractional derivatives starting at a point $a > 0$, of orders p and q, respectively, $0 < p, q \leq 1$, $f\, J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$.

In the present paper, we cover the multi-valued case of the problems addressed in [**26**]. In precise terms, we investigate the inclusion problems:

$$
(3) \qquad \begin{cases} {}^C D^p({}^H D^q x)(t) \in F(t, x(t)), & t \in J := [a, b], \\ \alpha_1 x(a) + \alpha_2({}^H D^q x)(a) = 0, & \beta_1 x(b) + \beta_2({}^H D^q x)(b) = 0, \end{cases}
$$

and

$$
(4) \qquad \begin{cases} {}^H D^q({}^C D^p x)(t) \in F(t, x(t)), & t \in J := [a, b], \\ \alpha_1 x(a) + \alpha_2({}^C D^p x)(a) = 0, & \beta_1 x(b) + \beta_2({}^C D^p x)(b) = 0, \end{cases}
$$

where $F\, J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$) and all other constants are the same as in problems (1) and (2).

We derive the existence results for the inclusion boundary value problems (3) and (4) with the aid of standard fixed point theorems for multi-valued maps. In case of convex valued right-hand side of the inclusions, we use Leray-Schauder nonlinear alternative for multi-valued maps. In the case when F is not necessarily convex valued, we apply nonlinear alternative of Leray-Schauder type for single-valued maps together with the selection theorem due to Bressan and Colombo for lower semi-continuous maps with decomposable values. Finally, in the case of nonconvex valued right hand side of the inclusions, we apply a fixed point theorem for multivalued contractions due to Covitz and Nadler.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and preliminary results about multi-valued maps and fractional calculus, related to our work. Section 3 contains the main results for the boundary value problems (3) and (4). Illustrative examples are presented in Section 4.

2. Preliminaries

In this section, we introduce notations and preliminary facts that are used throughout this paper.

2.1. Fractional Calculus

Here, we recall the definitions of the Riemann-Liouville fractional primitive and derivative [**18**], [**25**].

Definition 2.1. For an at least n-times differentiable function $g\,[a,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined

$$^{C}D^{q}g(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} g^{(n)}(s)\mathrm{d}s, \qquad n-1 < q < n, \quad n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q of a function $g\,[a,\infty) \to \mathbb{R}$ is defined

$$^{RL}I^{q}g(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{g(s)}{(t-s)^{1-q}}\mathrm{d}s, \quad q > 0,$$

provided the integral exists.

Definition 2.3. For an at least n-times differentiable function $g\,[a,\infty) \to \mathbb{R}$, the Caputo-type Hadamard derivative of fractional order q is defined

$$^{H}D^{q}g(t) = \frac{1}{\Gamma(n-q)} \int_a^t \Big(\log\frac{t}{s}\Big)^{n-q-1} \delta^{n} g(s)\frac{\mathrm{d}s}{s}, \qquad n-1 < q < n, \quad n = [q]+1,$$

where $\delta = t\dfrac{\mathrm{d}}{\mathrm{d}t}$, $\log(\cdot) = \log_e(\cdot)$.

Definition 2.4. The Hadamard fractional integral of order q is defined

$$^{H}I^{q}g(t) = \frac{1}{\Gamma(q)} \int_a^t \Big(\log\frac{t}{s}\Big)^{q-1} g(s)\frac{\mathrm{d}s}{s}, \qquad q > 0,$$

provided the integral exists.

Lemma 2.5 ([**18**])**.** *For $q > 0$, the general solution of the fractional differential equation $^{C}D^{q}u(t) = 0$ is given by*

$$u(t) = c_0 + c_1(t-a) + \ldots + c_{n-1}(t-a)^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ $(n = [q]+1)$.

In view of Lemma 2.5, it follows that

$$^{RL}I^{q}(^{C}D^{q}u)(t) = u(t) + c_0 + c_1(t-a) + \ldots + c_{n-1}(t-a)^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $(n = [q]+1)$.

Lemma 2.6 ([**17**])**.** *Let $u \in AC_\delta^n[a,b]$ or $C_\delta^n[a,b]$, and $q \in \mathbb{C}$, where $C_\delta^n[a,b] = \{g\,[a,b] \to \mathbb{C} : \delta^{n-1}g(t) \in AC[a,b]\}$. Then*

$$^{H}I^{q}(^{H}D^{q})u(t) = u(t) - \sum_{k=0}^{n-1} c_k(\log(t/a))^k,$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $(n = [q]+1)$.

In order to define the solution of the boundary value problem (1), we consider its linear variant

$$(5)\qquad \begin{cases} {}^{C}D^{p}({}^{H}D^{q}x)(t) = y(t), & t \in J, \\ \alpha_1 x(a) + \alpha_2({}^{H}D^{q}x)(a) = 0, & \beta_1 x(b) + \beta_2({}^{H}D^{q}x)(b) = 0, \end{cases}$$

where $y \in C(J, \mathbb{R})$.

Lemma 2.7 ([**26**]). *Let*

$$(6)\qquad \Omega := \beta_1\alpha_2 - \alpha_1\Big(\beta_1 \frac{(\log(b/a))^{q}}{\Gamma(q+1)} + \beta_2\Big) \neq 0.$$

Then the unique solution to the separated boundary value problem of sequential Caputo and Hadamard fractional differential equation (5) *is given by the integral equation*

$$(7)\qquad \begin{aligned} x(t) = {} & \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^{q}}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^{H}I^{q}({}^{RL}I^{p}y)(b) + \beta_2\ {}^{RL}I^{p}y(b)\Big] \\ & + {}^{H}I^{q}({}^{RL}I^{p}y)(t), \qquad t \in J. \end{aligned}$$

Next, we consider the linear variant of problem (2)

$$(8)\qquad \begin{cases} {}^{H}D^{q}({}^{C}D^{p}x)(t) = z(t), & t \in J, \\ \alpha_1 x(a) + \alpha_2({}^{C}D^{p}x)(a) = 0, & \beta_1 x(b) + \beta_2({}^{C}D^{p}x)(b) = 0, \end{cases}$$

where $z \in C(J, \mathbb{R})$.

Lemma 2.8. [**26**] *Let*

$$(9)\qquad \Omega^{*} := \beta_1\alpha_2 - \alpha_1\Big(\beta_1 \frac{(b-a)^{p}}{\Gamma(p+1)} + \beta_2\Big) \neq 0.$$

Then the unique solution of the separated boundary value problem of sequential Caputo and Hadamard fractional differential equation (8) *is given by*

$$(10)\qquad \begin{aligned} x(t) = {} & \frac{1}{\Omega^{*}}\Big(\alpha_1 \frac{(t-a)^{p}}{\Gamma(p+1)} - \alpha_2\Big)\Big[\beta_1\ {}^{RL}I^{p}({}^{H}I^{q}z)(b) + \beta_2\ {}^{H}I^{q}z(b)\Big] \\ & + {}^{RL}I^{p}({}^{H}I^{q}z)(t), \qquad t \in J. \end{aligned}$$

2.2. Multivalued Analysis

Let $C(J, \mathbb{R})$ denote the Banach space of continuous functions x from J into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)| : t \in J\}$. By $L^1(J, \mathbb{R})$, we denote the Banach space of Lebesgue integrable functions $y J \to \mathbb{R}$ endowed with the norm $\|y\|_{L^1} = \int_a^b |y(t)|\mathrm{d}t$.

Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $F\, X \to \mathcal{P}(X)$:

(i) is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$,
(ii) is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all bounded set B of X, i.e., $\sup_{x\in B}\{\sup\{|y| : y \in F(x)\}\} < \infty$,

(iii) is called upper semi-continuous (u.s.c for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $F(\mathcal{V}) \subseteq \mathcal{U}$,

(iv) is said to be completely continuous if $F(B)$ is relatively compact for every bounded subset B of X,

(v) has a fixed point if there exists $x \in X$ such that $x \in F(x)$.

For each $y \in C(J, \mathbb{R})$, the set

$$S_{F,y} = \{f \in L^1(J, \mathbb{R}) : f(t) \in F(t, y) \text{ for a.e. } \ t \in J\}$$

is known as the set of selections for the multi-valued map F.

In the following, by $\mathcal{P}_p$, we mean a set of all nonempty subsets of X which have the property "p," where "p" stands for bounded (b), closed (cl), convex (c), compact (cp), etc. Thus $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. Let the graph of G be defined by the set $Gr(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall two useful results concerning closed graphs and upper-semicontinuity.

Lemma 2.9 ([**9**, Proposition 1.2])**.** *If $G\ X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$, i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if $x_n \to x_*$, $y_n \to y_*$, and $y_n \in G(x_n)$ when $n \to \infty$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 2.10 ([**21**])**.** *Let X be a separable Banach space. Let $F\ [0, T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\Theta \circ S_F\ C(J, X) \to \mathcal{P}_{cp,c}(C(J, X)), \qquad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

For more details on multi-valued maps and the proof of the known results cited in this section, we refer the interested reader to the books by Deimling [**9**], Gorniewicz [**12**], and Hu and Papageorgiou [**16**].

3. Existence results

Before stating and proving our main existence results for problems (3) and (4), we give the definition of its solution.

Definition 3.1. A function $x \in C(J, \mathbb{R})$ is said to be a solution of the problem (3) [of the problem (4)] if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, x)$ a.e. on J such that x satisfies the differential equation ${}^C D^p ({}^H D^q x)(t) = v(t)$ $[{}^H D^q ({}^C D^p x)(t) = v(t)]$ on J and the boundary conditions $\alpha_1 x(a) + \alpha_2 ({}^H D^q x)(a) = 0$, $\beta_1 x(b) + \beta_2 ({}^H D^q x)(b) = 0$ $[\alpha_1 x(a) + \alpha_2 ({}^C D^p x)(a) = 0$, $\beta_1 x(b) + \beta_2 ({}^C D^p x)(b) = 0]$, respectively.

Now, we set some notations for sequential Riemann-Liouville and Hadamard fractional integrals of a function with two variables as

$$^{H}I^{q}(^{RL}I^{p}(f_x))(u) = \frac{1}{\Gamma(q)\Gamma(p)}\int_a^u\int_a^s\left(\log\frac{u}{s}\right)^{q-1}(s-\tau)^{p-1}f(\tau,x(\tau))\mathrm{d}\tau\frac{\mathrm{d}s}{s},$$

and

$$^{RL}I^{p}(^{H}I^{q}(f_x))(u) = \frac{1}{\Gamma(p)\Gamma(q)}\int_a^u\int_a^s(u-s)^{p-1}\left(\log\frac{s}{\tau}\right)^{q-1}f(\tau,x(\tau))\frac{\mathrm{d}\tau}{\tau}\mathrm{d}s,$$

where $u \in \{t, b\}$. Also, we use these notations for single Riemann-Liouville and Hadamard fractional integrals of orders p and q, respectively.

For computational convenience, we put

$$\begin{aligned}\Omega_1 =& \frac{1}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)}+|\alpha_2|\right)\left[|\beta_1|\ {}^{H}I^{q}(^{RL}I^{p}(1))(b)+|\beta_2|\ {}^{RL}I^{p}(1)(b)\right]\\ &+{}^{H}I^{q}(^{RL}I^{p}(1))(b),\end{aligned} \tag{11}$$

and

$$\begin{aligned}\Omega_1^* =& \frac{1}{|\Omega^*|}\left(|\alpha_1|\frac{(b-a)^p}{\Gamma(q+1)}+|\alpha_2|\right)\left[|\beta_1|\ {}^{RL}I^{p}(^{H}I^{q}(1))(b)+|\beta_2|\ {}^{H}I^{q}(1)(b)\right]\\ &+{}^{RL}I^{p}(^{H}I^{q}(1))(b).\end{aligned} \tag{12}$$

3.1. The Upper Semicontinuous case

Our first result, dealing with the convex-valued F, is based on Leray-Schauder nonlinear alternative.

Lemma 3.2 (Nonlinear alternative for Kakutani maps [**14**]). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F\,\overline{U} \to \mathcal{P}_{c,cv}(C)$ is a upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is $u \in \partial U$ and $\mu \in (0,1)$ with $u \in \mu F(u)$.*

Definition 3.3. A multivalued map $F\,J\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$, and
(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in J$.
(iii) Further a Carathéodory function F is called L^1-Carathéodory if for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J,\mathbb{R}^+)$ such that $\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\rho(t)$ for all $x \in \mathbb{R}$ with $\|x\| \le \rho$ and for a.e. $t \in J$.

Theorem 3.4. *Assume that:*

(H_1) *$F\ J\times\mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory*
(H_2) *there exists a continuous nondecreasing function $\psi\ [0,\infty) \to (0,\infty)$ and a function $p \in C(J,\mathbb{R}^+)$ such that*
$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|)$ *for each $(t,x) \in J\times\mathbb{R}$,*

(H_3) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\Omega_1 \psi(M)\|p\|} > 1,$$

where Ω_1 *is given by* (11).

Then the boundary value problem (3) *has at least one solution on* J.

Proof. Firstly, we transform the problem (3) into a fixed point problem. Consider the multi-valued map: $N\, C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined by

$$N(x) = \begin{cases} h \in C(J,\mathbb{R}) : \\ h(t) = \dfrac{1}{\Omega}\Big(\alpha_1 \dfrac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^{H}I^q({}^{RL}I^p v)(b) \\ \quad + \beta_2\ {}^{RL}I^p v(b)\Big] + {}^{H}I^q({}^{RL}I^p v)(t) \end{cases}$$

for $v \in S_{F,x}$.

It is clear that fixed points of N are solutions of problem (3). So, we need to show that the operator N satisfies all condition of Lemma 3.2. The proof is given in several steps.

Step 1. *$N(x)$ is convex for each $x \in C(J,\mathbb{R})$.*

Indeed, if h_1, h_2 belong to $N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$h_i(t) = \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^{H}I^q({}^{RL}I^p v_i)(b) + \beta_2\ {}^{RL}I^p v_i(b)\Big] + {}^{H}I^q({}^{RL}I^p v_i)(t), \qquad i = 1,2.$$

Let $0 \le \theta \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} &[\theta h_1 + (1-\theta)h_2](t) \\ &= \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^{H}I^q({}^{RL}I^p[\theta v_1(s) + (1-\theta)v_2(s)])(b) \\ &\quad + \beta_2\ {}^{RL}I^p[\theta v_1(s) + (1-\theta)v_2(s)])(b)\Big] + {}^{H}I^q({}^{RL}I^p[\theta v_1(s) + (1-\theta)v_2(s)])(t). \end{aligned}$$

Since F has convex values ($S_{F,x}$ is convex), therefore, $\theta h_1 + (1-\theta)h_2 \in N(x)$.

Step 2. *$N(x)$ maps bounded sets (balls) into bounded sets in $C(J,\mathbb{R})$.*

For a positive number r, let $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C(J,\mathbb{R})$. Then, for each $h \in N(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^{H}I^q({}^{RL}I^p v)(b) + \beta_2\ {}^{RL}I^p v(b)\Big] + {}^{H}I^q({}^{RL}I^p v)(t).$$

In view of (H_2), for each $t \in J$, we have

$$|h(t)| \le \frac{1}{|\Omega|}\Big(|\alpha_1| \frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\Big)\Big[|\beta_1|\ {}^{H}I^q({}^{RL}I^p|v(s)|)(b) + |\beta_2|\ {}^{RL}I^p|v(s)|(b)\Big] + {}^{H}I^q({}^{RL}I^p|v(s)|)(t)$$

$$\begin{aligned}&\leq \left\{\frac{1}{|\Omega|}\Big(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)}+|\alpha_2|\Big)\Big[|\beta_1|\ {}^{H}I^q({}^{RL}I^p(1))(b)+|\beta_2|\ {}^{RL}I^p(1))(b)\Big]\right.\\&\quad\left.+\ {}^{H}I^q({}^{RL}I^p(1))(b)\right\}\|p\|\psi(\|x\|)\\&\leq \Omega_1\|p\|\psi(r),\end{aligned}$$

which yields

$$\|h\| \leq \Omega_1\|p\|\psi(r).$$

<u>Step 3</u>. *$N(x)$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.*

Let x be any element in B_r and $h \in N(x)$. Then there exists a function $v \in S_{F,x}$ such that for each $t \in J$, we have

$$\begin{aligned}h(t) = {}&\frac{1}{\Omega}\Big(\alpha_1\frac{(\log(t/a))^q}{\Gamma(q+1)}-\alpha_2\Big)\Big[\beta_1\ {}^{H}I^q({}^{RL}I^p v)(b)+\beta_2\ {}^{RL}I^p v(b)\Big]\\&+{}^{H}I^q({}^{RL}I^p v)(t).\end{aligned}$$

Let $\tau_1,\ \tau_2 \in J,\ \tau_1 < \tau_2$. Then

$$\begin{aligned}&|h(\tau_2)-h(\tau_1)|\\&\leq \frac{|\alpha_1|}{|\Omega|\Gamma(q+1)}\left|(\log(\tau_2/a))^q-(\log(\tau_1/a))^q\right|\Big\{|\beta_1|{}^{H}I^q({}^{RL}I^p(|v|))(b)\\&\quad+|\beta_2|\ {}^{RL}I^p(|v|)(b)\Big\}+\left|{}^{H}I^q({}^{RL}I^p(v))(\tau_2)-{}^{H}I^q({}^{RL}I^p(v))(\tau_1)\right|\\&\leq \frac{\|p\|\psi(r)|\alpha_1|}{|\Omega|\Gamma(q+1)}\left|(\log(\tau_2/a))^q-(\log(\tau_1/a))^q\right|\Big\{|\beta_1|\ {}^{H}I^q({}^{RL}I^p(1))(b)\\&\quad+|\beta_2|{}^{RL}I^p(1)(b)\Big\}+\|p\|\psi(r)\left|{}^{H}I^q({}^{RL}I^p(1))(\tau_2)-{}^{H}I^q({}^{RL}I^p(1))(\tau_1)\right|.\end{aligned}$$

The right hand of the above inequality tends to zero independently of $x \in B_r$ as $\tau_1 \to \tau_2$.

As a consequence of Steps 1–3 together with Arzelá-Ascoli theorem, we conclude that $N\ C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous.

Since $N(x)$ is completely continuous, in order to prove that it is u.s.c., it is enough to prove that it has a closed graph by Lemma 2.9, which is done in the next step.

<u>Step 4</u>. *N has a closed graph.*

Let $x_n \to x_*$, $h_n \in N(x_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(x_*)$. Now $h_n \in N(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that for each $t \in J$,

$$\begin{aligned}h_n(t) = {}&\frac{1}{\Omega}\Big(\alpha_1\frac{(\log(t/a))^q}{\Gamma(q+1)}-\alpha_2\Big)\Big[\beta_1\ {}^{H}I^q({}^{RL}I^p v_n)(b)+\beta_2\ {}^{RL}I^p v_n(b)\Big]\\&+{}^{H}I^q({}^{RL}I^p v_n)(t).\end{aligned}$$

Therefore, we must show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in J$,

$$\begin{aligned}h_*(t) = {}&\frac{1}{\Omega}\Big(\alpha_1\frac{(\log(t/a))^q}{\Gamma(q+1)}-\alpha_2\Big)\Big[\beta_1\ {}^{H}I^q({}^{RL}I^p v_*)(b)+\beta_2\ {}^{RL}I^p v_*(b)\Big]\\&+{}^{H}I^q({}^{RL}I^p v_*)(t).\end{aligned}$$

Consider the continuous linear operator $\Theta\ L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$v \to \Theta(v)(t) = \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^H I^q({}^{RL}I^p v)(b) + \beta_2\ {}^{RL}I^p v(b)\Big] + {}^H I^q({}^{RL}I^p v)(t).$$

Observe that $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$, and thus it follows by Lemma 2.10 that $\Theta \circ S_{F,x}$ is a closed graph operator.
Moreover, we have

$$h_n \in \Theta(S_{F,x_n}).$$

Since $x_n \to x_*$, Lemma 2.10 implies that

$$h_*(t) = \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^H I^q({}^{RL}I^p v_*)(b) + \beta_2\ {}^{RL}I^p v_*(b)\Big] + {}^H I^q({}^{RL}I^p v_*)(t)$$

for some $v_* \in S_{F,x_*}$

Step 5. We show there exists an open set $U \subseteq C(J,\mathbb{R})$ with $x \notin \lambda N(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$.

Let $\lambda \in (0,1)$ and $x \in \lambda N(x)$. Then there exists $v \in L^1(J,\mathbb{R})$ with $v \in S_{F,x}$ such that for $t \in J$, we have

$$x(t) = \lambda\frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^H I^q({}^{RL}I^p v)(b) + \beta_2\ {}^{RL}I^p v(b)\Big] + \lambda\ {}^H I^q({}^{RL}I^p v)(t).$$

In view of (H_2), for each $t \in J$, we have

$$\begin{aligned}
|x(t)| &\le \frac{1}{|\Omega|}\Big(|\alpha_1| \frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\Big)\Big[|\beta_1|\ {}^H I^q({}^{RL}I^p|v(s)|)(b) \\
&\quad + |\beta_2|\ {}^{RL}I^p|v(s)|(b)\Big] + {}^H I^q({}^{RL}I^p|v(s)|)(t) \\
&\le \Big\{\frac{1}{|\Omega|}\Big(|\alpha_1| \frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\Big)\Big[|\beta_1|\ {}^H I^q({}^{RL}I^p(1))(b) \\
&\quad + |\beta_2|\ {}^{RL}I^p(1)(b)\Big] + {}^H I^q({}^{RL}I^p(1))(b)\Big\}\|p\|\psi(\|x\|) \\
&= \Omega_1\|p\|\psi(\|x\|),
\end{aligned}$$

which can alternatively be written as

$$\frac{\|x\|}{\Omega_1\psi(\|x\|)\|p\|} \le 1.$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(J,\mathbb{R}) : \|x\| < M\}.$$

The operator $N\overline{U} \to \mathcal{P}(C(J,\mathbb{R}))$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.2), we deduce that N has a fixed point $x \in \overline{U}$ which is a solution of the boundary value problem (3). This completes the proof. □

Corollary 3.5. *Assume that* (H_1), (H_2), *and the following condition hold:*

$(H_3)^*$ *There exists a constant* $M^* > 0$ *such that*

$$\frac{M^*}{\Omega_1^* \psi(M)\|p\|} > 1,$$

where Ω_1^* *is given by* (12).

Then the boundary value problem (4) *has at least one solution on* J.

3.2. The lower semicontinuous case

In this subsection, we study the case when F is not necessarily convex valued by applying the nonlinear alternative of Leray-Schauder type together with a selection theorem by Bressan and Colombo [**6**] for lower semi-continuous maps with decomposable values.

Definition 3.6. Let A be a subset of $I \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in I and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

Definition 3.7. A subset $\mathcal{A}$ of $L^1(I,\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{A}$ and measurable $\mathcal{J} \subset I$, the function $u\chi_{\mathcal{J}} + v\chi_{I-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Lemma 3.8 ([**6**]). *Let* Y *be a separable metric space and let* $N\, Y \to \mathcal{P}(L^1(I,\mathbb{R}))$ *be a lower semi-continuous (l.s.c.) multivalued operator with nonempty closed and decomposable values. Then* N *has a continuous selection, that is, there exists a continuous function (single-valued)* $h\, Y \to L^1(I,\mathbb{R})$ *such that* $h(x) \in N(x)$ *for every* $x \in Y$.

Theorem 3.9. *Assume that* (H_2), (H_3), *and the following condition hold:*

(H_4) $F\, J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*
- (a) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*
- (b) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in J$.

Then the boundary value problem (3) *has at least one solution on* J.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type [**11**]. Then, by Lemma 3.8, there exists a continuous function $v\, C^1(J,\mathbb{R}) \to L^1(J,\mathbb{R})$ such that $v(x) \in \mathcal{F}(x)$ for all $v \in C(J,\mathbb{R})$, where $\mathcal{F}\, C(J \times \mathbb{R}) \to \mathcal{P}(L^1(J,\mathbb{R}))$ is the Nemytskii operator associated with F defined by

$$\mathcal{F}(v) = \{w \in L^1(J,\mathbb{R}) : w(t) \in F(t, v(t)) \text{ for a.e. } t \in J\}.$$

Consider the problem

$$\begin{cases} {}^C D^p({}^H D^q x)(t) = f(x(t)), & t \in J, \\ \alpha_1 x(a) + \alpha_2({}^H D^q x)(a) = 0, & \beta_1 x(b) + \beta_2({}^H D^q x)(b) = 0. \end{cases} \tag{13}$$

Note that x is a solution to the boundary value problem (3) if $x \in C^1(J, \mathbb{R})$ is a solution of (13). In order to transform the problem (13) into a fixed point problem, we define an operator $\overline{N}$

$$\overline{N}x(t) = \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1 \ {}^H I^q({}^{RL}I^p f(x(s))(b) + \beta_2 \ {}^{RL}I^p f(x(s))(b)\Big] + {}^H I^q({}^{RL}I^p f(x(s))(t).$$

It can easily shown that $\overline{N}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.4. So, we omit it. This completes the proof. □

Corollary 3.10. *Assume that* $(H_2), (H_3)^*$, *and* (H_4) *hold. Then the boundary value problem* (4) *has at least one solution on* J.

3.3. The Lipschitz case

Here, we prove the existence of solutions for the boundary value problem (3) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [**8**].

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d \, \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a; b)$ and $d(a, B) = \inf_{b\in B} d(a; b)$. Then $(\mathcal{P}_{b,cl}(X),\ H_d)$ is a metric space (see [**19**]).

Definition 3.11. A multivalued operator $N \, X \to \mathcal{P}_{cl}(X)$ is called

(a) θ-Lipschitz if and only if there exists $\theta > 0$ such that

$$H_d(N(x), N(y)) \leq \theta d(x, y) \qquad \text{for each } x, y \in X,$$

(b) a contraction if and only if it is θ-Lipschitz with $\theta < 1$.

Lemma 3.12 ([**8**]). *Let* (X, d) *be a complete metric space. If* $N \, X \to \mathcal{P}_{cl}(X)$ *is a contraction, then* $Fix N \neq \emptyset$.

Theorem 3.13. *Assume that the following conditions hold:*

(H_5) $F \, J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x) \, J \to \mathcal{P}_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$,

(H_6) $H_d(F(t, x), F(t, \bar{x})) \leq m(t)|x - \bar{x}|$ *for almost all* $t \in J$ *and* $x, \bar{x} \in \mathbb{R}$ *with* $m \in C(J, \mathbb{R}^+)$ *and* $d(0, F(t, 0)) \leq m(t)$ *for almost all* $t \in J$.

Then the boundary value problem (3) *has at least one solution on* J *if*

$$\Omega_1 \|m\| < 1,$$

where Ω_1 *is given by* (11).

Proof. We transform the boundary value problem (3) into a fixed point problem by considering the operator $N\ C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined in the beginning of the proof of Theorem 3.2. The proof will be complete once it is shown that the operator N satisfies the assumptions of Lemma 3.12. This established in two steps.
Step I. *$N(x)$ is nonempty and closed for every $v \in S_{F,x}$.*

Since the set-valued map $F(\cdot, x(\cdot))$ is measurable by the measurable selection theorem (e.g., [**7**, Theorem III.6]), it admits a measurable selection $v\ J \to \mathbb{R}$. Moreover, by the assumption (H_6), we have

$$|v(t)| \leq m(t) + m(t)|x(t)|,$$

i.e., $v \in L^1(J,\mathbb{R})$, and hence F is integrably bounded. Therefore, $S_{F,x} \neq \emptyset$.

Next, we show that $N(x)$ is closed for each $x \in C(J,\mathbb{R})$. Let $\{u_n\}_{n\geq 0} \in N(x)$ be such that $u_n \to u$, $(n \to \infty)$ in $C(J,\mathbb{R})$. Then $u \in C(J,\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that for each $t \in J$,

$$\begin{aligned} u_n(t) = {} & \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^{H}I^{q}({}^{RL}I^{p} v_n)(b) + \beta_2\ {}^{RL}I^{p} v_n(b)\Big] \\ & + {}^{H}I^{q}({}^{RL}I^{p} v_n)(t). \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus $v \in S_{F,x}$ and for each $t \in J$, we have

$$\begin{aligned} u_n(t) \to v(t) = {} & \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^{H}I^{q}({}^{RL}I^{p} v)(b) + \beta_2\ {}^{RL}I^{p} v(b)\Big] \\ & + {}^{H}I^{q}({}^{RL}I^{p} v)(t). \end{aligned}$$

Hence $u \in N(x)$.
Step II. *There exists $0 < \theta < 1$ ($\theta = \Omega_1\|m\|$) such that*

$$H_d(N(x), N(\bar{x})) \leq \theta\|x - \bar{x}\| \qquad \text{for each } x, \bar{x} \in C(J,\mathbb{R}).$$

Let $x, \bar{x} \in C(J,\mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in J$

$$\begin{aligned} h_1(t) = {} & \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1\ {}^{H}I^{q}({}^{RL}I^{p} v_1)(b) + \beta_2\ {}^{RL}I^{p} v_1(b)\Big] \\ & + {}^{H}I^{q}({}^{RL}I^{p} v_1)(t). \end{aligned}$$

By (H_6), we have

$$H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w(t) \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|, \qquad t \in J.$$

Define $U\ J \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable ([**7**, Proposition III.4]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in J$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in J$, let us define

$$h_2(t) = \frac{1}{\Omega}\Big(\alpha_1 \frac{(\log(t/a))^q}{\Gamma(q+1)} - \alpha_2\Big)\Big[\beta_1 \ {}^{H}I^q({}^{RL}I^p v_2)(b) + \beta_2 \ {}^{RL}I^p v_2(b)\Big]$$
$$+ {}^{H}I^q({}^{RL}I^p v_2)(t).$$

In consequence, we get

$$\begin{aligned}|h_1(t) - h_2(t)| \leq & \frac{1}{|\Omega|}\Big(|\alpha_1| \frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\Big)\Big[|\beta_1| \ {}^{H}I^q({}^{RL}I^p|v_2(s) - v_1(s)|)(b) \\ & + |\beta_2| \ {}^{RL}I^p|v_2(s) - v_1(s)|(b)\Big] + {}^{H}I^q({}^{RL}I^p|v_2(s) - v_1(s)|)(b) \\ \leq & \left\{\frac{1}{|\Omega|}\left(|\alpha_1| \frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right)\left[|\beta_1| \ {}^{H}I^q({}^{RL}I^p(1))(b)\right.\right. \\ & \left.\left. + |\beta_2| \ {}^{RL}(1))(b)\right] + {}^{H}I^q({}^{RL}(1))(b)\right\}\|m\|\|x - \overline{x}\|.\end{aligned}$$

Hence

$$\|h_1 - h_2\| \leq \Omega_1\|m\|\|x - \overline{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(N(x), N(\bar{x})) \leq \Omega_1\|m\|\|x - \overline{x}\|.$$

Since N is a contraction, it follows by Lemma 3.12 that N has a fixed point x which is a solution of (3). This completes the proof. $\square$

Corollary 3.14. *Assume that* (H_5) *and* (H_6) *are satisfied. Then the boundary value problem* (4) *has at least one solution on* J, *provided that* $\Omega_1^*\|m\| < 1$, *where* Ω_1^* *is given by* (12).

4. Examples

Example 4.1. Consider the following Caputo and Hadamard fractional differential inclusion with separated boundary conditions

$$\begin{cases} {}^{C}D^{1/2}\left({}^{H}D^{2/3}x\right) \in F(t, x(t)), & t \in \Big[\frac{1}{3}, \frac{8}{3}\Big], \\ \frac{3}{4}x\Big(\frac{1}{3}\Big) + \frac{2}{5}\left({}^{H}D^{2/3}x\right)\left(\frac{1}{3}\right) = 0, & \frac{1}{6}x\Big(\frac{8}{3}\Big) + \frac{5}{7}\left({}^{H}D^{2/3}x\right)\left(\frac{8}{3}\right) = 0. \end{cases} \tag{14}$$

Here $p = 1/2$, $q = 2/3$, $a = 1/3$, $b = 8/3$, $\alpha_1 = 3/4$, $\alpha_2 = 2/5$, $\beta_1 = 1/6$, and $\beta_2 = 5/7$. Using the given values, we find that $|\Omega| \approx 0.6946321201$, ${}^{RL}I^{1/2}(1)(8/3) \approx 1.723627649$, ${}^{H}I^{2/3}\left({}^{RL}I^{1/2}(1)\right)(8/3) \approx 1.908401883$, and $\Omega_1 \approx 5.819227275$.

(i) Consider the multivalued map $F\ [1/3, 8/3] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$x \to F(t,x) = \Big[\frac{\mathrm{e}^{-x^2}}{2(25+t^2)}, \frac{1}{17+3t}\Big(\frac{|x|^3}{2(1+|x|)} + \frac{x^2}{1+|x|} + \frac{1}{2}\Big)\Big]. \tag{15}$$

Clearly the multivalued map F satisfies condition (H_1) and that

$$\|F(t,x)\|_{\mathcal{P}} \leq \frac{1}{17+3t}\left(\frac{1}{2}x^2 + |x| + \frac{1}{2}\right) := p(t)\psi(|x|),$$

which yields $\|p\| = 1/18$ and $\psi(\|x\|) = (1/2)\|x\|^2 + \|x\| + (1/2)$. Therefore, the condition (H_2) is fulfilled. By direct computation, there exists a constant $M \in (0.254319013, 3.932069367)$ satisfying condition (H_3). Hence all assumptions of Theorem 3.4 hold and hence the problem (14) with F given by (15) has at least one solution on $[1/3, 8/3]$.

(ii) Let the multivalued map $F\ [1/3, 8/3] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be defined by

$$x \to F(t,x) = \left[0, \frac{1}{8+12t}\Big(\frac{x^2+2|x|}{1+|x|}\Big) + \frac{1}{40}\right]. \tag{16}$$

Choosing $m(t) = 2/(8+12t)$, we find that $H_d(F(t,x), F(t,y)) \leq m(t)|x-y|$ and $d(0, F(t,0)) = 1/40 \leq m(t)$ for almost all $t \in [1/3, 8/3]$. In addition we get $\|m\| = 1/6$ which leads to $\Omega_1\|m\| \approx 0.9698712125 < 1$. By the conclusion of Theorem 3.13, the problem (14) with F given by (16), has at least one solution on $[1/3, 8/3]$.

References

1. Agarwal R. P., Ahmad B. and Alsaedi A., *Fractional-order differential equations with anti-periodic boundary conditions: a survey*, Bound. Value Probl. **2017** (2017), 173.

2. Agarwal R. P., Ahmad B. and Alsaedi A., *New existence results for nonlinear fractional differential equations with three-point integral boundary conditions*, Adv. Difference Equ. **2011** (2011), Art. ID 107384.

3. Ahmad B. and Ntouyas S. K., *Nonlinear fractional differential equations and inclusions of arbitrary order and multi-strip boundary conditions*, Electron. J. Differ. Equ. **2012** (2012), Art. ID 98.

4. Ahmad B., Alsaedi A., Ntouyas S. K. and Tariboon J., *Hadamard-type Fractional Differential Equations, Inclusions and Inequalities*, Springer, Cham, Switzerland, 2017.

5. Bai Z. B. and Sun W., *Existence and multiplicity of positive solutions for singular fractional boundary value problems*, Comput. Math. Appl. **63** (2012), 1369–1381.

6. Bressan A. and Colombo G., *Extensions and selections of maps with decomposable values*, Studia Math. **90** (1988), 69–86.

7. Castaing C. and Valadier M., *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

8. Covitz H. and Nadler S. B. Jr., *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math. **8** (1970), 5–11.

9. Deimling K., *Multivalued Differential Equations*, De Gruyter, Berlin, 1992.

10. Diethelm K. and Ford N. J., *Analysis of fractional differential equations*, J. Math. Anal. Appl. **265** (2002), 229–248.

11. Frigon M., *Théorèmes d'existence de solutions d'inclusions différentielles, Topological Methods in Differential Equations and Inclusions*, (A. Granas and M. Frigon, eds.), NATO ASI Series C, Vol. 472, Kluwer Acad. Publ., Dordrecht, 1995, 51–87.

12. Górniewicz L., *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and its Applications 495, Kluwer Academic Publishers, Dordrecht, 1999.

13. Graef J. R., Kong L. and Kong Q., *Application of the mixed monotone operator method to fractional boundary value problems*, Fract. Calc. Differ. Calc **2** (2011), 554–567.

14. Granas A. and Dugundji J., *Fixed Point Theory,* Springer-Verlag, New York, 2005.

15. Heymans N. and Podlubny I., *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives*, Rheologica Acta **45**(5) 2006, 765–772.

16. Hu S. and Papageorgiou N., *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer Academic Publishers, Dordrecht, 1997.

17. Jarad F., Abdeljawad T. and Baleanu D., *Caputo-type modification of the Hadamard fractional derivatives*, Adv. Difference Equ. **2012** (2012), Art. ID 142.

18. Kilbas A. A., Srivastava H. M. and Trujillo J. J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies 204, Elsevier Science B.V., Amsterdam, 2006.

19. Kisielewicz M., *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.

20. Kiryakova V., *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics Series 301, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1994.

21. Lasota A. and Opial Z., *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13** (1965), 781–786.

22. Mainardi F., *Fractional calculus: Some basic problems in continuum and statistical mechanis*, in: Fractals and Fractional Calculus in Continuum Mechanics (A. Carpinteri and F. Mainardi, eds.), Springer-Verlag, Wien, 1997,

23. Miller K. S. and Ross B., *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

24. Ntouyas S. K., and Tariboon J. and Sudsutad W., *Boundary value problems for Riemann-Liouville fractional differential inclusions with nonlocal Hadamard fractional integral conditions*, Meditter. J. Math. **13** (2016), 939–954.

25. Podlubny I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.

26. Tariboon J., Cuntavepani A., Ntouyas S. K. and Nithiarayaphaks W., *Separated boundary value problems of sequential Caputo and Hadamard fractional differential equations*, J. Funct. Spaces **2018** Art. ID 6974046.

27. Samko S. G., Kilbas A. A. and Marichev O. I., *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

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