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ON $\varphi - |A, \delta|_k$ SUMMABILITY OF ORTHOGONAL SERIES

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Abstract. We prove two theorems on $\varphi - |A, \delta|_k$ summability of orthogonal series. In addition, several known and new results are deduced as corollaries of the main results.

1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums s_n, (C, α) the Cesàro matrix with index α. If σ_n^{α} denotes the nth term of the $(C, \alpha)-$transform of $s := \{s_n\}$, then Flett [**3**] define absolute summability of order $k \geq 1$ as follows. A series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$ if,

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^{\alpha} - \sigma_{n-1}^{\alpha}|^k < \infty. \tag{1}$$

Let $\{p_n\}$ be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \qquad (P_{-i} = p_{-i} = 0,\ i \geq 1).$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence $\{t_n\}$ of the Riesz means of the sequence $\{s_n\}$, generated by the sequence of the coefficients $\{p_n\}$ (see [**4**]), where the sequence $\{s_n\}$ is a sequence of partial sums of $\sum_{n=0}^{\infty} a_n$.

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|R, p_n|_k$, $k \geq 1$, if (see [**2**])

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{2}$$

Let $A := (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of non-zero diagonal entries. Then A defines the sequence-to-sequence transformation,

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mapping the sequence $s := \{s_n\}$ to $As := \{A_n(s)\}$, where

$$A_n(s) := \sum_{v=0}^{n} a_{nv} s_v, \qquad n = 0, 1, 2, \dots$$

The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be absolutely summable $|A|_k$ ($|A, \delta|_k$), $k \geq 1$, $\delta \geq 0$, if (see [**3**])

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\Delta} A_n(s)|^k \quad \left(\sum_{n=1}^{\infty} n^{\delta k + k - 1} |\bar{\Delta} A_n(s)|^k \right)$$

converges, where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s),$$

and, we write

$$\sum_{n=0}^{\infty} a_n \in |A|_k \qquad \big(\in |A, \delta|_k \big),$$

respectively.

Let $\{\psi_n(x)\}$ be an orthonormal system defined in the interval (a, b). We assume that $f(x)$ belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x), \tag{3}$$

where $c_n = \int_a^b f(x) \psi_n(x) \mathrm{d}x$, $(n = 0, 1, 2, \dots)$.

Let p_n be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \qquad \text{as } n \to \infty.$$

The sequence-to-sequence transformation

$$\overline{t_n} = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v$$

defines the sequence $\{\overline{t_n}\}$ of the Nörlund means of the sequence $\{s_n\}$, generated by the sequence of the coefficients $\{p_n\}$, where the sequence $\{s_n\}$ is a sequence of partial sums of $\sum_{n=0}^{\infty} a_n$.

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} (P_n / p_n)^{k-1} \, |\overline{t_n} - \overline{t_{n-1}}|^k < \infty.$$

Among others, Y. Okuyama [**6**] concerning the $|N, p_n|_k$, $1 \leq k \leq 2$, summability of orthogonal series (3) proved the following two theorems.

Theorem 1.1. *Let $1 \leq k \leq 2$ and $\{\lambda_n\}$ be a positive sequence. If $\{p_n\}$ is a positive sequence and the series*

$$\sum_{n=0}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{j=1}^{n} p_{n-j}^2 \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big)^2 \lambda_j^2 |c_j|^2 \right\}^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} \lambda_n c_n \psi_n(x)$$

is summable $|N,p_n|_k$ almost everywhere.

For the second one, first of all he write

$$w^{(k)}(j) = j^{-1} \sum_{n=j}^{\infty} \frac{n^{\frac{2}{k}} p_n^{\frac{2}{k}} p_{n-j}^2}{P_n^{2+\frac{2}{k}}} \Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big)^2.$$

Theorem 1.2. *Let $1 \le k \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If $\{p_n\}$ is a positive non-increasing sequence and the series $\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) w^{(k)}(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n \psi_n(x)$ is $|N,p_n|_k$ summable almost everywhere.*

Theorem 1.1 includes a result of Singh [**10**] which is an extension, for trigonometric series due to Pati [**9**], Ul'yanov [**11**], and Wang [**12**], until Theorem 1.2 generalizes a theorem of Okuyama [**7**].

Given a normal matrix $A := (a_{nv})$, we associate two lower semi matrices $\bar{A} := (\bar{a}_{nv})$ and $\hat{A} := (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} := \sum_{i=v}^{n} a_{ni}, \qquad n, i = 0, 1, 2, \dots$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \qquad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \qquad n = 1, 2, \dots$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively.

We have generalized Theorems 1.1–1.2 proving the following (since, if we take $a_{nv} = \frac{p_v}{P_v}$, $\delta = 0$, then $|A,\delta|_k$ summability is the same as $|R,p_n|_k$ summability).

Theorem 1.3 ([**5**]). *If for $1 \le k \le 2$, the series*

$$\sum_{n=1}^{\infty} \left[n^{2(\delta+1-1/k)} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 \lambda_j^2 |c_j|^2 \right]^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} \lambda_n c_n \psi_n(x) \tag{4}$$

is $|A,\delta|_k$ summable almost everywhere.

Theorem 1.4 ([**5**]). *Let $1 \le k \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If* *the following series* $\sum_{n=1}^{\infty}|c_n|^2\Omega^{\frac{2}{k}-1}(n)w^{(k)}(A,\delta;n)$ *converges, then the orthogonal series* $\sum_{n=0}^{\infty}c_n\psi_n(x)\in|A,\delta|_k$ *almost everywhere, where* $w^{(k)}(A,\delta;n)$ *is defined by*

$$w^{(k)}(A,\delta;j):=\frac{1}{j^{\frac{2}{k}-1}}\sum_{n=j}^{\infty}n^{2(\delta+1)}|\hat{a}_{n,j}|^2.$$

Let (φ_n) be a sequence of positive real numbers. The infinite series $\sum_{n=0}^{\infty}a_n$ is said to be summable $\varphi-|A,\delta|_k$, $k\geq 1$, $\delta\geq 0$, if (see [**8**])

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}|\bar{\Delta}A_n(s)|^k<\infty,$$

and in brief, we write

$$\sum_{n=0}^{\infty}a_n\in\varphi-|A,\delta|_k.$$

The main purpose of the present paper is to generalize further Theorems 1.3–1.4 for $\varphi-|A,\delta|_k$ summability of the orthogonal series (3), where $1\leq k\leq 2$, $\delta\geq 0$.

Due to B. Levi (see, for example [**1**]), the following lemma is often used in the theory of functions. It will help us to prove main results.

Lemma 1.5. *If* $h_n(t)\in L(U)$ *are non-negative functions and*

$$\sum_{n=1}^{\infty}\int_U h_n(t)\mathrm{d}t<\infty,$$

then the series

$$\sum_{n=1}^{\infty}h_n(t)$$

converges (absolutely) almost everywhere on U *to a function* $h(t)\in L(U)$.

Throughout this paper, K denotes a positive constant that it may depend only on k, and be different in different relations.

2. Main results

We prove the following two theorems.

Theorem 2.1. *If for* $1\leq k\leq 2$, *the series*

$$\sum_{n=1}^{\infty}\left[\psi_n^{2(\delta+1-1/k)}\sum_{j=0}^{n}|\hat{a}_{n,j}|^2\lambda_j^2|c_j|^2\right]^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty}\lambda_n c_n\psi_n(x) \tag{5}$$

is $\varphi-|A,\delta|_k$ *summable almost everywhere.*

Proof. For the matrix transform $A_n(s)(x)$ of the partial sums of the orthogonal series $\sum_{n=0}^{\infty}\lambda_n c_n\psi_n(x)$, we have

$$\begin{aligned}A_n(s)(x) &= \sum_{v=0}^{n} a_{nv}s_v(x) = \sum_{v=0}^{n} a_{nv}\sum_{j=0}^{v}\lambda_j c_j\psi_j(x)\\ &= \sum_{j=0}^{n}\lambda_j c_j\psi_j(x)\sum_{v=j}^{n} a_{nv} = \sum_{j=0}^{n}\bar{a}_{nj}\lambda_j c_j\psi_j(x),\end{aligned}$$

where $\sum_{j=0}^{v}\lambda_j c_j\psi_j(x)$ is the partial sum of order v of the series (5). Hence

$$\begin{aligned}\bar{\Delta}A_n(s)(x) &= \sum_{j=0}^{n}\bar{a}_{nj}\lambda_j c_j\psi_j(x) - \sum_{j=0}^{n-1}\bar{a}_{n-1,j}\lambda_j c_j\psi_j(x)\\ &= \bar{a}_{nn}\lambda_n c_n\psi_n(x) + \sum_{j=0}^{n-1}\left(\bar{a}_{n,j}-\bar{a}_{n-1,j}\right)\lambda_j c_j\psi_j(x)\\ &= \hat{a}_{nn}\lambda_n c_n\psi_n(x) + \sum_{j=0}^{n-1}\hat{a}_{n,j}\lambda_j c_j\psi_j(x) = \sum_{j=0}^{n}\hat{a}_{n,j}\lambda_j c_j\psi_j(x).\end{aligned}$$

Using Hölder's inequality with $p=\frac{2}{k}>1$ and orthogonality to the latter equality, we have that

$$\begin{aligned}\int_a^b|\bar{\Delta}A_n(s)(x)|^k\mathrm{d}x &\le (b-a)^{1-\frac{k}{2}}\left(\int_a^b|A_n(s)(x)-A_{n-1}(s)(x)|^2\mathrm{d}x\right)^{\frac{k}{2}}\\ &= (b-a)^{1-\frac{k}{2}}\left(\int_a^b\Big|\sum_{j=0}^{n}\hat{a}_{n,j}\lambda_j c_j\psi_j(x)\Big|^2\mathrm{d}x\right)^{\frac{k}{2}}\\ &= (b-a)^{1-\frac{k}{2}}\left[\sum_{j=0}^{n}|\hat{a}_{n,j}|^2\lambda_j^2|c_j|^2\right]^{\frac{k}{2}}.\end{aligned}$$

Thus, the series

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\int_a^b|\bar{\Delta}A_n(s)(x)|^k\mathrm{d}x \le K\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\left[\sum_{j=0}^{n}|\hat{a}_{n,j}|^2\lambda_j^2|c_j|^2\right]^{\frac{k}{2}} \tag{6}$$

converges by the assumption. According to the Lemma of Beppo-Lévi, the proof of the theorem ends. □

If, we put

$$\Phi^{(k)}(A,\delta;j) := \frac{1}{\varphi_j^{\frac{2}{k}-1}}\sum_{n=j}^{\infty}\varphi_n^{2\left(\delta+\frac{1}{k}\right)}|\hat{a}_{n,j}|^2, \tag{7}$$

then the following theorem holds true.

Theorem 2.2. *Let $1\le k\le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/\psi_n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{\psi_n\Omega(n)}$ converges. If* *the following series* $\sum_{n=1}^{\infty}|c_n|^2\Omega^{\frac{2}{k}-1}(n)\Phi^{(k)}(A,\delta;n)$ *converges, then the orthogonal series* $\sum_{n=0}^{\infty}c_n\psi_n(x)\in\varphi-|A,\delta|_k$ *almost everywhere, where* $\Phi^{(k)}(A,\delta;n)$ *is defined by* (7).

Proof. Using (6) and applying Hölder's inequality with $p=\frac{2}{2-k}>1$, we get that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\int_a^b|\bar{\Delta}A_n(s)(x)|^k\mathrm{d}x\\
&\leq K\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\left[\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right]^{\frac{k}{2}}\\
&=K\sum_{n=1}^{\infty}\frac{1}{(\varphi_n\Omega(n))^{\frac{2-k}{2}}}\left[(\varphi_n\Omega(n))^{\frac{2}{k}-1}\varphi_n^{2(\delta+1-1/k)}\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right]^{\frac{k}{2}}\\
&\leq K\left(\sum_{n=1}^{\infty}\frac{1}{(\varphi_n\Omega(n))}\right)^{\frac{2-k}{2}}\left[\sum_{n=1}^{\infty}(\varphi_n\Omega(n))^{\frac{2}{k}-1}\varphi_n^{2(\delta+1-1/k)}\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right]^{\frac{k}{2}}\\
&\leq K\left\{\sum_{j=1}^{\infty}|c_j|^2\sum_{n=j}^{\infty}\left(\frac{\varphi_n^2\Omega(n)}{\varphi_n}\right)^{\frac{2}{k}-1}\varphi_n^{2(\delta+1-1/k)}|\hat{a}_{n,j}|^2\right\}^{\frac{k}{2}}\\
&\leq K\left\{\sum_{j=1}^{\infty}|c_j|^2\left(\frac{\Omega(j)}{\varphi_j}\right)^{\frac{2}{k}-1}\sum_{n=j}^{\infty}\varphi_n^{\frac{4}{k}-2}\varphi_n^{2(\delta+1-1/k)}|\hat{a}_{n,j}|^2\right\}^{\frac{k}{2}}\\
&\leq K\left\{\sum_{j=1}^{\infty}|c_j|^2\left(\frac{\Omega(j)}{\varphi_j}\right)^{\frac{2}{k}-1}\sum_{n=j}^{\infty}\varphi_n^{2(\delta+1/k)}|\hat{a}_{n,j}|^2\right\}^{\frac{k}{2}}\\
&=K\left\{\sum_{j=1}^{\infty}|c_j|^2\Omega^{\frac{2}{k}-1}(j)\Phi^{(k)}(A,\delta;j)\right\}^{\frac{k}{2}},
\end{aligned}$$

which is finite by virtue of the hypothesis of the theorem, and this completes the proof of the theorem. □

Remark. If, we take $\psi_n=n$, then Theorems 1.3–1.4 are implied as immediate consequences of the the main results.

Remark. Let us show that Theorem 1.1 is included in Theorem 2.1. Namely, for $a_{n,v}=\frac{p_{n-v}}{P_n}$, we get

$$\begin{aligned}
\hat{a}_{n,j}&=\bar{a}_{n,j}-\bar{a}_{n-1,j}\\
&=\frac{1}{P_n}\sum_{i=j}^{n}p_{n-i}-\frac{1}{P_{n-1}}\sum_{i=j}^{n-1}p_{n-1-i}\\
&=\frac{1}{P_nP_{n-1}}\left(P_{n-1}P_{n-j}-P_nP_{n-1-j}\right)
\end{aligned}$$

$$= \frac{1}{P_n P_{n-1}} \left((P_n - p_n)P_{n-j} - P_n(P_{n-j} - p_{n-j})\right)$$
$$= \frac{p_n}{P_n P_{n-1}} \left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right) p_{n-j}.$$

Whence, putting this equality to Theorem 2.1, for $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, we immediately obtain Theorem 1.1.

The following corollaries follow also from the main results ($\delta = 0$, $\psi_n = n$).

Corollary 2.3 ([**5**]). *If for $1 \le k \le 2$, the series*

$$\sum_{n=1}^{\infty} \left[n^{2(1-1/k)} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 \lambda_j^2 |c_j|^2 \right]^{\frac{k}{2}}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} \lambda_n c_n \psi_n(x)$$

is $|A|_k$ summable almost everywhere.

Corollary 2.4 ([**5**]). *Let $1 \le k \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If the following series $\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) w^{(k)}(A;n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n \psi_n(x) \in |A|_k$ almost everywhere, where*

$$w^{(k)}(A;n) = \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^2 |\hat{a}_{n,j}|^2.$$

Also, taking $\delta = 0$, $\psi_n = n$, and $k = 1$ in our main results, we obtain the following corollary

Corollary 2.5 ([**5**]). *If the series*

$$\sum_{n=1}^{\infty} \left(\sum_{j=0}^{n} |\hat{a}_{n,j}|^2 \lambda_j^2 |c_j|^2 \right)^{\frac{1}{2}}$$

converges, then the orthogonal series $\sum_{n=0}^{\infty} \lambda_n c_n \psi_n(x)$ is $|A|$ summable almost everywhere.

Corollary 2.6 ([**5**]). *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series*

$$\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$$

converges. If the following series

$$\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) w(A;n)$$

converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n\psi_n(x) \in |A|$ *almost everywhere, where* $w(A;n)$ *is defined by*

$$w(A;j) := j^{-1}\sum_{n=j}^{\infty} n^2|\hat{a}_{n,j}|^2.$$

Remark. Finally, it should be also noted that our main results, as special cases, contain the results on $|C,1|$, $|C,1|_k$, $|C,\alpha|$, $|C,\alpha|_k$, $|H,p|$, $|H,p|_k$, $|\bar{N},p_n|$, $|\bar{N},p_n|_k$ $|N,p_n,q_n|$, and $|N,p_n,q_n|_k$ summabilities of orthogonal series $(1 \le k \le 2,\ \delta \ge 0)$.

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