DEGREE OF CONVERGENCE OF AN INTEGRAL OPERATOR

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ABSTRACT. In this paper, we define an integral operator on L^p and obtain its degree of convergence in the appropriate norm. By specializing the kernel of the integral operator, we get many known results as corollaries. We also apply our results to obtain results on singular integral operators.

1. INTRODUCTION

Let $L^p \equiv L^p(\mathbb{R})$ with fixed $1 \le p \le \infty$ be the space of all real-valued functions Lebesgue integrable to the $p-$th power over $\mathbb{R}$ if $1 \le p < \infty$, and uniformly continuous and bounded on $\mathbb{R}$ if $p = \infty$. We define the norm in L^p, as usual, by the formula

$$(1) \qquad \|f\|_p \equiv \|f(\cdot)\|_p := \begin{cases} \left\{ \int_{\mathbb{R}} |f(x)|^p \mathrm{d}x \right\}^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \sup_{x \in \mathbb{R}} |f(x)| & \text{if } p = \infty. \end{cases}$$

By $\omega(f; \cdot)_p$ denote the modulus of continuity of $f \in L^p$, i.e.,

$$\omega(f; t)_p := \sup_{0 \le h \le t} \|\Delta_h f(\cdot)\|_p, \qquad t \ge 0,$$

where $\Delta_h f(x) = f(x + h) - f(x)$.

Consider the family of integral operators

$$(2) \qquad F_\lambda(f; x) := \lambda \int_{\mathbb{R}} f(t) \mathcal{K}(\lambda(t - x)) \mathrm{d}t, \qquad \lambda > 0,$$

with Fejér type kernel $\mathcal{K}$ [**1**, p. 126]:

$$(3) \qquad \mathcal{K}(-x) = \mathcal{K}(x),$$

$$(4) \qquad \int_{\mathbb{R}} \mathcal{K}(x) \mathrm{d}x = 1,$$

$$(5) \qquad \sup_{-1 \le x \le 1} |\mathcal{K}(x)| < \infty,$$

$$(6) \qquad \sup_{x \in \mathbb{R}} x^2 |\mathcal{K}(x)| < \infty.$$

Under these conditions, the integral (2) represents a linear operator acting from L^p to L^p.

For fixed $m \in \mathbb{N} \cup \{0\}$ and $1 \le p \le \infty$ by L_m^p, we denote the set of all $f \in L^p$ whose derivatives $f', f'', \ldots, f^{(m)}$ also belong to L^p. The norm in these L_m^p is defined by (1), i.e., for $f \in L_m^p$, we have $\|f\|_{p,\,m} = \|f\|_p$, where $\|f\|_p$ is defined by (1). It is clear that $L_0^p \equiv L^p$.

Definition 1.1. Let $f \in L_m^p$ for fixed $m \in \mathbb{N} \cup \{0\}$ and $1 \le p \le \infty$. We define a family of modified integral operators by the formula

$$(7) \qquad F_{\lambda,\,m}(f;x) := \lambda \int_{\mathbb{R}} \sum_{j=0}^m \frac{f^{(j)}(t)}{j!}(x-t)^j \mathcal{K}(\lambda(t-x))\mathrm{d}t$$

for $x \in \mathbb{R}$ and $\lambda > 0$.

In particular, we have $F_{\lambda,\,0}(f;\cdot) \equiv F_\lambda(f;\cdot)$ for $f \in L^p$.

It is obvious that the formula (7) can be rewritten in the following form

$$F_{\lambda,\,m}(f;x) := \sum_{j=0}^m \frac{(-1)^j}{j!}\lambda \int_{\mathbb{R}} f^{(j)}(t+x)t^j \mathcal{K}(\lambda t)\mathrm{d}t$$

for every $f \in L_m^p$, $x \in \mathbb{R}$, and $\lambda > 0$.

If (3) holds and for any $j = 0, 1, 2, \ldots, m$,

$$\int_0^\infty u^j \, |\mathcal{K}(u)| < \infty,$$

then for fixed $m \in \mathbb{N} \cup \{0\}$ and $\lambda > 0$, the integral (7) is a linear operator from space L_m^p into L^p (see Remark 2).

By $H^{\omega^*,p}$ denote the set of all functions $f \in L^p$ $(1 \le p \le \infty)$ satisfying the condition

$$\sup_{h \ne 0} h^{\omega^*,\,p}(f;h) < \infty,$$

where

$$h^{\omega^*,p}(f;h) := \frac{\|\Delta_h f(\cdot)\|_p}{\omega^*(|h|)}, \qquad h^{0,p}(f;h) = 0,$$

and for $t \ge 0$, $\omega^*(t)$ is a nondecreasing function. We can show that $H^{\omega^*,\,p}$ is a Banach space with respect to the generalized Hölder norm

$$(8) \qquad \|f\|_{\omega^*,p} := \|f\|_p + \sup_{h \ne 0} h^{\omega^*,p}(f;h).$$

Suppose that $H^{\omega,p}$ is the set of functions $f \in L^p$ $(1 \le p \le \infty)$ satisfying the condition

$$\sup_{h \ne 0} h^{\omega,p}(f;h) = \sup_{h \ne 0} \frac{\|\Delta_h f(\cdot)\|_p}{\omega(|h|)} < \infty$$

and contained in the space $H^{\omega^*,p}$, $H^{\omega,p} \subset H^{\omega^*,p}$, where for $t \ge 0$, $\omega(t)$ is a nondecreasing function. In particular, setting

$$\omega(t) = t^\alpha \quad \omega^*(t) = t^\beta, \qquad t \ge 0 \ \text{ and } \ 0 \le \beta < \alpha \le 1,$$

for $H^{\omega^*,\,p}$, we obtain the spaces

$$H^{\beta,p} := \left\{ f \in L^p : \omega\left(f,t\right)_p \leq C_1\, t^\beta \right\}$$

with Hölder norm

$$\|f\|_{\beta,p} := \|f\|_p + \sup_{h \neq 0} \frac{\|\Delta_h f(\cdot)\|_p}{|h|^\beta},$$

and for the set $H^{\omega,p}$, we have

$$H^{\alpha,p} := \left\{ f \in L^p : \omega\left(f,t\right)_p \leq C_2\, t^\alpha \right\},$$
$$H^{\alpha,p} \subset H^{\beta,p}.$$

For fixed $m \in \mathbb{N} \cup \{0\}$ and $1 \leq p \leq \infty$, by $H_m^{\omega^*,p}$ (or $H_m^{\omega,p}$) we denote the set of all $f \in H^{\omega^*,p}$ ($f \in H^{\omega,p}$) whose derivatives $f', f'', \ldots, f^{(m)}$ also belong to $H^{\omega^*,p}$ (or $H^{\omega,p}$), where for $t \geq 0$, $\omega^*(t)$ (or $\omega(t)$) is a nondecreasing function. The norm in these $H_m^{\omega^*,p}$ is defined by (8), i.e., for $f \in H_m^{\omega^*,p}$, we have $\|f\|_{\omega^*,p,\,m} = \|f\|_{\omega^*,p}$, where $\|f\|_{\omega^*,p}$ is defined by (8). It is clear that $H_0^{\omega^*,p} \equiv H^{\omega^*,p}$.

Throughout the paper, we use the following notation:

$$E_\lambda(x) = E_\lambda(f;x) = F_\lambda(f;x) - f(x),$$
$$E_\lambda\left(x+h,x\right) = E_\lambda\left(f;x+h,x\right) = E_\lambda(x+h) - E_\lambda(x),$$

and

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

The object of this paper is to study degree of convergence of the integral operator $F_{\lambda,m}(f)$ to f in the appropriate norm and to deduce many interesting results as corollaries. We also apply our results to obtain degree of convergence of singular integrals.

2. Statement of the results

Theorem 2.1. *Suppose that $0 \leq \beta < \eta \leq 1$ and (3)–(6) hold. Then, for any $f \in H^{\omega,p}$ ($1 \leq p \leq \infty$), the following relation is true*

$$(9) \qquad \|F_\lambda(f) - f\|_{\omega^*,p} = O(1) \sup_{h \neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \left\{ \frac{1}{\lambda} \int_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}} \mathrm{d}u \right\},$$

where $\lambda > 1$.

Remark 1. For $p = \infty$ (9), was proved in [**6**, Theorem 1].

Theorem 2.2. *Suppose that $0 \leq \beta < \eta \leq 1$, $\lambda \geq \lambda_0 > 0$, and (3)–(5) hold. If,*

$$(10) \qquad \int_1^\infty u\left|\mathcal{K}(u)\right| \mathrm{d}u < \infty,$$

then for $f \in H^{\omega,p} \, (1 \le p \le \infty)$

$$\|F_\lambda(f) - f\|_{\omega^*,p} = O(1) \sup_{h \ne 0} \frac{\left(\omega(|h|)\right)^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \left(\omega\!\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Remark 2. Let $m \in \mathbb{N} \cup \{0\}$, $1 \le p \le \infty$, and (3) hold. If for any $j = 0, 1, 2, \ldots, m$,

(11) $$\int_0^\infty u^j \, |\mathcal{K}(u)| \, \mathrm{d}u < \infty,$$

then for every $f \in L_m^p$ and $\lambda > 0$, we have

$$\|F_{\lambda,\,m}(f)\|_p = O(1) \sum_{j=0}^m \frac{\left\|f^{(j)}\right\|_p}{j!\lambda^j}.$$

Moreover, if $f \in H_m^{\omega^*,p}$ and $\lambda > 0$, then

$$\|F_{\lambda,\,m}(f)\|_{\omega^*,p} = O(1) \sum_{j=0}^m \frac{\left\|f^{(j)}\right\|_{\omega^*,p}}{j!\lambda^j}.$$

Theorem 2.3. *Suppose that* $0 \le \beta < \eta \le 1$, $m \in \mathbb{N}$, $\lambda \ge \lambda_0 > 0$, *and* (3)–(4) *hold. If for any* $j = 1, 2, \ldots, m+1$,

(12) $$\int_0^\infty u^j \, |\mathcal{K}(u)| \, \mathrm{d}u < \infty,$$

then for $f \in H_m^{\omega,p} \, (1 \le p \le \infty)$,

$$\|F_{\lambda,m}(f) - f\|_{\omega^*,p} = O(1) \sup_{h \ne 0} \frac{\left(\omega(|h|)\right)^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \frac{1}{\lambda^m} \left(\omega\!\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Setting
$$\omega(t) = t^\alpha, \quad \omega^*(t) = t^\beta, \qquad 0 \le \beta < \alpha \le 1, \quad \eta = \alpha$$
in the assumptions of Theorem 2.1, 2.2, and 2.3, we obtain the following corollaries:

Corollary 2.4. *Suppose that* $0 \le \beta < \alpha \le 1$, *and* (3)–(6) *hold. Then, for any* $f \in H^{\alpha,p} \, (1 \le p \le \infty)$, *the following relation is true*

$$\|F_\lambda(f) - f\|_{\beta,p} = \begin{cases} O\left(\lambda^{\beta-\alpha}\right) & \text{if } \alpha - \beta < 1, \\ O\left(\frac{\ln \lambda}{\lambda}\right) & \text{if } \alpha - \beta = 1, \end{cases}$$

where $\lambda > 1$.

Corollary 2.5. *Suppose that* $0 \le \beta < \alpha \le 1$, $\lambda \ge \lambda_0 > 0$, *and* (3)–(5) *hold. If*

$$\int_1^\infty u \, |\mathcal{K}(u)| \, \mathrm{d}u < \infty,$$

then for $f \in H^{\alpha,p} \, (1 \le p \le \infty)$,

$$\|F_\lambda(f) - f\|_{\beta,p} = O\left(\lambda^{\beta-\alpha}\right).$$

Corollary 2.6. *Suppose that* $0 \leq \beta < \alpha \leq 1$, $m \in \mathbb{N}$, $\lambda \geq \lambda_0 > 0$, *and* (3)–(4) *hold. If for any* $j = 1, 2, \ldots, m+1$,

$$\int_0^\infty u^j \, |\mathcal{K}(u)| \, \mathrm{d}u < \infty,$$

then for $f \in H_m^{\alpha,p} \, (1 \leq p \leq \infty)$,

$$\|F_{\lambda,\,m}(f) - f\|_{\beta,p} = O\left(\lambda^{\beta-\alpha-m}\right).$$

3. Examples

3.1. The Fejér means of the Fourier series

Suppose that the kernel $\mathcal{K}$ satisfies the conditions (3)–(6). Then, as is well known (see [**1**, p. 132]), if we consider 2π periodic function $f \in L^p$ with the Fourier series

$$S(f) = \sum_{k=-\infty}^{\infty} c_k(f)\mathrm{e}^{\mathrm{i}kx},$$

the family of operators of Fejér type (2) can be transformed into sequences of linear means of series of the function f

$$F_n(f;x) = \sum_{k=-\infty}^{\infty} \varphi\left(\frac{k}{n}\right) c_k(f)\mathrm{e}^{\mathrm{i}kx}, \qquad n \in \mathbb{N},$$

where the values of $\varphi\left(\frac{k}{n}\right)$ coincide for $t = \frac{k}{n}$, with the values of the function $\varphi(t)$, which is the Fourier transform of the kernel $\mathcal{K}$.

Consider the Fejér kernel

$$\mathcal{K}_\sigma(t) := \frac{2}{\pi} \left(\frac{\sin(t/2)}{t}\right)^2,$$

and consequently, the Fejér means of the Fourier series $S(f)$ is given by

$$\sigma_n(f;x) := \sum_{k=-\infty}^{\infty} \varphi\left(\frac{\kappa}{n}\right) c_k(f)\mathrm{e}^{\mathrm{i}kx} = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n}\right) c_k(f)\mathrm{e}^{\mathrm{i}kx}.$$

It is clear that the kernel $\mathcal{K}_\sigma(t)$ satisfies the conditions (3), (4), (5), and (6). Hence, by Theorem 2.1, we obtain the following corollaries ([**3**], [**6**], and [**10**]):

Corollary 3.1. *Suppose that* $0 \leq \beta < \eta \leq 1$. *Then, for any* 2π *periodic function* $f \in H^{\omega,p} \, (1 \leq p \leq \infty)$, *the following relation is true*

$$\|\sigma_n(f) - f\|_{\omega^*,p} = O(1) \sup_{h \neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \left\{\frac{1}{n} \sum_{k=1}^{n} \left(\omega\left(\frac{1}{k}\right)\right)^{1-\frac{\beta}{\eta}}\right\}.$$

Corollary 3.2. *Suppose that* $0 \leq \beta < \alpha \leq 1$. *Then, for any* 2π *periodic function* $f \in H^{\alpha,p} \, (1 \leq p \leq \infty)$, *the following relation holds*

$$\|\sigma_n(f) - f\|_{\beta,p} = \begin{cases} O\left(n^{\beta-\alpha}\right) & \text{if } \alpha - \beta < 1, \\ O\left(\frac{\ln(n+1)}{n}\right) & \text{if } \alpha - \beta = 1. \end{cases}$$

3.2. The Poisson operator

Let

$$\mathcal{K}_{\overline{P}}(t) = \frac{1}{\pi}\frac{1}{1+t^2}, \quad \lambda = \frac{1}{\varepsilon}, \quad \varepsilon > 0.$$

Then from (2), we obtain the Poisson singular integral of a function $f \in L^p$, i.e,

$$\overline{P}_\varepsilon(f;x) = \frac{\varepsilon}{\pi}\int_{\mathbb{R}} f(x+t)\frac{1}{\varepsilon^2+t^2}\mathrm{d}t$$

The approximation properties of this integral were given for example in [**5**]. It is clear that the kernel $\mathcal{K}_{\overline{P}}$ satisfies conditions (3)–(6). Hence, by Theorem 2.1, we obtain the following assertion.

Corollary 3.3. *Suppose that $0 \leq \beta < \eta \leq 1$. Then, for any 2π periodic function $f \in H^{\omega,p}$ ($1 \leq p \leq \infty$), the following relation is true*

$$\left\|\overline{P}_\varepsilon(f) - f\right\|_{\omega^*,p} = O(1)\sup_{h\neq 0}\frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)}\left\{\varepsilon\int_1^{1/\varepsilon}\left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}}\mathrm{d}u\right\},$$

where $0 < \varepsilon < 1$.

Corollary 3.4. *Suppose that $0 \leq \beta < \alpha \leq 1$. Then, for any 2π periodic function $f \in H^{\alpha,p}$ ($1 \leq p \leq \infty$), the following relation is true*

$$\left\|\overline{P}_\varepsilon(f) - f\right\|_{\beta,p} = \begin{cases} O\left(\varepsilon^{\beta-\alpha}\right) & if\ \alpha - \beta < 1, \\ O\left(\varepsilon\ln\left(1/\varepsilon\right)\right) & if\ \alpha - \beta = 1, \end{cases}$$

as $\varepsilon \to 0^+$.

3.3. The Picard and Gauss-Weierstrass operators

Taking

$$\mathcal{K}_P(t) = \frac{1}{2}\exp\left(-|t|\right), \quad \lambda = \frac{1}{r}, \quad r > 0,$$

and

$$\mathcal{K}_W(t) = \frac{1}{\sqrt{\pi}}\exp\left(-t^2\right), \quad \lambda = \frac{1}{\sqrt{2r}}, \quad r > 0,$$

we obtain from (2), respectively, the Picard singular integral and the Gauss-Weierstrass singular integrals of a function $f \in L^p$, i.e.,

$$P_r(f;x) = \frac{1}{2r}\int_{\mathbb{R}} f(x+t)\exp\left(\frac{-|t|}{r}\right)\mathrm{d}t,$$

$$W_r(f;x) = \frac{1}{\sqrt{4\pi r}}\int_{\mathbb{R}} f(x+t)\exp\left(-\frac{t^2}{4r}\right)\mathrm{d}t.$$

The limit properties (as $r \to 0^+$) of these integrals were given in many papers and monographs (see, e.g., [**2**, **3**, **4**]).

It is clear (see [**6**, Lemma 1]) that for every $m \in \mathbb{N} \cup \{0\}$,

$$\int_0^\infty u^m\left|\mathcal{K}_P(u)\right|\mathrm{d}u = \frac{m!}{2}$$

and

$$\int_0^\infty u^m \left|\mathcal{K}_W(u)\right| \mathrm{d}u = \begin{cases} \frac{1}{2} & \text{if } m = 0, \\ \frac{(2k-1)!!}{2^{k+1}} & \text{if } m = 2k \geq 2, \\ \frac{k!}{2\sqrt{\pi}} & \text{if } m = 2k+1 \geq 1. \end{cases}$$

Hence, by Theorems 2.2 and 2.3, we obtain the following assertion.

Corollary 3.5. *Let* $0 \leq \beta < \eta \leq 1$, $m \in \mathbb{N} \cup \{0\}$, *and* $f \in H_m^{\omega,p}$ $(1 \leq p \leq \infty)$. *Then*

$$\|P_r(f) - f\|_{\omega^*,p} = O(1) \sup_{h \neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} r^m \left(\omega(r)\right)^{1-\frac{\beta}{\eta}}$$

and

$$\|W_r(f) - f\|_{\omega^*,p} = O(1) \sup_{h \neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} r^{m/2} \left(\omega\left(\sqrt{r}\right)\right)^{1-\frac{\beta}{\eta}}$$

as $r \to 0^+$.

Corollary 3.6. *Let* $0 \leq \beta < \alpha \leq 1$, $m \in \mathbb{N} \cup \{0\}$, *and* $f \in H_m^{\alpha,p}$ $(1 \leq p \leq \infty)$. *Then*

$$\|P_r(f) - f\|_{\beta,p} = O\left(r^{m+\alpha-\beta}\right)$$

and

$$\|W_r(f) - f\|_{\beta,p} = O\left(r^{(m+\alpha-\beta)/2}\right)$$

as $r \to 0^+$.

4. Proofs of the theorems

4.1. Proof of Theorem 2.1

Proof. For $p = \infty$, (9) was proved in [**6**, Theorem 1]. Let $1 \leq p < \infty$. Then using (3) and (4), we get

$$E_\lambda(x) = \lambda \int_0^\infty \phi_x(t) \mathcal{K}(\lambda t) \, \mathrm{d}t$$

and

$$E_\lambda(x+h, x) = \lambda \int_0^\infty (\phi_{x+h}(t) - \phi_x(t)) \mathcal{K}(\lambda t) \, \mathrm{d}t.$$

Applying the Fubini inequality [**11**], we have
(13)

$$
\begin{aligned}
\|E_\lambda(\cdot + h, \cdot)\|_p &= \lambda \left\{ \int_\mathbb{R} \left| \int_0^\infty (\phi_{x+h}(t) - \phi_x(t)) \mathcal{K}(\lambda t) \, \mathrm{d}t \right|^p \mathrm{d}x \right\}^{\frac{1}{p}} \\
&\leq \lambda \int_0^\infty |\mathcal{K}(\lambda t)| \left\{ \int_\mathbb{R} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \right\}^{\frac{1}{p}} \mathrm{d}t \\
&= \lambda \left(\int_0^{1/\lambda} + \int_{1/\lambda}^1 + \int_1^\infty \right) |\mathcal{K}(\lambda t)| \left\{ \int_\mathbb{R} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \right\}^{\frac{1}{p}} \mathrm{d}t \\
&= I_1 + I_1 + I_3.
\end{aligned}
$$

It is clear that for $1 \le p < \infty$,

(14)
$$\left\{ \int_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \right\}^{\frac{1}{p}} \le 4\omega \left(f, |h| \right)_p$$

and for $t \ge 0$

(15)
$$\left\{ \int_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \right\}^{\frac{1}{p}} \le 4\omega \left(f, t \right)_p .$$

Then, in view of the property (5) of the kernel $\mathcal{K}$, and inequalities (14) and (15), we obtain for $f \in H^{\omega, p}$

(16)
$$\begin{aligned}
I_1 &= \lambda \int_0^{1/\lambda} |\mathcal{K}(\lambda t)| \left\{ \int_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \right\}^{\frac{1}{p}\frac{\beta}{\eta}} \\
&\quad \times \left\{ \int_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \right\}^{\frac{1}{p}\left(1 - \frac{\beta}{\eta}\right)} \mathrm{d}t \\
&= O(1) \lambda \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \int_0^{1/\lambda} |\mathcal{K}(\lambda t)| \left(\omega\left(f, t \right)_p \right)^{1 - \frac{\beta}{\eta}} \mathrm{d}t \\
&= O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda} \right) \right)^{1 - \frac{\beta}{\eta}} .
\end{aligned}$$

Further, in view of property (6) of the kernel $\mathcal{K}$, by (14) and (15), we have for $f \in H^{\omega, p}$,

(17)
$$\begin{aligned}
I_2 &= O(1) \lambda \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \int_{1/\lambda}^1 |\mathcal{K}(\lambda t)| \left(\omega\left(f, t \right)_p \right)^{1 - \frac{\beta}{\eta}} \mathrm{d}t \\
&= O(1) \lambda \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \int_{1/\lambda}^1 \left(\omega\left(f, t \right)_p \right)^{1 - \frac{\beta}{\eta}} \frac{1}{(\lambda t)^2} \mathrm{d}t \\
&= O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \frac{1}{\lambda} \int_1^\lambda \left(\omega\left(f; \frac{1}{u} \right)_p \right)^{1 - \frac{\beta}{\eta}} \mathrm{d}u \\
&= O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \frac{1}{\lambda} \int_1^\lambda \left(\omega\left(\frac{1}{u} \right) \right)^{1 - \frac{\beta}{\eta}} \mathrm{d}u.
\end{aligned}$$

Applying (6), (14), we get for $f \in H^{\omega, p}$

$$\begin{aligned}
I_3 &= \lambda \int_0^{1/\lambda} |\mathcal{K}(\lambda t)| \left\{ \int_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \right\}^{\frac{1}{p}\frac{\beta}{\eta}} \\
&\quad \times \left\{ \int_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \right\}^{\frac{1}{p}\left(1 - \frac{\beta}{\eta}\right)} \mathrm{d}t \\
&= O(1) \lambda \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \int_1^\infty |\mathcal{K}(\lambda t)| \left(8\|f\|_p \right)^{1 - \frac{\beta}{\eta}} \mathrm{d}t = O(1) \lambda \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \int_1^\infty \frac{1}{(\lambda t)^2} \mathrm{d}t \\
&= O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \frac{1}{\lambda} = O(1) \left(\omega(|h|) \right)^{\frac{\beta}{\eta}} \frac{1}{\lambda^{\frac{\beta}{\eta}}} \frac{1}{\lambda^{1 - \frac{\beta}{\eta}}} .
\end{aligned}$$

If $\lambda > 1$, then $\omega\left(f;1\right)_p \le 2\lambda\omega\left(f,\frac{1}{\lambda}\right)$. Thus

$$(18) \quad I_3 = O(1)\left(\omega(|h|)\right)^{\frac{\beta}{\eta}}\left(\omega\left(f,\frac{1}{\lambda}\right)_p\right)^{1-\frac{\beta}{\eta}}\frac{1}{\lambda^{\frac{\beta}{\eta}}} = O(1)\left(\omega(|h|)\right)^{\frac{\beta}{\eta}}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

From relation (13), (16), (17), and (18), we obtain

$$\left\|E_\lambda\left(\cdot + h, \cdot\right)\right\|_p = O(1)\left(\omega(|h|)\right)^{\frac{\beta}{\eta}}\left\{\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}} + \frac{1}{\lambda}\int_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}}\mathrm{d}u\right\}.$$

Since

$$\frac{1}{\lambda}\int_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}}\mathrm{d}u \ge \frac{1}{\lambda}\int_{\frac{\lambda+1}{2}}^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}}\mathrm{d}u$$

$$\ge \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}\frac{\lambda-1}{2\lambda},$$

therefore,

$$\left\|E_\lambda(\cdot + h, \cdot)\right\|_p = O(1)(\omega(|h|))^{\frac{\beta}{\eta}}\left\{\frac{1}{\lambda}\int_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}}\mathrm{d}u\right\}.$$

Hence

$$(19) \quad \sup h \ne 0 \frac{\left\|E_\lambda(\cdot + h, \cdot)\right\|_p}{\omega^*(|h|)} = O(1)\frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)}\left\{\frac{1}{\lambda}\int_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}}\mathrm{d}u\right\}.$$

We can easily see that

$$(20) \quad \left\|E_\lambda\left(\cdot\right)\right\|_p = O(1)\frac{1}{\lambda}\int_1^\lambda \omega\left(\frac{1}{u}\right)\mathrm{d}u = O(1)\frac{1}{\lambda}\int_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}}\mathrm{d}u.$$

From (19) and (20), we finally obtain

$$\left\|E_\lambda\left(\cdot\right)\right\|_{\omega^*,p} = \left\|E_\lambda\left(\cdot\right)\right\|_p + \sup_{h\ne 0}\frac{\left\|E_\lambda\left(\cdot + h, \cdot\right)\right\|_p}{\omega^*(|h|)}$$

$$= O(1)\frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)}\left\{\frac{1}{\lambda}\int_1^\lambda \left(\omega\left(\frac{1}{u}\right)\right)^{1-\frac{\beta}{\eta}}\mathrm{d}u\right\}.$$

This completes the proof of Theorem 2.1. $\square$

4.2. Proof of Theorem 2.2

Proof. Let $p = \infty$. Then by (3) and (4), we get

$$\left|E_\lambda\left(x + h, x\right)\right| \le \lambda \int_0^\infty \left|\mathcal{K}\left(\lambda t\right)\right|\left|\phi_{x+h}(t) - \phi_x(t)\right|\mathrm{d}t$$

$$(21)$$

$$= \lambda\left(\int_0^{1/\lambda} + \int_{1/\lambda}^\infty\right)\left|\mathcal{K}\left(\lambda t\right)\right|\left|\phi_{x+h}(t) - \phi_x(t)\right|\mathrm{d}t = J_1 + J_2.$$

It is clear that

$$(22) \quad \begin{aligned} \left|\phi_{x+h}(t) - \phi_x(t)\right| &\le 4\omega\left(f;|h|\right)_\infty, \qquad \text{and} \\ \left|\phi_{x+h}(t) - \phi_x(t)\right| &\le 4\omega\left(f;t\right)_\infty. \end{aligned}$$

Using this and (5), we have for $f \in H^{\omega,p}$,
(23)
$$J_1 = \lambda \int_0^{1/\lambda} |\mathcal{K}(\lambda t)| \left(|\phi_{x+h}(t) - \phi_x(t)|\right)^{\frac{\beta}{\eta}} \left(|\phi_{x+h}(t) - \phi_x(t)|\right)^{1-\frac{\beta}{\eta}} dt$$
$$= O(1)\lambda \left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \int_0^{1/\lambda} \left(\omega\left(f;t\right)_\infty\right)^{1-\frac{\beta}{\eta}} dt = O(1) \left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Further, by (22) and (10), we obtain for $f \in H^{\omega,p}$,

$$J_2 = \lambda \int_{1/\lambda}^{\infty} |\mathcal{K}\left(\lambda t\right)| \left(|\phi_{x+h}(t) - \phi_x(t)|\right)^{\frac{\beta}{\eta}} \left(|\phi_{x+h}(t) - \phi_x(t)|\right)^{1-\frac{\beta}{\eta}} dt$$

$$= O(1) \left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \lambda \int_{1/\lambda}^{\infty} \left(\omega\left(f;t\right)_\infty\right)^{1-\frac{\beta}{\eta}} |\mathcal{K}\left(\lambda t\right)| dt$$

$$= O(1) \left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \lambda \int_{1/\lambda}^{\infty} \left(\frac{\omega\left(f;t\right)_\infty}{t}\right)^{1-\frac{\beta}{\eta}} t^{1-\frac{\beta}{\eta}} |\mathcal{K}\left(\lambda t\right)| dt$$

(24) $\qquad = O(1) \left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \left(\omega\left(f;\frac{1}{\lambda}\right)_\infty\right)^{1-\frac{\beta}{\eta}} \lambda^{2-\frac{\beta}{\eta}} \int_{1/\lambda}^{\infty} t^{1-\frac{\beta}{\eta}} |\mathcal{K}\left(\lambda t\right)| dt$

$$= O(1) \left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}} \lambda^2 \int_{1/\lambda}^{\infty} t |\mathcal{K}\left(\lambda t\right)| dt$$

$$= O(1) \left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}} \int_1^{\infty} u |\mathcal{K}(u)| du$$

$$= O(1) \left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Similarly, we can prove that

(25) $\qquad \|E_\lambda\left(\cdot\right)\|_\infty = O(1)\omega\left(f;\frac{1}{\lambda}\right)_\infty = O(1)\left(\omega\left(f;\frac{1}{\lambda}\right)_\infty\right)^{1-\frac{\beta}{\eta}} \left(\omega\left(f;\frac{1}{\lambda_0}\right)_\infty\right)^{\frac{\beta}{\eta}}$

$$= O(1)\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Hence, by (21), (23), (24), and (25),

$$\|E_\lambda\left(\cdot\right)\|_{\omega^*,\infty} = \|E_\lambda\left(\cdot\right)\|_\infty + \sup_{h \neq 0} \frac{\|E_\lambda\left(\cdot + h, \cdot\right)\|_\infty}{\omega^*(|h|)}$$

$$= O(1)\sup_{h \neq 0} \frac{\left(\omega(|h|)\right)^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Suppose $1 \leq p < \infty$. Then using (3), (4), and the Fubini inequality [**11**], we get

$$\|E_\lambda(\cdot + h, \cdot)\|_p = \lambda \Big\{ \int_{\mathbb{R}} \Big| \int_0^\infty (\phi_{x+h}(t) - \phi_x(t)) \, \mathcal{K}(\lambda t) \, \mathrm{d}t \Big|^p \mathrm{d}x \Big\}^{\frac{1}{p}}$$

$$\leq \lambda \int_0^\infty |\mathcal{K}(\lambda t)| \Big\{ \int_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \Big\}^{\frac{1}{p}} \mathrm{d}t$$

(26)

$$= \lambda \Big(\int_0^{1/\lambda} + \int_{1/\lambda}^\infty \Big) |\mathcal{K}(\lambda t)| \Big\{ \int_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \Big\}^{\frac{1}{p}} \mathrm{d}t$$

$$= S_1 + S_2.$$

In view of property (5) of the kernel $\mathcal{K}$, and inequalities (14) and (15), we obtain for $f \in H^{\omega, p}$,

$$S_1 = \lambda \int_0^{1/\lambda} |\mathcal{K}(\lambda t)| \Big\{ \int_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \Big\}^{\frac{1}{p} \frac{\beta}{\eta}}$$

(27)

$$\times \Big\{ \int_{\mathbb{R}} |\phi_{x+h}(t) - \phi_x(t)|^p \, \mathrm{d}x \Big\}^{\frac{1}{p}\left(1 - \frac{\beta}{\eta}\right)} \mathrm{d}t$$

$$= O(1) \lambda \left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \int_0^{1/\lambda} |\mathcal{K}(\lambda t)| \left(\omega(f, t)_p\right)^{1 - \frac{\beta}{\eta}} \mathrm{d}t$$

$$= O(1)(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1 - \frac{\beta}{\eta}}.$$

Further, in view of property (10) of the kernel $\mathcal{K}$, by (14), and (15) we have for $f \in H^{\omega, p}$,

$$S_2 = O(1) \lambda (\omega(|h|))^{\frac{\beta}{\eta}} \int_{1/\lambda}^\infty |\mathcal{K}(\lambda t)| \left(\omega(f, t)_p\right)^{1 - \frac{\beta}{\eta}} \mathrm{d}t$$

$$= O(1)(\omega(|h|))^{\frac{\beta}{\eta}} \lambda \int_{1/\lambda}^\infty \left(\frac{\omega(f; t)_p}{t}\right)^{1 - \frac{\beta}{\eta}} t^{1 - \frac{\beta}{\eta}} |\mathcal{K}(\lambda t)| \, \mathrm{d}t$$

$$= O(1)(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(f; \frac{1}{\lambda}\right)_p\right)^{1 - \frac{\beta}{\eta}} \lambda^{2 - \frac{\beta}{\eta}} \int_{1/\lambda}^\infty t^{1 - \frac{\beta}{\eta}} |\mathcal{K}(\lambda t)| \, \mathrm{d}t$$

(28)

$$= O(1)(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1 - \frac{\beta}{\eta}} \lambda^2 \int_{1/\lambda}^\infty t |\mathcal{K}(\lambda t)| \, \mathrm{d}t$$

$$= O(1)(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1 - \frac{\beta}{\eta}} \int_1^\infty u |\mathcal{K}(u)| \, \mathrm{d}u$$

$$= O(1)(\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1 - \frac{\beta}{\eta}}.$$

We can easily see that

$$\|E_\lambda(\cdot)\|_p = O(1)\omega\left(f; \frac{1}{\lambda}\right)_p = O(1)\left(\omega\left(f; \frac{1}{\lambda_0}\right)\right)^{\frac{\beta}{\eta}} \left(\omega\left(f; \frac{1}{\lambda}\right)_p\right)^{1 - \frac{\beta}{\eta}}$$

(29)

$$= O(1)\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1 - \frac{\beta}{\eta}}.$$

Hence, by (26)–(29),

$$\|E_\lambda(\cdot)\|_{\omega^*,p} = \|E_\lambda(\cdot)\|_p + \sup_{h \neq 0} \frac{\|E_\lambda(\cdot + h, \cdot)\|_p}{\omega^*(|h|)}$$

$$= O(1)\sup_{h \neq 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)} \left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

The proof is complete. □

4.2.1. Proof of Remark 2.

Proof. Let $p = \infty$. Then by (3) and (11), we get

(30)

$$\|F_{\lambda,\,m}(f)\|_\infty \leq \sum_{j=0}^m \frac{\|f^{(j)}\|_\infty}{j!} \lambda \int_{\mathbb{R}} \left|t^j \mathcal{K}(\lambda t)\right| \mathrm{d}t = 2\sum_{j=0}^m \frac{\|f^{(j)}\|_\infty}{j!} \lambda \int_0^\infty t^j |\mathcal{K}(\lambda t)|\, \mathrm{d}t$$

$$= 2\sum_{j=0}^m \frac{\|f^{(j)}\|_\infty}{j!\lambda^j} \int_0^\infty u^j |\mathcal{K}(u)|\, \mathrm{d}u = O(1)\sum_{j=0}^m \frac{\|f^{(j)}\|_\infty}{j!\lambda^j}.$$

Suppose that $1 \leq p < \infty$. Then using the Fubini inequality [**11**], (3), and (11), we obtain

$$\|F_{\lambda,m}(f)\|_p = \left\|\sum_{j=0}^m \frac{(-1)^j}{j!} \lambda \int_{\mathbb{R}} f^{(j)}(t + \cdot)\, t^j \mathcal{K}(\lambda t)\, \mathrm{d}t\right\|_p$$

$$\leq \sum_{j=0}^m \frac{\lambda}{j!} \left\{\int_{\mathbb{R}} \left|\int_{\mathbb{R}} f^{(j)}(t+x) t^j \mathcal{K}(\lambda t)\, \mathrm{d}t\right|^p \mathrm{d}x\right\}^{\frac{1}{p}}$$

(31)

$$\leq \sum_{j=0}^m \frac{\lambda}{j!} \int_{\mathbb{R}} \left|t^j \mathcal{K}(\lambda t)\right| \left\{\int_{\mathbb{R}} \left|f^{(j)}(t+x)\right|^p \mathrm{d}x\right\}^{\frac{1}{p}}$$

$$= \sum_{j=0}^m \frac{\|f^{(j)}\|_p}{j!} \lambda \int_0^\infty t^j |\mathcal{K}(\lambda t)|\, \mathrm{d}t = O(1)\sum_{j=0}^m \frac{\|f^{(j)}\|_p}{j!\lambda^j}.$$

Using (30) and (31), we get

$$\|F_{\lambda,\,m}(f)\|_{\omega^*,p} = \|F_{\lambda,\,m}(f)\|_p + \sup_{h \neq 0} \frac{\|\Delta_h F_{\lambda,\,m}(f; \cdot)\|_p}{\omega^*(|h|)}$$

$$= \|F_{\lambda,\,m}(f)\|_p + \sup_{h \neq 0} \frac{\|F_{\lambda,m}(\Delta_h f; \cdot)\|_p}{\omega^*(|h|)}$$

$$= O(1)\left\{\sum_{j=0}^m \frac{\|f^{(j)}\|_p}{j!\lambda^j} + \sup_{h \neq 0} \sum_{j=0}^m \frac{\|\Delta_h f^{(j)}(\cdot)\|_p}{j!\lambda^j \omega^*(|h|)}\right\}$$

$$= O(1)\sum_{j=0}^m \frac{\|f^{(j)}\|_{\omega^*,p}}{j!\lambda^j}.$$

This ends our proof. □

4.3. Proof of Theorem 2.3

Proof. We use the following modified Taylor formula for $f \in L_m^p$, with $m \in \mathbb{N}$

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(t)}{j!}(x-t)^j + \frac{(x-t)^m}{(m-1)!} \int_0^1 (1-u)^{m-1} \left\{ f^{(m)}(t+u(x-t)) - f^{(m)}(t) \right\} du$$

for a fixed $t \in \mathbb{R}$ and every $x \in \mathbb{R}$.

By (3), we get

$$f(x) = \lambda \int_{\mathbb{R}} f(x) \mathcal{K}(\lambda(t-x)) dt$$

$$= \lambda \int_{\mathbb{R}} \sum_{j=0}^{m} \frac{f^{(j)}(t)}{j!}(x-t)^j \mathcal{K}(\lambda(t-x)) dt + \lambda \int_{\mathbb{R}} \mathcal{K}(\lambda(t-x)) \frac{(x-t)^m}{(m-1)!}$$

$$\times \left(\int_0^1 (1-u)^{m-1} \left\{ f^{(m)}(t+u(x-t)) - f^{(m)}(t) \right\} du \right) dt$$

$$= F_{\lambda,m}(f;x) + \lambda \int_{\mathbb{R}} \mathcal{K}(\lambda(t-x)) \frac{(x-t)^m}{(m-1)!}$$

$$\times \left(\int_0^1 (1-u)^{m-1} \left\{ f^{(m)}(t+u(x-t)) - f^{(m)}(t) \right\} du \right) dt.$$

Therefore, by (3),

$$f(x) - F_{\lambda,m}(f;x)$$

(32)
$$= \lambda \int_{\mathbb{R}} \left(\frac{(x-t)^m}{(m-1)!} \int_0^1 (1-u)^{m-1} \Delta_{u(x-t)} f^{(m)}(t) du \right) \mathcal{K}(\lambda(t-x)) dt$$

$$= \lambda \int_{\mathbb{R}} \left(\frac{t^m}{(m-1)!} \int_0^1 (1-u)^{m-1} \Delta_{ut} f^{(m)}(x-t) du \right) \mathcal{K}(\lambda t) dt.$$

Set

$$E_{\lambda,m}(x) = E_{\lambda,m}(f;x) := f(x) - F_{\lambda,m}(f;x)$$

and

$$E_{\lambda,m}(x+h,x) = E_{\lambda,m}(f;x+h,x) := E_{\lambda,m}(x+h) - E_{\lambda,m}(x).$$

Let $p = \infty$. Then

$$|E_{\lambda,m}(x+h,x)| \le \lambda \int_{\mathbb{R}} |\mathcal{K}(\lambda t)| \frac{|t|^m}{(m-1)!}$$

$$\times \left(\int_0^1 (1-u)^{m-1} \left| \Delta_{ut} f^{(m)}(x+h-t) - \Delta_{ut} f^{(m)}(x-t) \right| du \right) dt.$$

It is clear that

$$\left| \Delta_{ut} f^{(m)}(x+h-t) - \Delta_{ut} f^{(m)}(x-t) \right| \le 2\omega \left(f^{(m)}; |ut| \right)_\infty$$

and

$$\left| \Delta_{ut} f^{(m)}(x+h-t) - \Delta_{ut} f^{(m)}(x-t) \right| \le 2\omega \left(f^{(m)}; |h| \right)_\infty.$$

This and the properties of the modulus of continuity yield for $f \in H_m^{\omega,p}$,

$$|E_{\lambda,m}\left(x+h,x\right)|$$

$$\leq \lambda\Big(\omega\Big(f^{(m)};|h|\Big)_\infty\Big)^{\frac{\beta}{\eta}} \int_{\mathbb{R}} \frac{|t|^m}{(m-1)!} |\mathcal{K}\left(\lambda t\right)|$$

$$\times \Big(\int_0^1 (1-u)^{m-1}\Big(\omega\Big(f^{(m)};|ut|\Big)_\infty\Big)^{1-\frac{\beta}{\eta}} \mathrm{d}u\Big)\mathrm{d}t$$

$$= O(1)\lambda(\omega(|h|))^{\frac{\beta}{\eta}}$$

$$\times \int_{\mathbb{R}} \Big(\frac{|t|^m}{(m-1)!} |\mathcal{K}\left(\lambda t\right)| \Big(\omega\big(f^{(m)};|t|\big)_\infty\Big)^{1-\frac{\beta}{\eta}} \int_0^1 (1-u)^{m-1} \,\mathrm{d}u\Big)\mathrm{d}t$$

$$= O(1)\left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \Big(\omega\Big(f^{(m)};\frac{1}{\lambda}\Big)_\infty\Big)^{1-\frac{\beta}{\eta}} \lambda \int_{\mathbb{R}} \frac{|t|^m}{m!} |\mathcal{K}\left(\lambda t\right)| \left(1+\lambda\left|t\right|\right)^{1-\frac{\beta}{\eta}} \mathrm{d}t.$$

Using (3) and (12),
(33)

$$= O(1)\left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \Big(\omega\Big(f^{(m)};\frac{1}{\lambda}\Big)_\infty\Big)^{1-\frac{\beta}{\eta}} \frac{\lambda}{m!} \int_0^\infty t^m\left(1+\lambda t\right) |\mathcal{K}\left(\lambda t\right)| \,\mathrm{d}t$$

$$= O(1)\left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \Big(\omega\Big(\frac{1}{\lambda}\Big)\Big)^{1-\frac{\beta}{\eta}} \Big(\lambda\int_0^\infty t^m |\mathcal{K}\left(\lambda t\right)| \,\mathrm{d}t + \lambda^2 \int_0^\infty t^{m+1} |\mathcal{K}\left(\lambda t\right)| \,\mathrm{d}t\Big)$$

$$= O(1)\left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \Big(\omega\Big(\frac{1}{\lambda}\Big)\Big)^{1-\frac{\beta}{\eta}} \frac{1}{\lambda^m} \Big(\int_0^\infty u^m |\mathcal{K}(u)| \,\mathrm{d}u + \int_0^\infty u^{m+1} |\mathcal{K}(u)| \,\mathrm{d}u\Big)$$

$$= O(1)\left(\omega(|h|)\right)^{\frac{\beta}{\eta}} \Big(\omega\Big(\frac{1}{\lambda}\Big)\Big)^{1-\frac{\beta}{\eta}} \frac{1}{\lambda^m}.$$

We can easily see that

$$(34) \qquad \|E_{\lambda,m}\left(\cdot\right)\|_\infty = O(1)\frac{1}{\lambda^m}\omega\Big(f^{(m)};\frac{1}{\lambda}\Big)_\infty = O(1)\frac{1}{\lambda^m}\Big(\omega\Big(\frac{1}{\lambda}\Big)\Big)^{1-\frac{\beta}{\eta}}.$$

Hence, by (33) and (34),

$$\|E_{\lambda,m}\left(\cdot\right)\|_{\omega^*,\infty} = \|E_{\lambda,m}(\cdot)\|_\infty + \sup_{h\neq 0}\frac{\|E_{\lambda,m}\|_\infty}{\omega^*(|h|)}$$

$$= O(1)\sup_{h\neq 0}\frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)}\frac{1}{\lambda^m}\Big(\omega\Big(\frac{1}{\lambda}\Big)\Big)^{1-\frac{\beta}{\eta}}.$$

Suppose $1 \leq p < \infty$. Using (32), the Fubini inequality [11], we get

$$\|E_{\lambda,m}\left(\cdot+h,\cdot\right)\|_p$$

$$= \frac{\lambda}{(m-1)!}\Big\{\int_{\mathbb{R}}\Big|\int_{\mathbb{R}} t^m\mathcal{K}(\lambda t)$$

$$\times \Big(\int_0^1 (1-u)^{m-1}\Big(\Delta_{ut}f^{(m)}\left(x+h-t\right) - \Delta_{ut}f^{(m)}(x-t)\Big)\,\mathrm{d}u\Big)\mathrm{d}t\Big|^p\mathrm{d}x\Big\}^{\frac{1}{p}}$$

$$\leq \frac{\lambda}{(m-1)!} \int_{\mathbb{R}} |t^m \mathcal{K}(\lambda t)|$$

$$\times \left\{ \int_{\mathbb{R}} \left| \int_0^1 (1-u)^{m-1} \left(\Delta_{ut} f^{(m)}(x+h-t) - \Delta_{ut} f^{(m)}(x-t) \right) du \right|^p dx \right\}^{\frac{1}{p}} dt$$

$$\leq \frac{\lambda}{(m-1)!} \int_{\mathbb{R}} |t^m \mathcal{K}(\lambda t)|$$

$$\times \left(\int_0^1 (1-u)^{m-1} \left\{ \int_{\mathbb{R}} \left| \Delta_{ut} f^{(m)}(x+h-t) - \Delta_{ut} f^{(m)}(x-t) \right|^p dx \right\}^{\frac{1}{p}} du \right) dt.$$

It is clear that for $1 \leq p < \infty$,

$$\left\{ \int_{\mathbb{R}} \left| \Delta_{ut} f^{(m)}(x+h-t) - \Delta_{ut} f^{(m)}(x-t) \right|^p dx \right\}^{\frac{1}{p}} \leq 2\omega \left(f^{(m)}; |ut| \right)_p$$

and

$$\left\{ \int_{\mathbb{R}} \left| \Delta_{ut} f^{(m)}(x+h-t) - \Delta_{ut} f^{(m)}(x-t) \right|^p dx \right\}^{\frac{1}{p}} \leq 2\omega \left(f^{(m)}; |h| \right)_p.$$

Using this, (3), (12), and the properties of the modulus of continuity, we get for $f \in H_m^{\omega,p}$,

$$\|E_{\lambda,m}(\cdot+h,\cdot)\|_p$$

$$\leq \frac{\lambda}{(m-1)!} (\omega(|h|))^{\frac{\beta}{\eta}}$$

$$\times \int_{\mathbb{R}} \left(|t|^m |\mathcal{K}(\lambda t)| \int_0^1 (1-u)^{m-1} \left(\omega(f^{(m)}; |ut|)_p \right)^{1-\frac{\beta}{\eta}} du \right)$$

$$= O(1) \frac{\lambda}{(m-1)!} (\omega(|h|))^{\frac{\beta}{\eta}}$$

$$\times \int_{\mathbb{R}} \left(|t|^m |\mathcal{K}(\lambda t)| \left(\omega(f^{(m)}; |t|)_p \right)^{1-\frac{\beta}{\eta}} \int_0^1 (1-u)^{m-1} du \right) dt$$

$$= O(1) \frac{\lambda}{(m-1)!} (\omega(|h|))^{\frac{\beta}{\eta}}$$

$$\times \int_{\mathbb{R}} \left(|t|^m |\mathcal{K}(\lambda t)| \left(\omega(f^{(m)}; |t|)_p \right)^{1-\frac{\beta}{\eta}} \int_0^1 (1-u)^{m-1} du \right) dt$$

$$= (\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega \left(f^{(m)}; \frac{1}{\lambda} \right)_p \right)^{1-\frac{\beta}{\eta}} \lambda \int_{\mathbb{R}} \frac{|t|^m}{m!} |\mathcal{K}(\lambda t)| (1+\lambda |t|)^{1-\frac{\beta}{\eta}} dt$$

$$= O(1) (\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega \left(\frac{1}{\lambda} \right) \right)^{1-\frac{\beta}{\eta}} \frac{1}{\lambda^m} \left(\int_0^\infty u^m |\mathcal{K}(u)| du + \int_0^\infty u^{m+1} |\mathcal{K}(u)| du \right)$$

$$= O(1) (\omega(|h|))^{\frac{\beta}{\eta}} \left(\omega \left(\frac{1}{\lambda} \right) \right)^{1-\frac{\beta}{\eta}} \frac{1}{\lambda^m}.$$

Similarly, we can prove

$$\|E_{\lambda,m}(\cdot)\|_p = O(1) \frac{1}{\lambda^m} \omega \left(f^{(m)}; \frac{1}{\lambda} \right)_p = O(1) \frac{1}{\lambda^m} \left(\omega \left(\frac{1}{\lambda} \right) \right)^{1-\frac{\beta}{\eta}}.$$

Hence, for $1 \le p < \infty$,

$$\|E_{\lambda,m}(\cdot)\|_{\omega^*,p} = \|E_{\lambda,m}(\cdot)\|_p + \sup_{h \ne 0} \frac{\|E_{\lambda,m}(\cdot + h, \cdot)\|_p}{\omega^*(|h|)}$$

$$= O(1)\sup_{h \ne 0} \frac{(\omega(|h|))^{\frac{\beta}{\eta}}}{\omega^*(|h|)}\left(\omega\left(\frac{1}{\lambda}\right)\right)^{1-\frac{\beta}{\eta}}.$$

Thus, the proof is completed. $\qquad\qquad\square$

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