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EXISTENCE AND STABILITY FOR NONLINEAR CAPUTO-HADAMARD FRACTIONAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we use the modified version of contraction mapping principle to obtain the existence and uniqueness of solutions for nonlinear Caputo-Hadamard fractional delay differential equations. We also use the method of successive approximations to show the stability of the equations. An example is given to illustrate this work.

1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including various fields of science and engineering such as applied sciences, physics, chemistry, biology, medicine, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [**1**]–[**5**], [**7**], [**8**], [**11**]–[**14**], [**16**], [**18**]–[**21**], and the references therein.

Recently, Kucche and Sutar [**13**] discussed the existence of solutions and stability results for the delay fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t)=f\left(t,x_{t}\right), & t\in[0,b]\,,\ b>0,\\ x\left(t\right)=\psi\left(t\right), & t\in[-r,0]\,,\end{cases}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $m-1<\alpha\leq m$. By employing the modified version of contraction principle and the successive approximation method, the authors obtained existence and stability results.

The implicit fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t)=f\left(t,x(t),{}^{C}D^{\alpha}x(t)\right), & t\in[0,b]\,,\ b>0,\\ x^{(k)}\left(0\right)=x_{k}\in\mathbb{R}^{n}, & k=0,1,\dots,m-1,\ m-1<\alpha\leq m,\end{cases}$$

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investigated in [**14**]. By using the modified version of contraction principle and the successive approximation method, the existence of solutions and stability results established.

In [**5**], Dhaigude and Bhairat investigated the existence of solutions and stability results of the following nonlinear implicit fractional differential equation

$$\begin{cases} \mathfrak{D}_1^{\alpha}x(t) = f\left(t, x(t), \mathfrak{D}_1^{\alpha}x(t)\right), & t \in [1,b], \ b > 1, \\ x^{(k)}(1) = x_k \in \mathbb{R}^n, & k = 0, 1, \ldots, m-1, \end{cases}$$

where $\mathfrak{D}_1^{\alpha}$ is the Caputo-Hadamard derivative of order $m-1 < \alpha \leq m$. By employing the modified version of contraction principle and the successive approximation method, the authors obtained existence and stability results.

In this paper, we are interested in the analysis of qualitative theory of the problems of the existence of solutions and stability results to delay fractional differential equations. Inspired and motivated by the works mentioned above and the references therein, we concentrate on the existence and uniqueness of solutions and stability results for the nonlinear delay fractional differential equation

$$\mathfrak{D}_1^{\alpha}x(t) = f(t, x_t), \qquad t \in [1,b], \ b > 1, \tag{1}$$

$$x(t) = \psi(t), \qquad t \in [1-r, 1], \tag{2}$$

where $f\ [1,b] \times C([1-r,1], \mathbb{R}^n) \to \mathbb{R}^n$ is a nonlinear continuous function, and $\mathfrak{D}_1^{\alpha}$ denotes the Caputo-Hadamard derivative of order $m-1 < \alpha \leq m$, $m \in \mathbb{N}$. To show the existence and uniqueness of solutions, we transform (1)–(2) into an integral equation and then use the modified version of contraction principle. Further, by the successive approximation method, we obtain Ulam-Hyers, Ulam-Hyers-Rassias, and E_{α}-Ulam-Hyers stability results of (1). Finally, we provide an example to illustrate our obtained results.

2. Preliminaries

In this section, we present some basic definitions, notations, and preliminaries. Basics of delay differential equations are considered from the monographs by Hale et al. [**9**] and Naito et al. [**15**]. Let $\mathbb{R}^n$ be an n-dimensional linear vector space over the reals with the norm

$$\|x\| = \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}, \qquad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Let $0 \leq r < \infty$ be a given real number, $C = C([1-r,1], \mathbb{R}^n)$ be the Banach space of continuous functions from $[1-r,1]$ into $\mathbb{R}^n$ with the norm

$$\|\psi\|_C = \sup_{1-r \leq \theta \leq 1} \|\psi(\theta)\|.$$

By $B = C([1-r,b], \mathbb{R}^n)$, $b > 1$, let us denote the Banach space of all continuous functions from $[1-r,b]$ into $\mathbb{R}^n$ endowed with supremum norm $\|\cdot\|_B$. For any $x \in B$ and any $t \in [1,b]$, by x_t, we denote the element of C defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [1-r,1]$.

We use the following results in our analysis.

Lemma 2.1 ([**15**]). *If $x \in C([1-r,b], \mathbb{R}^n)$, then x_t is continuous with respect to $t \in [1,b]$.*

Lemma 2.2 ([**15**]). *Let $x\ [1-r,b) \to \mathbb{R}^n$ be a continuous function with $x_1 = \psi$. If*

$$\|x(t)\| \le \|\psi(1)\| + m(t), \qquad t \in [1,b)$$

where m is a nondecreasing function, then

$$\|x_t\|_C \le \|\psi\|_C + m(t), \qquad t \in [1,b).$$

For fundamentals of fractional calculus, we refer the research monographs [**7, 12, 16**].

Definition 2.3 ([**12**]). The one parameter Mittag-Leffler function is defined by

$$E_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k+1)}, \qquad z \in \mathbb{R},\ \gamma > 0, \tag{3}$$

where $\Gamma(x) = \int_0^\infty \exp(-t)\, t^{x-1}$, $x > 0$, is the Gamma function.

Definition 2.4 ([**12**]). The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $g\ [1,+\infty) \to \mathbb{R}$, is defined as

$$\mathfrak{I}_1^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} g(s) \frac{\mathrm{d}s}{s}, \qquad \alpha > 0. \tag{4}$$

Definition 2.5 ([**12**]). The Caputo-Hadamard fractional derivative of order α for a continuous function $g\ [1,+\infty) \to \mathbb{R}$, is defined as

$$\mathfrak{D}_1^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} \delta^n(g)(s) \frac{\mathrm{d}s}{s}, \qquad n-1 < \alpha < n, \tag{5}$$

where $\delta^n = \left(t\dfrac{\mathrm{d}}{\mathrm{d}t}\right)^n$, $n \in \mathbb{N}$.

Lemma 2.6 ([**12**]). *Let $m-1 < \alpha < m$, $m \in \mathbb{N}$, and $g \in C^m[1,b]$. Then*

$$\mathfrak{I}_1^\alpha [\mathfrak{D}_1^\alpha g(t)] = g(t) - \sum_{k=0}^{m-1} \frac{g^{(k)}(1)}{\Gamma(k+1)} (\log t)^k.$$

Lemma 2.7 ([**10**]). *For any $t \in [1,b]$,*

$$u(t) \le a(t) + c(t) \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} u(s) \frac{\mathrm{d}s}{s},$$

where all the functions are not negative and continuous. The constant $\alpha > 0$, c is a bounded and monotonic increasing function on $[1,b)$, then

$$u(t) \le a(t) + \int_1^t \left[\sum_{n=1}^{\infty} \frac{(c(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\log\frac{t}{s}\right)^{n\alpha-1} a(s)\right] \frac{\mathrm{d}s}{s}, \qquad t \in [1,b).$$

Remark. Under the hypothesis of Lemma 2.7, let a be a nondecreasing function on $[1,b)$. Then
$$u(t)\leq a(t)E_{\alpha}\left(c(t)\Gamma\left(\alpha\right)\left(\log t\right)^{\alpha}\right).$$

Lemma 2.8 ([**12**])**.** *For all $\mu>0$ and $\nu>-1$,*
$$\frac{1}{\Gamma\left(\mu\right)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\mu-1}\left(\log s\right)^{\nu}\frac{\mathrm{d}s}{s}=\frac{\Gamma\left(\nu+1\right)}{\Gamma\left(\mu+\nu+1\right)}\left(\log t\right)^{\mu+\nu}.$$

3. Existence and uniqueness results

To obtain existence and uniqueness of solution to the initial value problem (1)–(2), we use the following lemma.

Lemma 3.1 (Modified version of contraction principle [**6, 17**])**.** *Let X be a Banach space and let D be an operator which maps the element of X into itself for which D^{r} is a contraction, where r is a positive integer, then D has a unique fixed point.*

Definition 3.2. A function x is said to be a solution of (1)–(2) if x satisfies nonlinear implicit fractional differential system of equations $\mathfrak{D}_{1}^{\alpha}x(t)=f\left(t,x_{t}\right)$ on $[1,b]$ and $x(t)=\psi(t)$ on $[1-r,1]$.

The proof of the following Lemma is close to the proof of Lemma 6.2 given in [**7**].

Lemma 3.3. *Let $f\ [1,b]\times C\rightarrow\mathbb{R}^{n}$ is continuous, then* (1)–(2) *is equivalent to the following fractional integral equation*
$$x(t)=\begin{cases}\psi(t), & t\in\left[1-r,1\right],\\ \sum\limits_{k=0}^{m-1}\dfrac{\psi^{(k)}(1)}{\Gamma\left(k+1\right)}(\log t)^{k}+\dfrac{1}{\Gamma\left(\alpha\right)}\int_{1}^{t}\left(\log\dfrac{t}{s}\right)^{\alpha-1}f\left(s,x_{s}\right)\dfrac{\mathrm{d}s}{s}, & t\in\left[1,b\right].\end{cases}$$

Next theorem guarantees existence and uniqueness of solution to initial value problem (1)–(2).

Theorem 3.4. *If $f\,[1,b]\times C\rightarrow\mathbb{R}^{n}$ a continuous function that satisfies Lipschitz condition with respect to the second variable*
$$\left\|f(t,u)-f(t,v)\right\|\leq M\left\|u-v\right\|_{C},\ t\in\left[1,b\right],\qquad u,v\in C,$$
then (1)–(2) *has unique solution $x\ [1-r,b]\rightarrow\mathbb{R}^{n}$.*

Proof. Consider the operator $F\ B\rightarrow B$ defined by
$$\left(Fx\right)(t)=\begin{cases}\psi(t),\ t\in\left[1-r,1\right],\\ \sum\limits_{k=0}^{m-1}\dfrac{\psi^{(k)}(1)}{\Gamma(k+1)}(\log t)^{k}+\dfrac{1}{\Gamma\left(\alpha\right)}\displaystyle\int_{1}^{t}\left(\log\dfrac{t}{s}\right)^{\alpha-1}f\left(s,x_{s}\right)\dfrac{\mathrm{d}s}{s}, \quad t\in\left[1,b\right].\end{cases}$$
Note that by definition of operator F, for any $x,z\in B$, we have
$$\left\|\left(F^{j}x\right)(t)-\left(F^{j}z\right)(t)\right\|=0\qquad\text{for all } t\in\left[1-r,1\right]\text{ and } j\in\mathbb{N}.\tag{6}$$

By using mathematical induction, for any $x, z \in B$ and $t \in [1, b]$, we prove that

$$\left\| \left(F^j x\right)(t) - \left(F^j z\right)(t) \right\| \le \frac{\left(M (\log t)^\alpha\right)^j}{\Gamma(j\alpha + 1)} \|x - z\|_B \qquad \text{for all } j \in \mathbb{N}. \tag{7}$$

By definition of operator F and using Lipschitz condition, for any $x, z \in B$ and $t \in [1, b]$, we have

$$\begin{aligned} \|(Fx)(t) - (Fz)(t)\| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|f(s, x_s) - f(s, z_s)\| \frac{\mathrm{d}s}{s} \\ &\le \frac{M}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|x_s - z_s\|_C \frac{\mathrm{d}s}{s}. \end{aligned}$$

For any $t \in [1, b]$ and $\theta \in [1 - r, 1]$, we have $1 - r \le t + \theta \le b$, and hence

$$\begin{aligned} \|x_t\|_C &= \sup\{x_t(\theta) : \theta \in [1 - r, 1]\} = \sup\{x(t + \theta) : \theta \in [1 - r, 1]\} \\ &\le \sup\{x(t + \theta) : 1 - r \le t + \theta \le b\} \le \|x\|_B. \end{aligned}$$

Thus

$$\begin{aligned} \|(Fx)(t) - (Fz)(t)\| &\le \frac{M}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|x - z\|_B \frac{\mathrm{d}s}{s} \\ &\le \frac{M}{\Gamma(\alpha)} \left(\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\mathrm{d}s}{s}\right) \|x - z\|_B. \end{aligned}$$

Therefore,

$$\|(Fx)(t) - (Fz)(t)\| \le \frac{M (\log t)^\alpha}{\Gamma(\alpha + 1)} \|x - z\|_B, \qquad t \in [1, b].$$

Thus the inequality (7) is true for $j = 1$. Let us suppose that the inequality (7) holds for $j = r \in \mathbb{N}$, hence

$$\|(F^r x)(t) - (F^r z)(t)\| \le \frac{\left(M (\log t)^\alpha\right)^r}{\Gamma(r\alpha + 1)} \|x - z\|_B, \qquad t \in [1, b]. \tag{8}$$

We prove that inequality (7) holds for $j = r + 1$. Let any $x, z \in B$ and denote $\hat{x} = F^r x$ and $\hat{z} = F^r z$. Then using definition of operator F and the Lipschitz condition of f, for any $t \in [1, b]$, we get

$$\begin{aligned} \left\| \left(F^{r+1} x\right)(t) - \left(F^{r+1} z\right)(t) \right\| &= \|F((F^r x)(t)) - F((F^r z)(t))\| \\ &= \|F(\hat{x}(t)) - F(\hat{z}(t))\| \\ &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|f(t, \hat{x}_s) - f(t, \hat{z}_s)\| \frac{\mathrm{d}s}{s} \\ &\le \frac{M}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|\hat{x}_s - \hat{z}_s\|_C \frac{\mathrm{d}s}{s}. \end{aligned} \tag{9}$$

From (8), we write

$$\|\hat{x}(t) - \hat{z}(t)\| = \|(F^r x)(t) - (F^r z)(t)\| \le \frac{\left(M (\log t)^\alpha\right)^r}{\Gamma(r\alpha + 1)} \|x - z\|_B.$$

An application of Lemma 2.2 gives

$$\left\|\hat{x}_t - \hat{z}_t\right\|_C \leq \frac{\left(M(\log t)^{\alpha}\right)^r}{\Gamma\left(r\alpha + 1\right)} \left\|x - z\right\|_B .$$

By using the above inequality in (9), and then applying Lemma 2.8, we get

$$\begin{aligned}
&\left\|\left(F^{r+1}x\right)(t) - \left(F^{r+1}z\right)(t)\right\| \\
&\leq \frac{M}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\left(M(\log t)^{\alpha}\right)^r}{\Gamma\left(r\alpha + 1\right)} \left\|x - z\right\|_B \frac{\mathrm{d}s}{s} \\
&= \frac{M^{r+1}}{\Gamma(\alpha)\Gamma(r\alpha+1)} \left(\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{\alpha r} \frac{\mathrm{d}s}{s}\right) \left\|x - z\right\|_B \\
&= \frac{M^{r+1}}{\Gamma(\alpha)\Gamma(r\alpha+1)} (\log t)^{(r+1)\alpha} \frac{\Gamma(\alpha)\Gamma(r\alpha+1)}{\Gamma((r+1)\alpha+1)} \left\|x - z\right\|_B \\
&= \frac{\left(M(\log t)^{\alpha}\right)^{r+1}}{\Gamma((r+1)\alpha+1)} \left\|x - z\right\|_B .
\end{aligned}$$

Thus

$$\left\|\left(F^{r+1}\right)x(t) - \left(F^{r+1}z\right)(t)\right\| \leq \frac{\left(M(\log t)^{\alpha}\right)^{r+1}}{\Gamma((r+1)\alpha+1)} \left\|x - z\right\|_B , \qquad t \in [1, b] .$$

We have proved that the inequality (7) holds for $j = r + 1$. By the principle of mathematical induction, the proof of inequality (7) is completed. Combining (6) and (7), we obtain

$$\left\|\left(F^j x\right)(t) - \left(F^j z\right)(t)\right\| \leq \frac{\left(M(\log t)^{\alpha}\right)^j}{\Gamma(j\alpha+1)} \left\|x - z\right\|_B \tag{10}$$

for $t \in [1 - r, b]$ and for all $j \in \mathbb{N}$.
This gives

$$\begin{aligned}
\left\|\left(F^j x\right) - \left(F^j z\right)\right\|_B &= \sup_{t\in[1-r,b]} \left\|\left(F^j x\right)(t) - \left(F^j z\right)(t)\right\| \\
&\leq \frac{\left(M(\log b)^{\alpha}\right)^j}{\Gamma(j\alpha+1)} \left\|x - z\right\|_B .
\end{aligned}$$

By definition of Mittag-Leffler function, we have

$$E_\alpha\left(M(\log b)^{\alpha}\right) = \sum_{j=0}^{\infty} \frac{\left(M(\log b)^{\alpha}\right)^j}{\Gamma(j\alpha+1)}.$$

Note that $\frac{(M(\log b)^{\alpha})^j}{\Gamma(j\alpha+1)}$ is the j^{th} term of the convergent series of nonnegative real numbers, this gives

$$\lim_{j\to\infty} \frac{\left(M(\log b)^{\alpha}\right)^j}{\Gamma(j\alpha+1)} = 0.$$

Thus we can choose $j \in \mathbb{N}$ such that $\frac{(M(\log b)^{\alpha})^j}{\Gamma(j\alpha+1)} < 1$ so that F^j is a contraction. Therefore, by modified version of contraction principle, F has a unique fixed point $x\,[1 - r, b] \to \mathbb{R}^n$ in B, which is the unique solution of (1)–(2). □

4. Ulam-Hyers stability

We adopt the definitions of Ulam-Hyers stability, generalized Ulam-Hyers stability, and Ulam-Hyers-Rassias stability given in [**20**].

Definition 4.1. We say that (1) has Ulam-Hyers stability if there exists a real number $K_f > 0$ such that for each $\varepsilon > 0$, if $y\ [1-r,b] \to \mathbb{R}^n$ in B satisfies

$$\|\mathfrak{D}_1^\alpha y(t) - f(t, y_t)\| \leq \varepsilon, \qquad t \in [1,b],$$

then there exists a solution $x\ [1-r,b] \to \mathbb{R}^n$ of (1) in B with

$$\|y(t) - x(t)\| \leq K_f \varepsilon, \qquad t \in [1-r,b].$$

Moreover, if $x^{(k)}(1) = y^{(k)}(1)$, $k = 0,1,2,\ldots,m-1$, (1) is Ulam-Hyers stable with initial conditions.

Definition 4.2. We say that (1) has generalized Ulam-Hyers stability if there exists $\varphi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\varphi_f(0) = 0$, such that for each $\varepsilon > 0$, if $y\ [1-r,b] \to \mathbb{R}^n$ in B satisfies

$$\|\mathfrak{D}_1^\alpha y(t) - f(t, y_t)\| \leq \varepsilon, \qquad t \in [1,b],$$

then there exists a solution $x\ [1-r,b] \to \mathbb{R}^n$ of (1) in B with

$$\|y(t) - x(t)\| \leq \varphi_f(\varepsilon), \qquad t \in [1-r,b].$$

Definition 4.3. We say that (1) has Ulam-Hyers-Rassias stability with respect to $\eta \in C([1,b], \mathbb{R}_+)$ if there exists a real number $K_{f,\eta} > 0$ such that for each $\varepsilon > 0$, if $y\ [1-r,b] \to \mathbb{R}^n$ in B satisfies

$$\|\mathfrak{D}_1^\alpha y(t) - f(t, y(t))\| \leq \varepsilon\eta(t), \qquad t \in [1,b],$$

then there exists a solution $x\ [1-r,b] \to \mathbb{R}^n$ of (1) in B with

$$\|y(t) - x(t)\| \leq \varepsilon K_{f,\eta}\eta(t), \qquad t \in [1-r,b].$$

Definition 4.4. We say that (1) has generalized Ulam-Hyers-Rassias stability with respect to $\eta \in C([1,b], \mathbb{R}_+)$, if there exists a real number $K_{f,\eta} > 0$ such that if $y\ [1-r,b] \to \mathbb{R}^n$ in B satisfies

$$\|\mathfrak{D}_1^\alpha y(t) - f(t, y(t))\| \leq \eta(t), \qquad t \in [1,b],$$

then there exists a solution $x\ [1-r,b] \to \mathbb{R}^n$ of (1) in B with

$$\|y(t) - x(t)\| \leq K_{f,\eta}\eta(t), \qquad t \in [1-r,b].$$

In the following theorem, by method of successive approximation, we prove that (1) is Ulam-Hyers stable.

Theorem 4.5. *Let $f\ [1,b] \times C([1-r,1], \mathbb{R}^n) \to \mathbb{R}^n$ be a continuous function that satisfies Lipschitz condition*

$$\|f(t,u) - f(t,v)\| \leq M \|u - v\|_C, \ t \in [1,b], \qquad u, v \in C.$$

For every $\varepsilon > 0$, if $y\ [1-r,b] \to \mathbb{R}^n$ in B satisfies

$$\|\mathfrak{D}_1^\alpha y(t) - f(t, y_t)\| \leq \varepsilon, \qquad t \in [1,b],$$

then there exists a unique solution $x\ [1-r,b] \to \mathbb{R}^n$ *of* (1) *in* B *with* $x^{(k)}(1) = y^{(k)}(1)$, $k = 0,1,2,\dots,m-1$, *that satisfies*

$$\|y(t) - x(t)\| \le \Big(\frac{E_\alpha\left(M\left(\log b\right)^{\alpha}\right) - 1}{M}\Big)\varepsilon, \qquad t \in [1-r,b]\,.$$

Proof. For every $\varepsilon > 0$, let $y\ [1-r,b] \to \mathbb{R}^n$ in B satisfy,

$$\|\mathfrak{D}_1^\alpha y(t) - f\left(t, y_t\right)\| \le \varepsilon, \qquad t \in [1,b] \tag{11}$$

Then there exists a function $\sigma_y \in B$ (depending on y) such that

$$\|\sigma_y(t)\| \le \varepsilon, \qquad t \in [1,b]$$

and

$$\mathfrak{D}_1^\alpha y(t) = f\left(t, y_t\right) + \sigma_y(t), \qquad t \in [1,b]\,. \tag{12}$$

In the light of Lemma 3.3, y satisfies the fractional integral equation

$$\begin{aligned} y(t) = &\sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^k + \frac{1}{\Gamma\left(\alpha\right)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f\left(s, y_s\right) \frac{\mathrm{d}s}{s} \\ &+ \frac{1}{\Gamma\left(\alpha\right)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \sigma_y(s) \frac{\mathrm{d}s}{s}, \qquad t \in [1,b] \end{aligned} \tag{13}$$

Define

$$x^0(t) = y(t), \qquad t \in [1-r,b]\,,$$

and consider the sequence $\left\{x^j\right\} \subseteq B$ defined by

$$x^j(t) = \begin{cases} y(t), & t \in [1-r,1]\,, \\ \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^k & \\ +\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f\left(s, x_s^{j-1}\right) \frac{\mathrm{d}s}{s}, & t \in [1,b]\,. \end{cases} \tag{14}$$

By using the principle of mathematical induction, we prove that

$$\left\|x^j(t) - x^{j-1}(t)\right\| \le \frac{\varepsilon\left(M(\log t)^\alpha\right)^j}{M\Gamma\left(j\alpha+1\right)}, \qquad j \in \mathbb{N},\ t \in [1,b]\,. \tag{15}$$

First we show that inequality (15) is true for $j = 1$. By definition of successive approximations, for any $t \in [1,b]$, we obtain

$$\begin{aligned} \|x^1(t) - x^0(t)\| &= \left\| \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} f\left(s, x_s^0\right) \frac{\mathrm{d}s}{s} - y(t) \right\| \\ &= \left\| \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} f\left(s, y_s\right) \frac{\mathrm{d}s}{s} - y(t) \right\| \\ &= \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} \sigma_y(s) \frac{\mathrm{d}s}{s} \right\| \\ &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \|\sigma_y(s)\| \frac{\mathrm{d}s}{s} \le \varepsilon \frac{(\log t)^\alpha}{\Gamma\left(\alpha+1\right)}, \end{aligned}$$

which proves the inequality (15) for $j = 1$. Let us suppose that the inequality (15) holds for $j = h \in \mathbb{N}$ prove it also holds for $j = h + 1 \in \mathbb{N}$.

By using the definition of successive approximations and Lipschitz condition of f, for any $t \in [1, b]$, we obtain

$$\begin{aligned}\left\|x^{h+1}(t) - x^h(t)\right\| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left\|f\left(s, x_s^h\right) - f\left(s, x_s^{h-1}\right)\right\| \frac{\mathrm{d}s}{s} \\ &\leq \frac{M}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left\|x_s^h - x_s^{h-1}\right\|_C \frac{\mathrm{d}s}{s}.\end{aligned} \tag{16}$$

Since (15) holds for $j = h$, we have

$$\left\|x^h(t) - x^{h-1}(t)\right\| \leq \frac{\varepsilon \left(M(\log t)^\alpha\right)^h}{M\Gamma\left(h\alpha + 1\right)}, \qquad t \in [1, b]\,.$$

Therefore, by using Lemma 2.2, we get

$$\left\|x_t^h - x_t^{h-1}\right\|_C \leq \frac{\varepsilon}{M} \frac{\left(M(\log t)^\alpha\right)^h}{\Gamma\left(h\alpha + 1\right)}, \qquad t \in [1, b]\,.$$

Thus the inequality (16) reduces to

$$\begin{aligned}\left\|x^{h+1}(t) - x^h(t)\right\| &\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\left(M(\log t)^\alpha\right)^h}{\Gamma\left(h\alpha + 1\right)} \frac{\mathrm{d}s}{s} \\ &= \frac{\varepsilon M^h}{\Gamma\left(h\alpha + 1\right)} \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log s\right)^{h\alpha} \frac{\mathrm{d}s}{s}\right).\end{aligned}$$

Using Lemma 2.8 in the above inequality, we get

$$\left\|x^{h+1}(t) - x^h(t)\right\| \leq \frac{\varepsilon M^h}{\Gamma\left(h\alpha + 1\right)} \frac{\Gamma\left(h\alpha + 1\right)}{\Gamma\left(\left(h + 1\right)\alpha + 1\right)} (\log t)^{(h+1)\alpha}.$$

Therefore,

$$\left\|x^{h+1}(t) - x^h(t)\right\| \leq \frac{\varepsilon}{M} \frac{\left(M(\log t)^\alpha\right)^{h+1}}{\Gamma\left(\left(h + 1\right)\alpha + 1\right)}, \qquad t \in [1, b]$$

which is the inequality (15) for $j = h + 1$. Using the principle of mathematical induction, the proof of the inequality (15) is completed.

Furthermore, for any $t \in [1, b]$, from inequality (15), we obtain

$$\sum_{j=1}^{\infty} \left\|x^j(t) - x^{j-1}(t)\right\| \leq \frac{\varepsilon}{M} \sum_{j=1}^{\infty} \frac{\left(M\left(\log b\right)^\alpha\right)^j}{\Gamma\left(j\alpha + 1\right)} = \frac{\varepsilon}{M} E_\alpha \left(M\left(\log b\right)^\alpha - 1\right). \tag{17}$$

Hence the series

$$x^0(t) + \sum_{j=1}^{\infty} \left[x^j(t) - x^{j-1}(t)\right]$$

converges absolutely and uniformly on $[1,b]$, with respect to the norm $\|\cdot\|$. Let us suppose

$$\tilde{x}(t)=x^{0}(t)+\sum_{j=1}^{\infty}\left[x^{j}(t)-x^{j-1}(t)\right], \qquad t\in[1,b]. \tag{18}$$

Then

$$x^{h}(t)=x^{0}(t)+\sum_{j=1}^{h}\left[x^{j}(t)-x^{j-1}(t)\right] \tag{19}$$

is the h^{th} partial sum of the series (18). Therefore, we can write,

$$\lim_{h\to\infty}\left\|x^{h}(t)-\tilde{x}(t)\right\|=0 \qquad \text{for all } t\in[1,b].$$

Further by definition of successive approximations, we have,

$$x^{h}(t)=y(t), \qquad t\in[1-r,1].$$

Therefore,

$$\lim_{h\to\infty}x^{h}(t)=y(t), \qquad t\in[1-r,1].$$

Define

$$x(t)=\begin{cases} y(t), & t\in[1-r,1], \\ \tilde{x}(t), & t\in[1,b]. \end{cases}$$

Clearly $x\in B$. We prove that this limit function is the solution of fractional integral equation

(20)

$$x(t)=\begin{cases} y(t), & t\in[1-r,1], \\ \sum\limits_{k=0}^{m-1}\dfrac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^{k}+\dfrac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\dfrac{t}{s}\right)^{\alpha-1}f\left(s,x_{s}\right)\dfrac{\mathrm{d}s}{s}, & t\in[1,b]. \end{cases}$$

Using definition of successive approximations for any $t\in[1,b]$, we have

$$\begin{aligned} &\left\|x(t)-\sum_{k=0}^{m-1}\frac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^{k}-\frac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}f\left(s,x_{s}\right)\frac{\mathrm{d}s}{s}\right\| \\ &=\left\|\tilde{x}(t)-\left(x^{h}(t)-\frac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}f\left(s,x_{s}^{h-1}\right)\frac{\mathrm{d}s}{s}\right)\right. \\ &\qquad\left.-\frac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}f\left(s,x_{s}\right)\frac{\mathrm{d}s}{s}\right\| \\ &\le\left\|\tilde{x}(t)-x^{h}(t)\right\|+\frac{1}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left\|f\left(s,x_{s}^{h-1}\right)-f\left(s,x_{s}\right)\right\|\frac{\mathrm{d}s}{s} \\ &\le\left\|\tilde{x}(t)-x^{h}(t)\right\|+\frac{M}{\Gamma(\alpha)}\int_{1}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left\|x_{s}^{h-1}-x_{s}\right\|_{C}\frac{\mathrm{d}s}{s}. \end{aligned} \tag{21}$$

Now for any $t \in [1, b]$, from equations (18) and (19), we write

$$\left\|\tilde{x}(t) - x^h(t)\right\| = \left\| \sum_{j=h+1}^{\infty} \left[x^j(t) - x^{j-1}(t)\right] \right\| \leq \sum_{j=h+1}^{\infty} \left\|x^j(t) - x^{j-1}(t)\right\|.$$

Using the inequality (15), we obtain

$$\left\|x(t) - x^h(t)\right\| = \left\|\tilde{x}(t) - x^h(t)\right\| \leq \sum_{j=h+1}^{\infty} \frac{\varepsilon}{M} \frac{(M(\log t)^\alpha)^j}{\Gamma(j\alpha + 1)}, \qquad t \in [1, b]. \tag{22}$$

Applying Lemma 2.2, we get

$$\left\|x_t - x_t^h\right\|_C \leq \sum_{j=h+1}^{\infty} \frac{\varepsilon}{M} \frac{(M(\log t)^\alpha)^j}{\Gamma(j\alpha + 1)}. \tag{23}$$

Using (22) and (23) in (21), we obtain

$$\begin{aligned}
&\left\| x(t) - \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x_s) \frac{\mathrm{d}s}{s} \right\| \\
&\leq \sum_{j=h+1}^{\infty} \frac{\varepsilon}{M} \frac{(M(\log t)^\alpha)^j}{\Gamma(j\alpha+1)} + \frac{M}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \sum_{j=h+1}^{\infty} \frac{\varepsilon}{M} \frac{(M(\log s)^\alpha)^j}{\Gamma(j\alpha+1)} \frac{\mathrm{d}s}{s} \\
&= \sum_{j=h+1}^{\infty} \frac{\varepsilon}{M} \frac{(M(\log t)^\alpha)^j}{\Gamma(j\alpha+1)} + \varepsilon \sum_{j=h+1}^{\infty} \frac{M^j}{\Gamma(j\alpha+1)} \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (\log s)^{j\alpha} \frac{\mathrm{d}s}{s} \right) \\
&= \sum_{j=h+1}^{\infty} \frac{\varepsilon}{M} \frac{(M(\log t)^\alpha)^j}{\Gamma(j\alpha+1)} + \varepsilon \sum_{j=h+1}^{\infty} \frac{M^j}{\Gamma(j\alpha+1)} \frac{\Gamma(j\alpha+1)}{\Gamma((j+1)\alpha+1)} (\log t)^{(j+1)\alpha} \\
&= \frac{\varepsilon}{M} \sum_{j=h+1}^{\infty} \frac{(M(\log t)^\alpha)^j}{\Gamma(j\alpha+1)} + \frac{\varepsilon}{M} \sum_{j=h+1}^{\infty} \frac{(M(\log t)^\alpha)^{(j+1)}}{\Gamma((j+1)\ alpha+1)}, \qquad t \in [1, b].
\end{aligned}$$

Since both series on the right hand side of the above inequality are convergent, by taking limit as $j \to \infty$, we obtain

$$\left\| x(t) - \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x_s) \frac{\mathrm{d}s}{s} \right\| \leq 0, \qquad t \in [1, b].$$

This implies

$$x(t) = \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x_s) \frac{\mathrm{d}s}{s}, \qquad t \in [1, b]. \tag{24}$$

Therefore, x is a solution of (1) with the initial condition

$$x^{(k)}(1) = y^{(k)}(1), \qquad k = 0, 1, 2, \ldots, m-1.$$

Further, from equations (17), (18), and (20), we have

$$\|y(t) - x(t)\| \leq \left(\frac{E_\alpha \left(M (\log b)^\alpha\right) - 1}{M} \right) \varepsilon, \qquad t \in [1 - r, b].$$

This proves that (1) is Ulam-Hyers stable. Moreover, as $x^{(k)}(1) = y^{(k)}(1)$, $k = 0, 1, \ldots, m-1$, (1) has Ulam-Hyers stability with the initial conditions.

It remains to prove the uniqueness of x. Assume $\bar{x}$ is another solution of (1) with the initial conditions $\bar{x}(1) = y^{(k)}(1)$, $k = 0, 1, \ldots, m-1$. Then

$$\bar{x}(t) = \begin{cases} y(t), & t \in [1-r, 1], \\ \sum\limits_{k=0}^{m-1} \dfrac{y^{(k)}(1)}{\Gamma(k+1)} (\log t)^k + \dfrac{1}{\Gamma(\alpha)} \int_1^t \left(\log \dfrac{t}{s} \right)^{\alpha-1} f(s, \bar{x}_s) \dfrac{\mathrm{d}s}{s}, & t \in [1, b]. \end{cases}$$

Note that

$$\|x(t) - \bar{x}(t)\| = 0, \qquad t \in [1-r, 1].$$

By using Lipschitz condition, we find that

$$\|x(t) - \bar{x}(t)\| \leq \frac{M}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \|x_s - \bar{x}_s\|_C \frac{\mathrm{d}s}{s}, \qquad t \in [1, b].$$

Using Lemma 2.2,

$$\|x_t - \bar{x}_t\|_C \leq \frac{M}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \|x_s - \bar{x}_s\|_C \frac{\mathrm{d}s}{s}, \qquad t \in [1, b].$$

By applying Lemma 2.7 to the above inequality with $u(t) = \|x_t - \bar{x}_t\|_C$ and $a(t) = 0$, we obtain

$$\|x_t - \bar{x}_t\|_C = 0, \qquad t \in [1, b].$$

Hence $\|x(t) - \bar{x}(t)\| = 0$ for all $t \in [1-r, b]$. This completes the proof. $\square$

Remark. If we set $\varphi_f(\varepsilon) = \left(\frac{E_\alpha(M(\log b)^\alpha) - 1}{M} \right) \varepsilon$, then $\varphi_f(0) = 0$. Hence (1) is generalized Ulam-Hyers stable with the initial conditions.

Next we obtain Ulam-Hyers-Rassias stability result for (1) by method of successive approximations.

Theorem 4.6. *Let $f\ [1, b] \times C \to \mathbb{R}^n$ be a continuous function that satisfies Lipschitz condition with respect to the second variable*

$$\|f(t, u) - f(t, v)\| \leq M \|u - v\|_C, \qquad t \in [1, b], \ u, v \in C.$$

For every $\varepsilon > 0$, if $y\ [1-r, b] \to \mathbb{R}^n$ in B satisfies

$$\|\mathfrak{D}_1^\alpha y(t) - f\| \leq \varepsilon \eta(t), \qquad t \in [1, b],$$

where $\eta \in C([1, b], \mathbb{R}_+)$ is a nondecreasing function such that

$$\left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \eta(s) \frac{\mathrm{d}s}{s} \right| \leq \lambda \eta(t), \qquad t \in [1, b],$$

and $\lambda > 0$ is a constant satisfying $0 < \lambda M < 1$, then there exists a unique solution $x\ [1-r, b] \to \mathbb{R}^n$ of (1) in B with $x^{(k)}(1) = y^{(k)}(1)$, $k = 0, 1, 2, \ldots, m-1$, that satisfies

$$\|y(t) - x(t)\| \leq \frac{\lambda}{(1 - \lambda M)} \varepsilon \eta(t), \qquad t \in [1-r, b].$$

Proof. For every $\varepsilon > 0$, let $y\ [1-r,b] \to \mathbb{R}^n$ in B satisfy

$$\|\mathfrak{D}_1^\alpha y(t) - f(t, y_t)\| \leq \varepsilon\eta(t), \qquad t \in [1,b]\,. \tag{25}$$

Then there exists a function $\sigma_y \in B$ (depending on y) such that

$$\|\sigma_y(t)\| \leq \varepsilon\eta(t), \qquad t \in [1,b]\,,$$

and

$$\mathfrak{D}_1^\alpha y(t) = f(t, y_t) + \sigma_y(t), \qquad t \in [1,b]\,.$$

By Lemma 3.3, y satisfies the fractional integral equation

$$\begin{aligned} y(t) = &\sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y_s) \frac{\mathrm{d}s}{s} \\ &+ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \sigma_y(s) \frac{\mathrm{d}s}{s}, \qquad t \in [1,b]\,. \end{aligned}$$

Let define the sequence of approximation $\{x^j\} \subseteq B$ as in proof of Theorem 4.5, starting with $x^0(t) = y(t)$ for $t \in [1-r,b]$. By using mathematical induction, we prove that

$$\left\|x^j(t) - x^{j-1}(t)\right\| \leq \frac{\varepsilon}{M} (\lambda M)^j \eta(t), \qquad j \in \mathbb{N},\ t \in [1,b]\,. \tag{26}$$

First we show that inequality (26) is true for $j = 1$. By definition of successive approximations, for any $t \in [1,b]$, we obtain

$$\begin{aligned} &\left\|x^1(t) - x^0(t)\right\| \\ &= \left\| \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f\left(s, x_s^0\right) \frac{\mathrm{d}s}{s} - y(t) \right\| \\ &= \left\| \sum_{k=0}^{m-1} \frac{y^{(k)}(1)}{\Gamma(k+1)}(\log t)^k + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y_s) \frac{\mathrm{d}s}{s} - y(t) \right\| \\ &= \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \sigma_y(s) \frac{\mathrm{d}s}{s} \right\| \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|\sigma_y(s)\| \frac{\mathrm{d}s}{s} \\ &\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \eta(t) \frac{\mathrm{d}s}{s} \leq \varepsilon\lambda\eta(t). \end{aligned}$$

Therefore,

$$\left\|x^1(t) - x^0(t)\right\| \leq \frac{\varepsilon}{M} (\lambda M)\, \eta(t), \qquad t \in [1,b]\,,$$

which is the inequality (26) for $j = 1$.

Let us suppose that the inequality (26) holds for $j = h \in \mathbb{N}$. Then by definition of successive approximations and Lipschitz condition of f for any $t \in [1,b]$, we obtain

$$\begin{aligned}
\left\|x^{h+1}(t)-x^{h}(t)\right\| &\leq \frac{1}{\Gamma(\alpha)} \int_{1}^{t}\left(\log \frac{t}{s}\right)^{\alpha-1}\left\|f\left(s, x_{s}^{h}\right)-f\left(s, x_{s}^{h-1}\right)\right\| \frac{\mathrm{d} s}{s} \\
&\leq \frac{M}{\Gamma(\alpha)} \int_{1}^{t}\left(\log \frac{t}{s}\right)^{\alpha-1}\left\|x_{s}^{h}-x_{s}^{h-1}\right\|_{C} \frac{\mathrm{d} s}{s} .
\end{aligned} \tag{27}$$

Since (26) holds for $j=h$, we have

$$\left\|x^{h}(t)-x^{h-1}(t)\right\| \leq \frac{\varepsilon}{M}(\lambda M)^{h} \eta(t), \qquad t \in[1, b] .$$

Therefore, by using Lemma 2.2, we get

$$\left\|x_{t}^{h}-x_{t}^{h-1}\right\|_{C} \leq \frac{\varepsilon}{M}(\lambda M)^{h} \eta(t), \qquad t \in[1, b] .$$

Thus the inequality (27) reduces to

$$\begin{aligned}
\left\|x^{h+1}(t)-x^{h}(t)\right\| &\leq \frac{M}{\Gamma(\alpha)} \int_{1}^{t}\left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\varepsilon}{M}(\lambda M)^{h} \eta(t) \frac{\mathrm{d} s}{s} \\
&=\varepsilon(\lambda M)^{h}\left(\frac{1}{\Gamma(\alpha)} \int_{1}^{t}\left(\log \frac{t}{s}\right)^{\alpha-1} \eta(t) \frac{\mathrm{d} s}{s}\right) \\
&\leq \varepsilon(\lambda M)^{h} \lambda \eta(t) \leq \frac{\varepsilon}{M}(\lambda M)^{h+1} \eta(t) .
\end{aligned}$$

Therefore,

$$\left\|x^{h+1}(t)-x^{h}(t)\right\| \leq \frac{\varepsilon}{M}(\lambda M)^{h+1} \eta(t), \qquad t \in[1, b],$$

which is the inequality (26) for $j=h+1$. Using principle of mathematical induction, the proof of the inequality (26) is completed.

Using the inequality (26) and the fact $0<\lambda M<1$, for any $t \in[1, b]$,

$$\sum_{j=1}^{\infty}\left\|x^{j}(t)-x^{j-1}(t)\right\| \leq \frac{\varepsilon}{M} \sum_{j=1}^{\infty}(\lambda M)^{h} \eta(t)=\frac{\varepsilon}{M} \frac{\lambda M}{(1-\lambda M)} \eta(t) . \tag{28}$$

Thus

$$\sum_{j=1}^{\infty}\left\|x^{j}(t)-x^{j-1}(t)\right\| \leq \frac{\lambda}{(1-\lambda M)} \varepsilon \eta(t), \qquad t \in[1, b] . \tag{29}$$

Since η is continuous on compact set $[1, b]$, it is bounded. Clearly, from the above inequality (29), it follows that the series

$$x^{0}(t)+\sum_{j=1}^{\infty}\left[x^{j}(t)-x^{j-1}(t)\right]$$

converges absolutely and uniformly on $[1, b]$, say to $\hat{x}(t)$ in the norm $\|\cdot\|$. Define

$$x(t)=\begin{cases} y(t), & t \in[1-r, 1], \\ \hat{x}(t), & t \in[1, b] .\end{cases}$$

Proceeding as in the proof of Theorem 4.5, one can show that x is a solution of (1) with $x^{(k)}(1) = y^{(k)}(1)$, $k = 0, 1, 2, \ldots, m-1$, that satisfies

$$\|y(t) - x(t)\| \leq \frac{\lambda}{(1 - \lambda M)} \varepsilon \eta(t), \qquad t \in [1 - r, b] .$$

Therefore, (1) is Ulam-Hyers-Rassias stable. $\square$

5. E_α-Ulam-Hyers stability

Now we consider the following definitions of E_α-Ulam-Hyers stability introduced by Wang and Li [**19**].

Definition 5.1. The equation (1) is E_α-Ulam-Hyers stable if there exists a real number $K_f > 0$ such that for each $\varepsilon > 0$ and each y $[1 - r, b] \to \mathbb{R}^n$ in B satisfies the inequality

$$\|\mathfrak{D}_1^\alpha y(t) - f(t, y_t)\| \leq \varepsilon, \qquad t \in [1, b] ,$$

then there exists a solution x $[1 - r, b] \to \mathbb{R}^n$ of (1) in B with

$$\|y(t) - x(t)\| \leq K_f E_\alpha \left(\delta_f (\log t)^\alpha\right) \varepsilon, \qquad \delta_f \geq 0,\ t \in [1 - r, b] .$$

Definition 5.2. The equation (1) is generalized E_α-Ulam-Hyers stable if there exists $\varphi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\varphi_f(0) = 0$ such that for each $\varepsilon > 0$, if y $[1 - r, b] \to \mathbb{R}^n$in B satisfies

$$\|\mathfrak{D}_1^\alpha y(t) - f(t, y_t)\| \leq \varepsilon, \qquad t \in [1, b] ,$$

then there exists a solution x $[1 - r, b] \to \mathbb{R}^n$ of (1) in B with

$$\|y(t) - x(t)\| \leq \varphi_f(\varepsilon) E_\alpha \left(\delta_f (\log t)^\alpha\right), \qquad \delta_f \geq 0,\ t \in [1 - r, b] .$$

Theorem 5.3. *Let f $[1, b] \times C \to \mathbb{R}^n$ be a continuous function that satisfies Lipschitz condition with respect to the second variable*

$$\|f(t, u) - f(t, v)\| \leq M \|u - v\|_C ,\ t \in [1, b] , \qquad u, v \in C.$$

For every $\varepsilon > 0$, if y $[1 - r, b] \to \mathbb{R}^n$ in B satisfies

$$\|\mathfrak{D}_1^\alpha y(t) - f(t, y_t)\| \leq \varepsilon, \qquad t \in [1, b] ,$$

then there exists unique solution x $[1 - r, b] \to \mathbb{R}^n$ of (1) in B with $x^{(k)}(1) = y^{(k)}(1)$, $k = 0, 1, 2, \ldots, m - 1$, that satisfies,

$$\|y(t) - x(t)\| \leq \frac{1}{M} E_\alpha \left(M (\log t)^\alpha\right) \varepsilon, \qquad t \in [1 - r, b] .$$

Proof. We define the sequence of approximations as in Theorem 4.5. Noting that $x^0(t) = y(t)$, and from (15), (18), and (20), we write

$$\begin{aligned}\|y(t) - x(t)\| &\leq \sum_{j=1}^{\infty} \left\|x^j(t) - x^{j-1}(t)\right\| \leq \frac{\varepsilon}{M} \sum_{j=0}^{\infty} \frac{\left(M (\log t)^\alpha\right)^j}{\Gamma(j\alpha + 1)} \\ &\leq \frac{1}{M} E_\alpha \left(M (\log t)^\alpha\right) \varepsilon, \qquad t \in [1 - r, b] .\end{aligned}$$

This means the problem (1) is E_α-Ulam-Hyers stable with the initial conditions. □

Remark. If we set $\varphi_f(\varepsilon) = \frac{\varepsilon}{M}$, then $\varphi_f(0) = 0$, the proves that the equation (1) is generalized E_α-Ulam-Hyers stable with the initial conditions.

6. An example

To illustrate existence and stability results for fractional delay differential equation obtained in this paper, we give the following example. Since any two norms on a finite dimensional linear spaces are equivalent, here we consider the example in $\mathbb{R}^2$ with the norm

$$\|x\| = |x_1| + |x_2|, \qquad x = (x_1, x_2) \in \mathbb{R}^2.$$

Consider the fractional delay differential equation of the form

$$\mathfrak{D}_1^{\frac{7}{2}} x(t) = f(t, x_t) = \Big(\frac{x_{t1}}{1 + x_{t1}}, \sin(x_{t2})\Big), \qquad t \in [1, e], \tag{30}$$

$$x(t) = (2, 1 + t), \qquad t \in [-1, 1], \tag{31}$$

where $x\ [-1, e] \to \mathbb{R}^2$ and $f\ [1, e] \times C\left([-1, 1], \mathbb{R}^2\right) \to \mathbb{R}^2$ is a nonlinear function. Let

$$f(t, \Psi) = f(t, (\Psi_1, \Psi_2)) \Big(\frac{\Psi_1}{1 + \Psi_1}, \sin(\Psi_2)\Big).$$

Then for any $\Psi, \phi \in C\left([-1, 1], \mathbb{R}^2\right)$, we find

$$\begin{aligned}
&\|f(t, \Psi) - f(t, \phi)\| \\
&= \|f(t, (\Psi_1, \Psi_2)) - f(t, (\phi_1, \phi_2))\| = \left\|\Big(\frac{\Psi_1}{1 + \Psi_1}, \sin(\Psi_2)\Big) - \Big(\frac{\phi_1}{1 + \phi_1}, \sin(\phi_2)\Big)\right\| \\
&= \left\|\Big(\frac{\Psi_1}{1 + \Psi_1} - \frac{\phi_1}{1 + \phi_1}, \sin(\Psi_2) - \sin(\phi_2)\Big)\right\| \\
&= \left|\frac{\Psi_1}{1 + \Psi_1} - \frac{\phi_1}{1 + \phi_1}\right| + |\sin(\Psi_2) - \sin(\phi_2)| \le |\Psi_1 - \phi_1| + |\Psi_2 - \phi_2| = \|\Psi - \phi\|.
\end{aligned}$$

Therefore, $\|f(t, \Psi) - f(t, \phi)\| \le \|\Psi - \phi\|$ for all $\Psi, \phi \in C\left([-1, 1], \mathbb{R}^2\right)$. This implies f satisfies Lipschitz condition with Lipschitz constant $M = 1$. Hence by Theorem 3.4, (30)–(31) has a unique solution. Further, if $y \in B = C\left([-1, e], \mathbb{R}^2\right)$ satisfies

$$\left\|\mathfrak{D}_1^{\frac{7}{2}} y(t) - f\right\| \le \varepsilon, \qquad t \in [1, e],$$

then as shown in Theorem 4.5, there exists a solution $x \in B$ of (30) such that

$$\|y(t) - x(t)\| \le \left(\frac{E_{\frac{7}{2}}\left(\left(\log e\right)^{\frac{7}{2}}\right) - 1}{1}\right)\varepsilon = \left(E_{\frac{7}{2}}(1) - 1\right)\varepsilon, \qquad t \in [-1, e].$$

Other stability results for (30) can be discussed similarly.

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