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RELATING TOTAL DOUBLE ROMAN DOMINATION TO 2-INDEPENDENCE IN TREES

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ABSTRACT. A *double Roman dominating function* (DRDF) on a graph $G = (V, E)$ is a function $f: V \to \{0, 1, 2, 3\}$ having the property that if $f(u) = 0$, then a vertex u has at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(u) = 1$, then a vertex u must have at least one neighbor w with $f(w) \geq 2$. A total double Roman dominating function (TDRDF) on a graph G with no isolated vertex is a DRDF f on G with the additional property that the subgraph of G induced by the set $\{v \in V : f(v) \neq 0\}$ has no isolated vertices. The weight of a total double Roman dominating function f is the value, $f(V) = \Sigma_{u\in V(G)} f(u)$. The *total double Roman domination number* $\gamma_{tdR}(G)$ is the minimum weight of a TDRDF on G. A subset S of V is a 2-independent set of G if every vertex of S has at most one neighbor in S. The maximum cardinality of a 2-independent set of G is the 2-independence number $\beta_2(G)$. In this paper, we show that if T is a tree, then $\gamma_{tdR}(T) \leq 2\beta_2(T)$, and we characterize all trees attaining the equality.

1. INTRODUCTION

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [**16, 17**]. In this paper, G is a simple graph with a vertex set $V = V(G)$ and a edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A vertex v with $\deg(v) = 1$ is called a leaf. The neighbor of a leaf is called a support vertex, while a support vertex having two or more adjacent leaves is called a strong support vertex. For a vertex v in a (rooted) tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively, and let $D[v] = D(v) \cup \{v\}$. Also, the *depth of* v, depth(v), is the largest distance from v to a vertex in $D[v]$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. We denote the set of leaves adjacent to a vertex v

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by L_v. The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is the greatest distance between two vertices of G. We write P_n for the *path* of order n and $K_{1,n-1}$ for the *star* of order n. A *double star* $DS_{p,q}$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other one to s leaves. For a subset $S \subseteq V(G)$ and a function $f\colon V(G) \to \mathbb{R}$, we define $f(S) = \sum_{x\in S} f(x)$.

A subset S of vertices of G is a *(total) dominating set* if $N[S] = V$ ($N(S) = V$). The *(total) domination number* $\gamma(G)$ ($\gamma_t(G)$) is the minimum cardinality of a (total) dominating set of G, and a (total) dominating set of minimum cardinality is called a γ-set (γ_t-set). For a positive integer k, the subset S is k-*dominating* if every vertex of $V - S$ has at least k neighbors in S. The k-*domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set of G. The literature on the subject of domination parameters in graphs surveyed in the three main books [**16, 17, 18**].

A subset $S \subseteq V(G)$ is said to be *independent* if $G[S]$ has no edges. The *independent domination number* (resp., the independence number) of G denoted by $i(G)$ (resp., $\beta(G)$) is the size of the smallest (resp., the largest) maximal independent set in G. It is well known that

$$\gamma(G) \le i(G) \le \beta(G).$$

In [**15**], Fink and Jacobson generalized the concept of independent sets as follows. Let k be a positive integer, a subset X of V is k-*independent* if $\Delta(G[X]) \le k-1$. The k-*independence number* $\beta_k(G)$ is the maximum cardinality among all k-independent sets of G. A k-independent set with maximum cardinality of a graph G is called a $\beta_k(G)$-set. For additional information we refer the reader to [**12**]. Relations between domination parameters and independence studied by several authors [**3, 11, 13, 14, 19, 21**].

A *Roman dominating function* on a graph G is a function $f\colon V \to \{0,1,2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. A *total Roman dominating function* of a graph G with no isolated vertex, abbreviated TRD-*function*, is a Roman dominating function $f = (V_0, V_1, V_2)$ on G with the additional property that the subgraph of G induced by the set $V_1 \cup V_2$ has no isolated vertices. The *weight* of f is defined by $\omega(f) = f(V(G))$. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight among all total dominating functions on G. A total Roman dominating function with minimum weight $\gamma_{tR}(G)$ in G is called a $\gamma_{tR}(G)$-*function*. For a total Roman dominating function f, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0,1,2$. Since these three sets determine f, we can equivalently write $f = (V_0, V_1, V_2)$. The total Roman domination was introduced by Liu and Chang [**20**] albeit in a more general setting, and studied by several authors [**1, 2, 4, 5, 6**].

In 2016, Beeler et al. [**9**] introduced the double Roman domination defined as follows. A function $f\colon V \to \{0,1,2,3\}$ is a *double Roman dominating function* (DRDF) on a graph G if the following conditions hold: (i) If $f(v) = 0$, then v must have either at least one neighbor with label 3 or at least two neighbors with label 2, and (ii) If $f(v) = 1$, then v must have at least one neighbor with label at least 2. The *weight* of DRDF f is defined by $\omega(f) = f(V(G))$. The *double Roman* *domination number* of a graph G, denoted by $\gamma_{dR}(G)$, is the minimum weight among all double Roman dominating functions of G. A double Roman dominating function with minimum weight $\gamma_{dR}(G)$ in G is called a $\gamma_{dR}(G)$-function. For a double Roman dominating function f, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2, 3$. Since these four sets determine f uniquely, we can equivalently write $f = (V_0, V_1, V_2, V_3)$. We observe that $\omega(f) = |V_1| + 2|V_2| + 3|V_3|$.

A *total double Roman dominating function* of a graph G with no isolated vertex, abbreviated *TDRD-function*, is a double Roman dominating function $f = (V_0, V_1, V_2, V_3)$ on G with the additional property that the subgraph of G induced by $V_1 \cup V_2 \cup V_3$ has no isolated vertices. The *total double Roman domination number* $\gamma_{tdR}(G)$ is the minimum weight among all total double Roman dominating functions on G. The concept of total double Roman domination was introduced by Amjadi et al. [**7**] and has been studied by several authors, eg., [**22**].

Hao et al. [**22**] proved that for any connected graph G of order $n \geq 2$, $\gamma_{tdR}(G) \leq 2\gamma_{tR}(G) - 1$ and Abdollahzadeh Ahangar [**1**] show that $\gamma_{tR}(G) \leq 2\gamma_t(G)$ for every graph G with no isolated vertex. In [**10**], the authors proved that for every graph G without isolated vertices, $\gamma_t(G) \leq \frac{3}{2}\gamma_2(G) - \frac{1}{2}$. Favaron [**14**] proved that for any graph G and positive integer k, $\gamma_k(G) \leq \beta_k(G)$. Combining these results, we obtain the next result.

Proposition 1.1. *For any graph G without isolated vertices, $\gamma_{tdR}(G) \leq 6\beta_2(G) - 3$.*

In this paper, we consider the problem of 2-independent set and total double Roman domination in trees and improve the above bound considerably.

We make use of the following result in this paper.

Observation 1.2. Let v be a support vertex of a graph G and u be a leaf neighbor of v. For any TDRDF of G, $f(u) + f(v) \geq 3$ and $f(v) \geq 1$.

Proposition 1.3 ([**8**]). *Let T' be a tree and let $u \in V(T')$. If T is a tree obtained from T' by adding a path $P_3 = x_1x_2x_3$ and joining u to x_3, then $\beta_2(T) = \beta_2(T') + 2$.*

Proposition 1.4 ([**13**]). *Let T' be a tree and $v \in V(T')$. If T is the tree obtained from T' by adding a path $P_4 = x_1x_2x_3x_4$ and joining v to x_3, then $\beta_2(T) = \beta_2(T') + 3$.*

Proposition 1.5. *Let T' be a tree and let $u \in V(T')$. If T is a tree obtained from T' by adding a path $P_3 = x_1x_2x_3$ and joining u to x_2, then $\beta_2(T) = \beta_2(T') + 2$.*

Proof. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding x_1, x_3, and so $\beta_2(T) \geq \beta_2(T') + 2$. On the other hand, for any $\beta_2(T)$-set S, we have $|S \cap \{x_1, x_2, x_3\}| \leq 2$, and since $S \setminus \{x_1, x_2, x_3\}$ is a 2-independent set of T', we have $\beta_2(T') \geq |S - \{x_1, x_2, x_3\}|$. This implies that $\beta_2(T') \geq \beta_2(T) - 2$. Thus $\beta_2(T) = \beta_2(T') + 2$. $\square$

Proposition 1.6. *Let T' be a tree and let $u \in V(T')$. If T is a tree obtained from T' by adding a path $P_3 = x_1x_2x_3$ and joining u to x_3 or x_2, then $\gamma_{tdR}(T) \leq \gamma_{tdR}(T') + 4$.*

Proof. Clearly, any $\gamma_{tdR}(T')$-function can be extended to a TDRDF of T by assigning 3 to x_2, 1 to x_1, and 0 to x_3, and so $\gamma_{tdR}(T) \le \gamma_{tdR}(T') + 4$. $\square$

Proposition 1.7. *Let T' be a tree and let $u \in V(T')$. If T is a tree obtained from T' by adding a path $P_4 = x_1x_2x_3x_4$ and joining u to x_3, then $\gamma_{tdR}(T) \le \gamma_{tdR}(T') + 6$.*

Proof. Clearly, any $\gamma_{tdR}(T')$-function, can be extended to a TDRDF of T by assigning 3 to x_2, x_3 and 0 to x_1, x_4, and so $\gamma_{tdR}(T) \le \gamma_{tdR}(T') + 6$. $\square$

2. Total double Roman domination and 2-independence

In this section, we show that if T is a tree, then $\gamma_{tdR}(T) \le 2\beta_2(T)$ and we provide a constructive characterization of all trees T with $\gamma_{tdR}(T) = 2\beta_2(T)$. We start with a definition.

Definition 2.1. Let u be a vertex of a graph G. A function $f\colon V(G) \to \{0,1,2,3\}$ is said to be an *almost total double Roman dominating function* (almost TDRDF) with respect to u if each element $v \in V(G) - \{u\}$ is total double Roman dominated under f, that is:

(i) if $v \in V - \{u\}$ and $f(v) = 0$, then v must have at least one neighbor with label 3 or at least two neighbors with label 2,

(ii) if $v \in V - \{u\}$ and $f(v) = 1$, then v must have at least one neighbor with label at least 2, and

(iii) if $v \in V - \{u\}$ and $f(v) \ge 1$, then v must have at least one neighbor with positive label.

Let

$$\gamma_{tdR}(G; u) = \min\{\omega(f) \mid f \text{ is an almost TDRDF with respect to } u\}.$$

Clearly, any total double Roman dominating function on G is an almost TDRDF with respect to each vertex of G. Hence $\gamma_{tdR}(G; u) \le \gamma_{tdR}(G)$ for each $u \in V(G)$. For a graph G, define W_G^1 and W_G^2 as follows:

$$W_G^1 = \{v \in V(G) \mid \gamma_{tdR}(G; v) = \gamma_{tdR}(G)\}$$

and

$$W_G^2 = \{v \in V(G) \mid \text{ for any } \gamma_{tdR}(G)\text{-function } f,\ f(v) \le 1\}.$$

Let $\mathcal{T}$ be the family of unlabeled trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_m$ ($m \ge 1$) of trees such that $T_1 \in \{P_3, P_4\}$, and if $m \ge 2$, T_{i+1} can be obtained recursively from T_i by the following operations for $1 \le i \le m-1$.

Operation $\mathcal{O}_1$. If $u \in W_{T_i}^1 \cap W_{T_i}^2$, then Operation $\mathcal{O}_1$ adds a path $P_3 = x_1x_2x_3$ and the edge ux_3 to obtain T_{i+1}.

Operation $\mathcal{O}_2$. If $u \in W_{T_i}^1$, then Operation $\mathcal{O}_2$ adds $P_3 = x_1x_2x_3$ and the edge ux_2 to obtain T_{i+1}.

Operation $\mathcal{O}_3$. If $u \in W_{T_i}^1$, then Operation $\mathcal{O}_3$ adds a path $P_4 = x_1x_2x_3x_4$ and the edge ux_3 to obtain T_{i+1}.

Lemma 2.2. *If T_i is a tree with $\gamma_{tdR}(T_i) = 2\beta_2(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $\gamma_{tdR}(T_{i+1}) = 2\beta_2(T_{i+1})$.*

Proof. By Propositions 1.3 and 1.6, $\beta_2(T_{i+1}) = \beta_2(T_i) + 2$ and $\gamma_{tdR}(T_{i+1}) \leq \gamma_{tdR}(T_i) + 4$. Assume that f is a $\gamma_{tdR}(T_{i+1})$-function. Clearly, $f(x_2) + f(x_1) \geq 3$. If $f(x_3) \geq 1$ or $f(x_2)+f(x_1) \geq 4$, then the function f restricted to T_i is an almost TDRDF of T_i with respect to u and we deduce from the assumption $u \in W^1_{T_i}$ that $\gamma_{tdR}(T_{i+1}) \geq \gamma_{tdR}(T_i; u) + 4 = \gamma_{tdR}(T_i) + 4$.

Assume that $f(x_3) = 0$ and $f(x_2)+f(x_1) = 3$. Then $f(u) \geq 2$ and the function f restricted to T_i is an TDRDF of T_i of weight $\gamma_{tdR}(T_{i+1}) - 3$ with $f(u) \geq 2$. We deduce from $u \in W^2_{T_i}$ that $\gamma_{tdR}(T_{i+1}) - 3 = \omega(f|_{T_i}) \geq \gamma_{tdR}(T_i) + 1$, and so $\gamma_{tdR}(T_{i+1}) \geq \gamma_{tdR}(T_i) + 4$. Therefore, $\gamma_{tdR}(T_{i+1}) = \gamma_{tdR}(T_i) + 4$. It follows from $\gamma_{tdR}(T_i) = 2\beta_2(T_i)$ that

$$2\beta_2(T_{i+1}) = 2\beta_2(T_i) + 4 = \gamma_{tdR}(T_i) + 4 = \gamma_{tdR}(T_{i+1}),$$

as desired. □

Lemma 2.3. *If T_i is a tree with $\gamma_{tdR}(T_i) = 2\beta_2(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma_{tdR}(T_{i+1}) = 2\beta_2(T_{i+1})$.*

Proof. The proof is similar to the proof of Lemma 2.2 and therefore, omitted. □

Lemma 2.4. *If T_i is a tree with $\gamma_{tdR}(T_i) = 2\beta_2(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_3$, then $\gamma_{tdR}(T_{i+1}) = 2\beta_2(T_{i+1})$.*

Proof. Let Operation $\mathcal{O}_3$ add a path $P_4 = x_4x_3x_2x_1$ and the edge ux_3 to obtain T_{i+1}. By Propositions 1.4 and 1.7, we have $\beta_2(T_{i+1}) = \beta_2(T_i)+3$ and $\gamma_{tdR}(T_{i+1}) \leq \gamma_{tdR}(T_i) + 6$. We now prove that $\gamma_{tdR}(T_{i+1}) \geq \gamma_{tdR}(T_i) + 6$-function. Let f be a $\gamma_{tdR}(T_{i+1})$. By Observation 1.2, $f(x_1) + f(x_2) \geq 3$ and $f(x_3) + f(x_4) \geq 3$. We may assume that $f(x_3) = f(x_2) = 3$ and $f(x_1) = f(x_4) = 0$. Then the function f restricted to T_i is an almost TDRDF of T_i with respect to u and we deduce from the assumption $u \in W^1_{T_i}$ that $\gamma_{tdR}(T_{i+1}) \geq \gamma_{tdR}(T_i; u) + 6 = \gamma_{tdR}(T_i) + 6$. Hence $\gamma_{tdR}(T_{i+1}) = \gamma_{tdR}(T_i) + 6$. As in the proof of Lemma 2.2, we obtain $\gamma_{tdR}(T_{i+1}) = 2\beta_2(T_{i+1})$. □

Lemma 2.5. *If $T \in \mathcal{T}$, then $\gamma_{tdR}(T) = 2\beta_2(T)$.*

Proof. Let $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that $T_1 \in \{P_3, P_4\}$ and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of Operations $\mathcal{O}_1, \mathcal{O}_2$, and $\mathcal{O}_3$ for $i = 1, 2, \ldots, k-1$.

We proceed by induction on the number of operations applied to construct T. Clearly, the statement is true for T_1, that is $\gamma_{tdR}(T_1) = 2\beta_2(T_1)$. Suppose that the result is true for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_{tdR}(T') = 2\beta_2(T')$. By construction, T is obtained from T' by using one of the operations $\mathcal{O}_1, \mathcal{O}_2$, or $\mathcal{O}_3$. It follows from Lemmas 2.2, 2.3 and 2.4 that $\gamma_{tdR}(T) = 2\beta_2(T)$, and the proof is complete. □

Next, we are ready to prove the main result of this paper.

Theorem 2.6. *For every tree T of order $n \geq 2$,*

$$\gamma_{tdR}(T) \leq 2\beta_2(T)$$

with equality if and only if $T \in \mathcal{T}$.

Proof. Let T be a tree of order $n \geq 2$. The proof is by induction on n. If $n = 2$, then $T = P_2$ and we have $\gamma_{tdR}(T) = 3 < 4 = 2\beta_2(T)$. If $n = 3$, then $T = P_3$ and we have $\gamma_{tdR}(T) = 4 = 2\beta_2(T)$. Let $n \geq 3$ and let the statement hold for any tree of order less than n. If $\mathrm{diam}(T) = 2$, then T is the star $K_{1,n-1}$ and we have $\gamma_{tdR}(T) = 4 \leq 2(n-1) = 2\beta_2(T)$ with equality if and only if $n = 3$, that is, $T = P_3 \in \mathcal{T}$. If $\mathrm{diam}(T) = 3$, then T is a double star $DS_{p,q}$ ($q \geq p \geq 1$), and we have $\gamma_{tdR}(T) = 6 \leq 2\beta_2(T)$ with equality if and only if $p = q = 1$, that is, $T = P_4 \in \mathcal{T}$. Assume that $\mathrm{diam}(T) \geq 4$ and let $v_1v_2\ldots v_{d+1}$ be a diametrical path in T such that $\deg(v_2)$ is as large as possible. Root T at v_{d+1}.

Assume first that $k = \deg(v_2) \geq 4$. Let $T' = T - T_{v_2}$. Clearly, every $\beta_2(T')$-set can be extended to a 2-independent set of T by adding all leaves adjacent to v_2, and this implies that $\beta_2(T) \geq \beta_2(T') + k - 1$. On the other hand, any $\gamma_{tdR}(T')$-function can be extended to a TDRDF of T by assigning the value 3 to v_2, 1 to v_1, and 0 to other leaves of T_{v_2}, and this implies that $\gamma_{tdR}(T) \leq \gamma_{tdR}(T') + 4$. It follows from the induction hypothesis that

$$\begin{aligned} 2\beta_2(T) &\geq 2\beta_2(T') + 2k - 2 \geq \gamma_{tdR}(T') + 2k - 2 \\ &\geq \gamma_{tdR}(T) - 4 + 2k - 2 > \gamma_{tdR}(T). \end{aligned}$$

Let $\deg(v_2) \leq 4$. We consider the following cases.

Case 1. $\deg(v_2) = 3$.
Let $L_{v_2} = \{v_1, u\}$ and $T' = T - T_{v_2}$. Clearly, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding two leaves of T_{v_2}, and so $\beta_2(T) \geq \beta_2(T') + 2$. On the other hand, any $\gamma_{tdR}(T')$-function g can be extended to a $TDRDF$ of T by assigning 3 to v_2, 1 to v_1, and 0 to u, implying that $\gamma_{tdR}(T) \leq \gamma_{tdR}(T') + 4$. It follows from the induction hypothesis that

$$2\beta_2(T) \geq 2\beta_2(T') + 4 \geq \gamma_{tdR}(T') + 4 \geq \gamma_{tdR}(T) - 4 + 4 = \gamma_{tdR}(T).$$

Let the equality hold. Then all inequalities occurring in the above inequality chain, become equalities and so $\beta_2(T) = \beta_2(T') + 2$, $\gamma_{tdR}(T) = \gamma_{tdR}(T') + 4$ and $\gamma_{tdR}(T') = 2\beta_2(T')$. We conclude from the induction hypothesis that $T' \in \mathcal{T}$. We show that $v_3 \in W^1_{T'}$. Suppose to the contrary that $v_3 \notin W^1_{T'}$ and let g be a almost $TDRDF$ of T' with respect to v_3 of weight at most $\gamma_{tdR}(T') - 1$. Define $h : V(T) \to \{0,1,2,3\}$ by $h(w) = g(w)$ for $w \in V(T'), h(v_2) = 3$, $h(v_1) = 1$, and $h(u) = 0$. Clearly, h is a $TDRDF$ of T of weight $\gamma_{tdR}(T') + 3$, which leads to a contradiction. Hence, $v_3 \in W^1_{T'}$. Now, T can be obtained from T' by Operation $\mathcal{O}_2$, and so $T \in \mathcal{T}$.

Case 2. $\deg(v_2) = 2$.
By the choice of diametrical path, all children of v_3 with depth 1, have degree 2.

Assume that v_3 has t children with depth 1. Then $t \geq 1$. We distinguish the following subcases.

Subcase 2.1. $\deg(v_3) = 2$.
Let $T' = T - T_{v_3}$. By Propositions 1.3 and 1.6, $\beta_2(T) \geq \beta_2(T') + 2$ and $\gamma_{tdR}(T) \leq \gamma_{tdR}(T') + 4$. It follows from the induction hypothesis that

$$2\beta_2(T) \geq 2\beta_2(T') + 4 \geq \gamma_{tdR}(T') + 4 \geq \gamma_{tdR}(T) - 4 + 4 = \gamma_{tdR}(T).$$

If the equality holds, then all inequalities occurring in the above inequality chain, become equalities, and so $\beta_2(T) = \beta_2(T') + 2$, $\gamma_{tdR}(T) = \gamma_{tdR}(T') + 4$, and $\gamma_{tdR}(T') = 2\beta_2(T')$. We conclude from the induction hypothesis that $T' \in \mathcal{T}$. We prove now that $v_4 \in W^1_{T'} \cap W^2_{T'}$. First we show that $v_4 \in W^1_{T'}$. Suppose to the contrary that $v_4 \notin W^1_{T'}$ and let g be an almost $TDRDF$ of T' with respect to v_4 of weight at most $\gamma_{tdR}(T') - 1$. Define $h\colon V(T) \to \{0, 1, 2, 3\}$ by $h(u) = g(u)$ for $u \in V(T')$, $h(v_2) = 3$, $h(v_3) = 1$, and $h(v_1) = 0$. Clearly, h is a $TDRDF$ of T of weight $\gamma_{tdR}(T') + 3$, which leads to a contradiction. Hence, $v_4 \in W^1_{T'}$. Next, we show that $v_4 \in W^2_{T'}$. Suppose to the contrary that $v_4 \notin W^2_{T'}$ and let g be a $\gamma_{tdR}(T')$-function with $g(v_4) \geq 2$. Define $h\colon V(T) \to \{0, 1, 2, 3\}$ by $h(u) = g(u)$ for $u \in V(T')$, $h(v_1) = 1, h(v_2) = 2$, and $h(v_3) = 0$. Obviously h is a $TDRDF$ of T of weight $\gamma_{tdR}(T') + 3$, which leads to a contradiction. Hence $v_4 \in W^2_{T'}$. Now, T can be obtained from T' by Operation $\mathcal{O}_1$, and so $T \in \mathcal{T}$.

Subcase 2.2. $t \geq 2$ and v_3 is not a support vertex.
Let $T' = T - T_{v_3}$. Obviously, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding all vertices of T_{v_3} except v_3, and so $\beta_2(T) \geq \beta_2(T') + 2t$. On the other hand, any $\gamma_{tdR}(T')$-function can be extended to a $TDRDF$ of T by assigning 1 to v_3, 3 to all children of v_3 with depth 1 and, 0 to all leaves of T_{v_3}, and so $\gamma_{tdR}(T) \leq \gamma_{tdR}(T') + 1 + 3t$. It follows from the induction hypothesis that

$$\begin{aligned}2\beta_2(T) \geq 2\beta_2(T') + 4t \geq \gamma_{tdR}(T') + 4t &\geq \gamma_{tdR}(T) - 1 - 3t + 4t \\ &= \gamma_{tdR}(T) + t - 1 > \gamma_{tdR}(T).\end{aligned}$$

Subcase 2.3. v_3 has $\ell \geq 1$ children with depth 0 and $t \geq 1$ children with depth 1. Assume that $T' = T - T_{v_3}$. First let $\ell + t \geq 3$. Obviously, any $\beta_2(T')$-set can be extended to a 2-independent set of T by adding $D(v_3)$, and so $\beta_2(T) \geq \beta_2(T') + 2t + l$. On the other hand, any $\gamma_{tdR}(T')$-function can be extended to a TDRDF of T by assigning 3 to v_3, and all children of v_3 with depth 1, and 0 to all leaves of T_{v_3}, and hence $\gamma_{tdR}(T) \leq \gamma_{tdR}(T') + 3 + 3t$. It follows from the induction hypothesis that

$$\begin{aligned}2\beta_2(T) &\geq 2\beta_2(T') + 4t + 2l \geq \gamma_{tdR}(T') + 4t + 2l \\ &\geq \gamma_{tdR}(T) - 3 - 3t + 4t + 2l = \gamma_{tdR}(T) + t + 2l - 3 > \gamma_{tdR}(T).\end{aligned}$$

Now let $\ell + t = 2$. Then we have $\ell = t = 1$. Suppose w is the leaf adjacent to v_3. As mentioned above, $\beta_2(T) \geq \beta_2(T') + 3$ and $\gamma_{tdR}(T) \leq \gamma_{tdR}(T') + 6$. It follows from the induction hypothesis that

$$2\beta_2(T) \geq 2\beta_2(T') + 6 \geq \gamma_{tdR}(T') + 6 \geq \gamma_{tdR}(T).$$

If the equality holds, then all inequalities occurring in above inequality chain, become equalities, and so $\beta_2(T) = \beta_2(T') + 3, \gamma_{tdR}(T) = \gamma_{tdR}(T') + 6$, and $\gamma_{tdR}(T') = 2\beta_2(T')$. We conclude from the induction hypothesis that $T' \in \mathcal{T}$. We prove now that $v_4 \in W^1_{T'}$. Suppose to the contrary that $v_4 \notin W^1_{T'}$ and let g be an almost TDRDF of T' with respect to v_4 of weight at most $\gamma_{tdR}(T') - 1$. Define $h\colon V(T) \to \{0,1,2,3\}$ by $h(u) = g(u)$ for $u \in V(T')$, $h(v_2) = h(v_3) = 3$, and $h(v_1) = h(w) = 0$. Clearly, h is a $TDRDF$ of T of weight $\gamma_{tdR}(T') + 3$, which leads to a contradiction. Hence, $v_4 \in W^1_{T'}$. Now, T can be obtained from T' by Operation $\mathcal{O}_3$, and so $T \in \mathcal{T}$.
This completes the proof. □

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